Title: Higher-order quantum computation and causality

Speakers: Mio Murao

Collection/Series: Causalworlds

Subject: Quantum Foundations, Quantum Information

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Abstract:

Supermaps are higher-order transformations that take maps as input. We explore quantum algorithms that implement supermaps of unitary operations using multiple calls to a black-box unitary operation. We investigate how the causal structure and spacetime symmetry of these unitary black-boxes affect their performance in implementing higher-order quantum operations. We analyze several tasks, inversion, complex conjugation, and transposition of black-box unitaries.

Higher-order quantum computation and causality

Mio Murao

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In collaboration with Satoshi Yoshida, Wataru Yokojima, Tim Forror, Philip Taranto, Jisho Miyazaki, Yutaka Hashimoto, Tatsuki Odake, Hlér Kristjánsson, Akihito Soeda, Marco Túlio Quintino, Jessica Bavaresco, Vanessa Brzic, Atushi Shimbo, Qingxiuxiong Dong, Shojun Nakayama, Matt Wilson, Dmitry Grinko, Michał Studzinski, Tomasz Młynik

(current and former members, and visitors of our group)

What is
higher-order quantum computation?

Well, starting from
(normal-order) quantum computers…

usual (**normal-order**) quantum computers:

Quantum computer for processing quantum **states**

variable

an arbitrary quantum **state**
(can be an **unknown** state)

ρ → physical system

→ $\rho' = f(\rho)$: output quantum state (can be unknown)

$f = \ldots$ command — A CPTP map, a quantum instrument, or a POVM

"quantum state processor" $\rho \to f(\rho)$

Higher-order quantum computers:

Quantum computer for **processing quantum maps**

a supermap: a map of a map

variables

An arbitrary quantum **map** (can be a **blackbox**) $-f-$ →  → $-F(f)-$

$F = \dots$

command

not a super-operator in general, it can be a super-super operator

Including a quantum tester

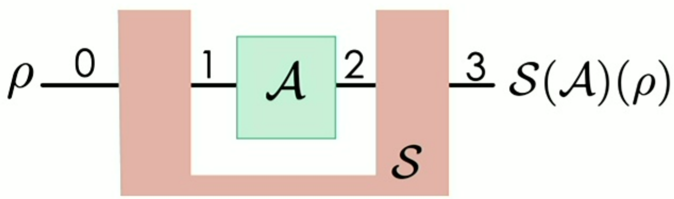
"Quantum **process** processor" $f \to F(f)$

Also known as
- = **superchannels and superinstruments** (Calgary group, our group)
- = **quantum supermaps, quantum testers** (Paiva group)
- = **Processes**, or **process matrices** (Vienna group)

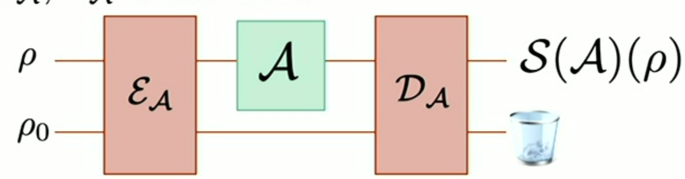
Tool: Formulation of higher-order quantum transformations (supermaps)

G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL (2008), PRA (2009)

The quantum comb formalism provides the condition to implement a supermap $\mathcal{S}$ by the conditions for the **Choi operator** of the quantum comb $\mathcal{J}_\mathcal{S}$ representing the supermap such that $\mathcal{S}(J_\mathcal{A}) = J_{\mathcal{S}(\mathcal{A})}$ $(\mathcal{J}_\mathcal{S} * J_\mathcal{A} = J_{\mathcal{S}(\mathcal{A})})$



$\mathcal{E}_\mathcal{A}, \mathcal{D}_\mathcal{A}$: unitaries



Single-slot quantum comb: $\mathcal{S}$ supermap

quantum circuit representation

Choi operator: $\mathcal{J}_\mathcal{S}$

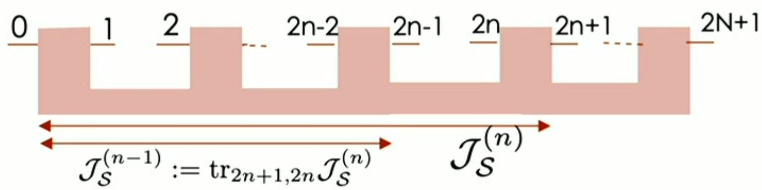
The Choi operator of a quantum comb:

$$\mathcal{J}_\mathcal{S} \in \mathcal{B}((\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_0 \otimes \mathcal{H}_4))$$

(1) $\mathcal{J}_\mathcal{S} \geq 0 \Leftrightarrow \mathcal{S}$ is completely CP preserving

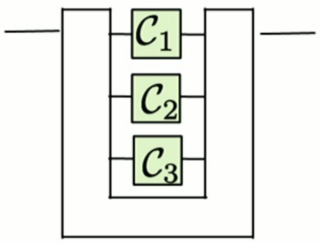
(2) $\mathrm{tr}_{123}\mathcal{J}_\mathcal{S} = I_0 \Leftrightarrow \mathcal{S}$ is trace preserving (deterministic)

(3) $\mathrm{tr}_3\mathcal{J}_\mathcal{S} = \mathrm{tr}_{23}\mathcal{J}_\mathcal{S} \otimes I_2$ ← the requirement of **causal order**

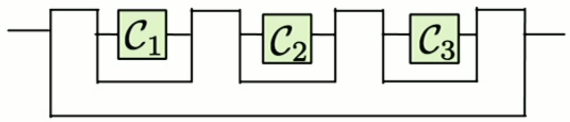


Multiple-slot case: $\mathrm{tr}_{2n+1}\mathcal{J}_\mathcal{S}^{(n)} = \mathrm{tr}_{2n,2n+1}\mathcal{J}_\mathcal{S}^{(n-1)} \otimes \dfrac{I_{2n}}{d_{2n}}$ for $n = 0, 1, \cdots, N$

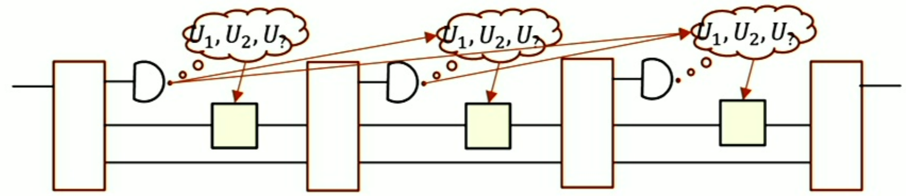
Causal order structures of higher-order quantum transformations



parallel



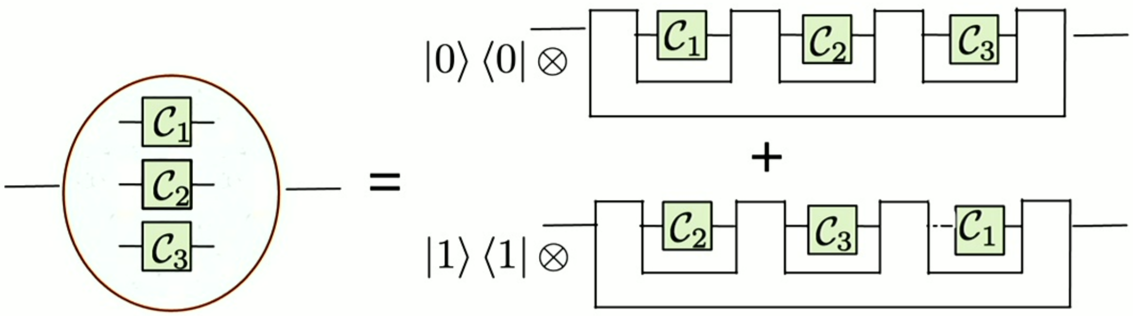
sequential



Dynamical ordering
(classical control of causal order)

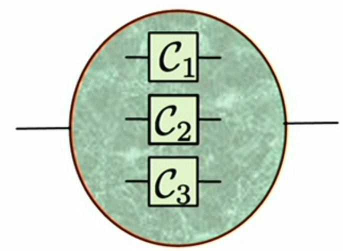
J. Wechs, H. Dourdent, A. A. Abbott, and C. Branciard, PRX Quantum (2021)

Indefinite causal order
cannot be written by a q. circuit



+
⋮ (superposition)

quantum switch



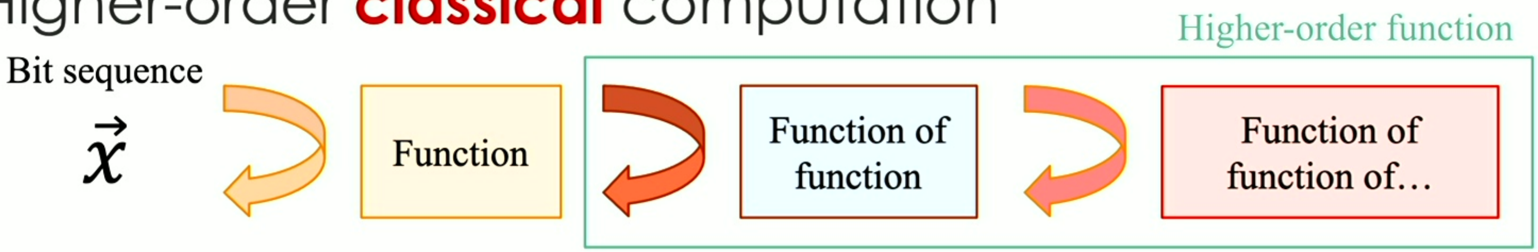
General Indefinite
causal order

O. Oreshkov, F. Costa, and C. Brukner, Nat Commun 3, 1092 (2012), G. Chilibella et al., Phys. Rev. A 88, 022318 (2013)

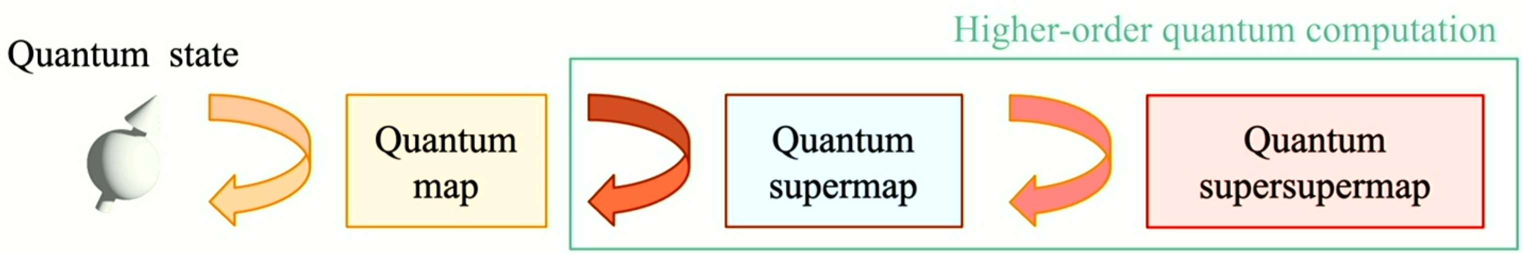
Two slot pure superchannel → quantum switch-type: W. Yokojima, M. T. Quintino, A. Soeda and M. Murao, Quantum (2021)

The concept of **higher-order** computation

- Higher-order **classical** computation



- Higher-order **quantum** computation

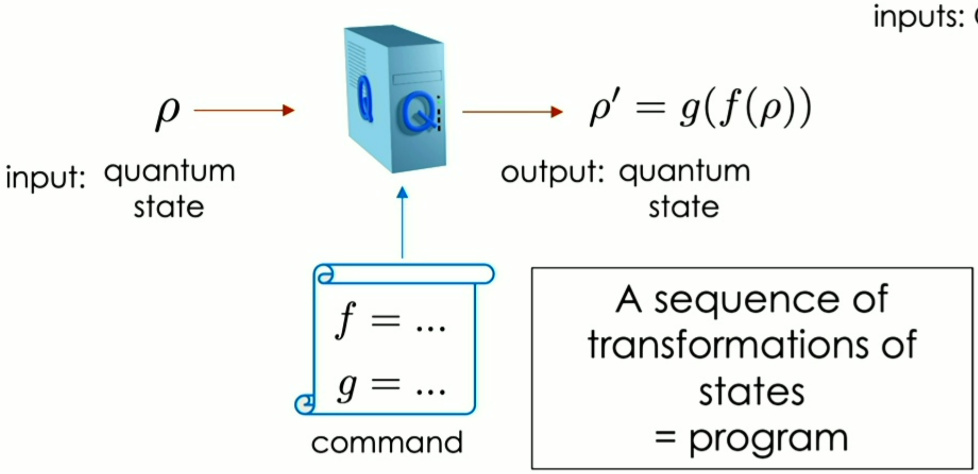


A. Bisio and P. Perinotti, Proc. R. Soc. A 475, 20180706 (2019).

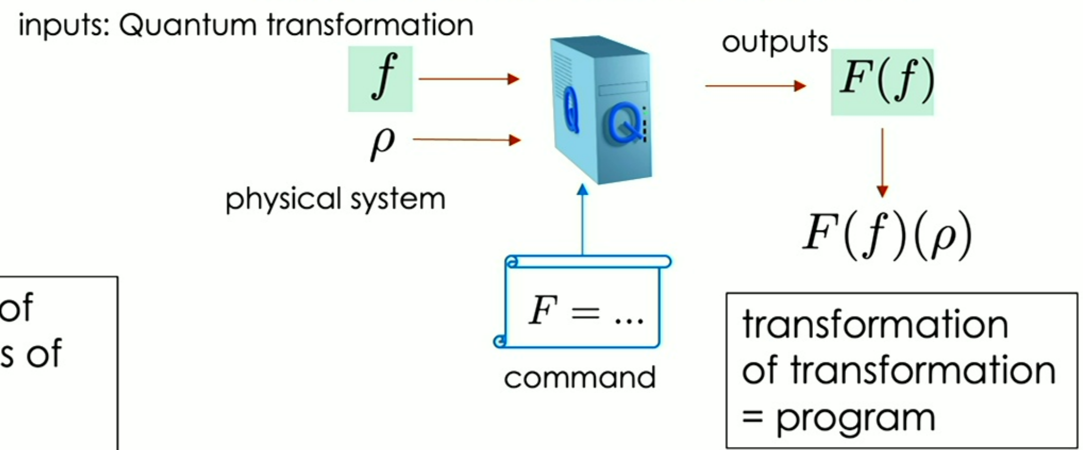
→ Functional Programming

Higher-order quantum computation

(normal-order) quantum computation



higher-order quantum computation



Seek **new quantum algorithms** and **applications** for quantum computers

Revealing **new characteristic properties** of programmable quantum systems

based on **higher-order quantum computation (functional quantum programming)**

A new frontier of quantum information and computation!

In our group, we investigate

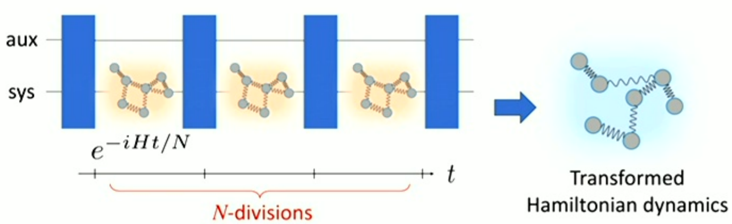
higher-order quantum computation

for developing a new paradigm of functional quantum programming

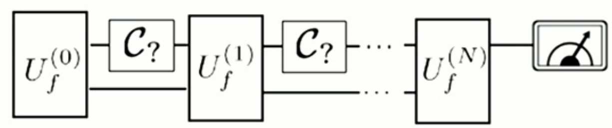
- *Aiming to develop a new framework of* ***functional quantum programming***
  - ➢ *Analyze* ***implementation algorithms*** *of higher-order quantum transformations*
  - ➢ *Formulate* ***compositions*** *of higher-order quantum transformations for programming quantum algorithms in a functional programming manner*

- *Analyze* ***causal order structures*** *in higher-order quantum computation*

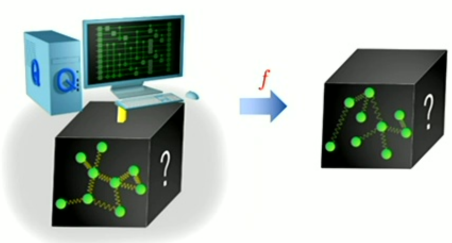
- *Seeking applications for quantum simulation, sensors, and process controllers/processor*



Quantum simulator



Quantum tester/sensor



Quantum dynamics processor
active quantum control

Quantum functional programming requires quantum functions of functions, higher-order quantum transformations!

Higher-order quantum transformations for **unitary operations** (black box)

Not a complete list...

- Examples useful transformations for quantum programming: $\mathcal{S} : U \to f(U) = V_U$
  - **Replication** $V_U = U \otimes U$ — G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL (2008)
  - **Inversion** $V_U = U^{-1} = U^\dagger$ **"undo"** — M.T. Quintino, Q. Dong, A. Shimbo, A. Soeda and M. Murao, PRL (2019); S. Yoshida, A. Soeda and M. Murao, PRL (2023); Y.-A. Chen, Y. Mo, Y. Liu, L. Zhang, and X. Wang, arXiv:2403.04704
  - **Complex conjugation** in terms of a fixed basis $V_U = U^*$ — J. Miyazaki, A. Soeda and M. Murao, PRR (2019); D. Ebler et al., IEEE Tran Info Theory (2023)
  - **Transposition** in terms of a fixed basis $V_U = U^T$ — M.T. Quintino, Q. Dong, A. Shimbo, A. Soeda and M. Murao, PRA (2019); D. Grinko and M. Ozols, arXiv:2207.05713
  - **Controllization** up to phase $V_U = |0\rangle\langle 0| \otimes I + e^{i\theta_U}|1\rangle\langle 1| \otimes U$ — Q. Dong, S. Nakayama, A. Soeda and M. Murao, arXiv:1911.01645v3
  - **Quantum switch** $V_{U_1,U_2} = |0\rangle\langle 0| \otimes U_1U_2 + |1\rangle\langle 1| \otimes U_2U_1$ — G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, PRA (2013)
  - **Neutralization** $V_U = I$ ← also known as refocusing, resetting, rewinding — D. Trillo, B. Dive, and M. Navascués, PRL (2023)
  - **Homomorphic and anti-homomorphic higher-order functions** — M.T. Quintino and D. Ebler, Quantum (2022)

Most of these cannot be universally implemented with a **single call** of U

Need to utilize **multiple calls** or **divisible calls*** of the black box

*Q. Dong, S. Nakayama, A. Soeda and M. Murao, arXiv:1911.01645v3 (some updates in 2021)
T. Odake, H. Kristjánsson A. Soeda M. Murao, Phys. Rev. Res. (2024)

What about **causality** in **higher-order** quantum computation?

Causality in higher-order quantum computation

Two different type of causality issues

- Manipulation of **causality of a quantum map itself**: what is the **cost** of manipulating causality of a quantum map?

$$U \to U^{-1}$$

Undo:
Negative-time evolution

$$U \to U^T$$

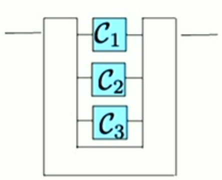
Input-out exchange*:
time-reversal

$$U \to U^*$$

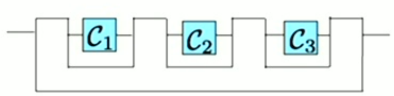
??:
time-reversal negative-time evolution

$$U = e^{-iHt},\ U^T = e^{-iH^*t},\ U^\dagger = e^{iHt},\ U^* = e^{iH^*t}$$

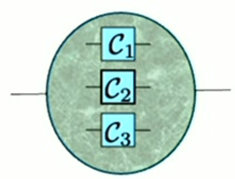
- How does **the causal structure** (parallel, sequential, quantum switch, general indefinite) **between the use of quantum maps** affect the performance of higher-order quantum computation?



parallel



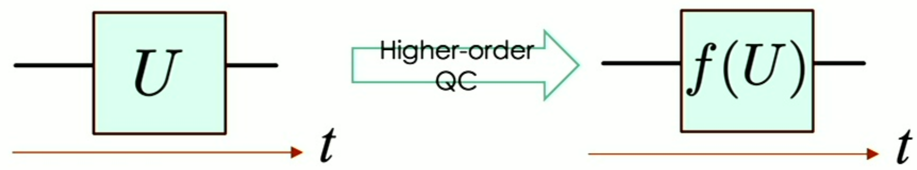
sequential



General Indefinite causal order

*G. Chiribella and Z. Liu, Communication Physics 5, 190 (2022)

In this talk,



- We show our results on higher-order quantum computation for unitaries concerning causality
- In particular, we show how to perform unitary inversion, transposition, and complex conjugation of an unknown unitary, which can be regarded as **causality manipulations**

unitary inversion	**unitary transposition**	**unitary complex conjugation**
$U \to U^{-1} = U^\dagger$	$U \to U^T$	$U \to U^*$
Undo: Negative-time evolution	Input-out exchange: time-reversal	??: time-reversal negative-time evolution

for Hamiltonian dynamics: $U = e^{-iHt},\ U^T = e^{-iH^*t},\ U^\dagger = e^{iHt},\ U^* = e^{iH^*t}$

How are they “difficult”? How much causal resource needed?

Evaluating "difficulty" of causality manipulations of unitaries

$U \to U^{-1}$ $U \to U^{T}$ $U \to U^{*}$

How does the cost of causal manipulations of unitaries depend on the **causal structure of multiple calls**?

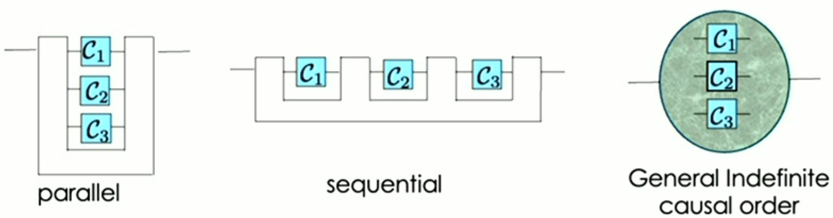
query complexity
+
causal complexity

- What is the **minimum number of calls** for **exact and deterministic** implementation? – but is it possible with a finite number of calls*?

Or, for a given number of calls,

- What is the **maximum probability** of success for exact implementation?
- What is the **minimum error** for approximate implementation?

What type of causal structure?



same unitaries in each slot
→ cannot be a quantum switch-type

*Note that if infinite calls of unitary are allowed, we can implement any supermap by first performing unitary tomography, then calculating the output unitary and implementing the output unitary

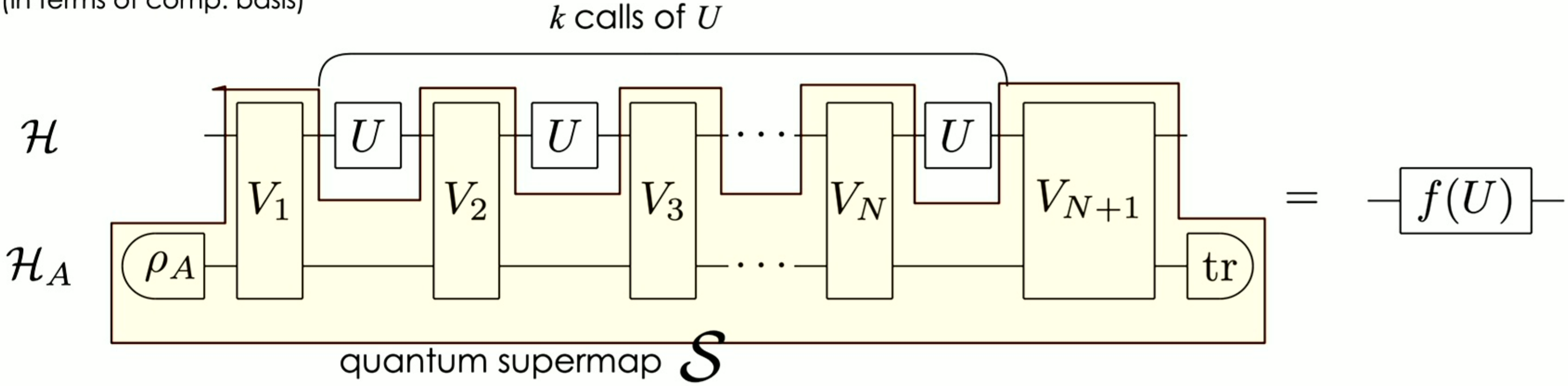
Higher-order quantum transformations of **blackbox unitaries**

Implementing $\mathcal{S}: U \to f(U)$ with multiple calls

- Inversion $f(U) = U^\dagger$ $\quad U^\dagger = (U^*)^T$
- Transposition $f(U) = U^T$ $\quad U^T = (U^\dagger)^*$
- Complex conjugation $f(U) = U^*$ $\quad U^* = (U^T)^\dagger$

Which is more difficult?

(in terms of comp. basis)



What is the minimum number of calls (queries) for implementing $f(U)$?

Evaluating "difficulty" of causality manipulations of unitaries

$$U \to U^{-1} \qquad\qquad U \to U^{T} \qquad\qquad U \to U^{*}$$

How does the cost of causal manipulations of unitaries depend on the **causal structure of multiple calls**?

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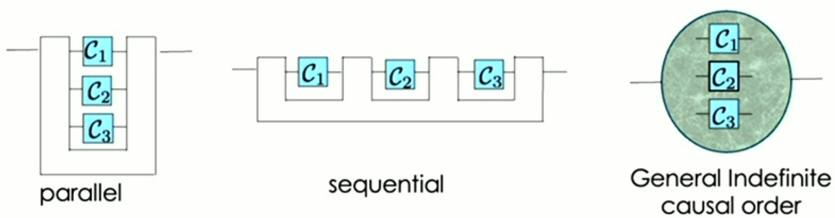


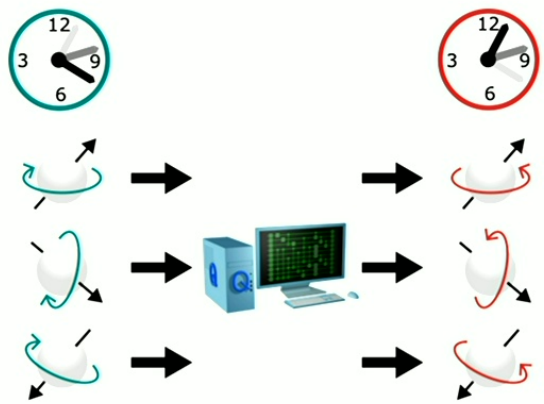
same unitaries in each slot
→ cannot be a quantum switch-type

*Note that if **infinite calls** of unitary are allowed, we can implement any supermap by first performing unitary tomography, then calculating the output unitary and implementing the output unitary

Unitary inversion

$$U \to U^{-1} = U^\dagger$$

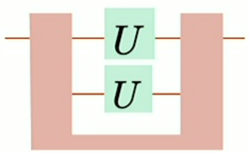
Undo:
Negative-time evolution



Quintino, Dong, Shimbo, Soeda and Murao, PRL2019

Unitary inversion $U \to U^{-1} = (U^*)^T$

Parallel vs. sequential uses of blackboxes



- k **parallel** calls – using port-based transposed teleportation + conjugation:

$$p \leq 1 - \frac{d^2 - 1}{k(d-1) + d^2 - 1} \Rightarrow p \leq 1 - O(1/k) \text{ for } k > d - 1$$

Sequential use of blackboxes greately helps!

By construction not optimal

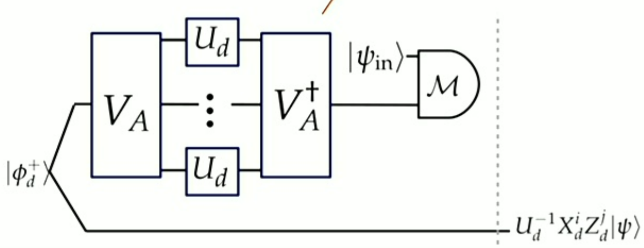
- k **sequential** calls: (not optimal)

$$p = 1 - (1 - \tfrac{1}{d^2})^{\lfloor \frac{k+1}{d} \rfloor} \Rightarrow p \geq 1 - O(\alpha^{-N}) \quad \text{for } k \geq d^2$$

Transposed-gate teleportation of conjugation $(U^*)^T Z^j X^i |\phi\rangle$ → reset to the initial state $|\phi\rangle$ by applying $Z^j X^i U$

Repeat until success!

$i, j = 0$ success! $U^{-1}|\phi\rangle$

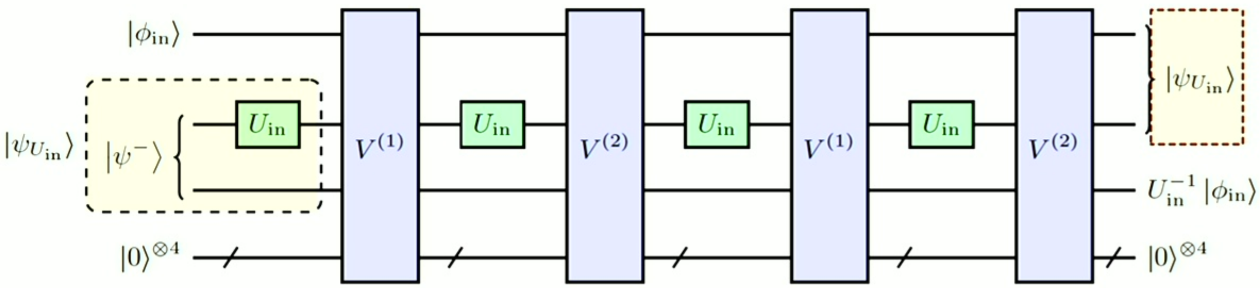


Sequential schemes cannot be used in the delayed protocol, the input state has to be inserted before the second use of the unitary.

S. Yoshida, A. Soeda and M. Murao, PRL 131, 120602 (2023)

Higher-order quantum algorithm for **qubit**-unitary inversion

- A **deterministic and exact** implementation of qubit-unitary inversion with **4 calls** of the unknown unitary operation is found



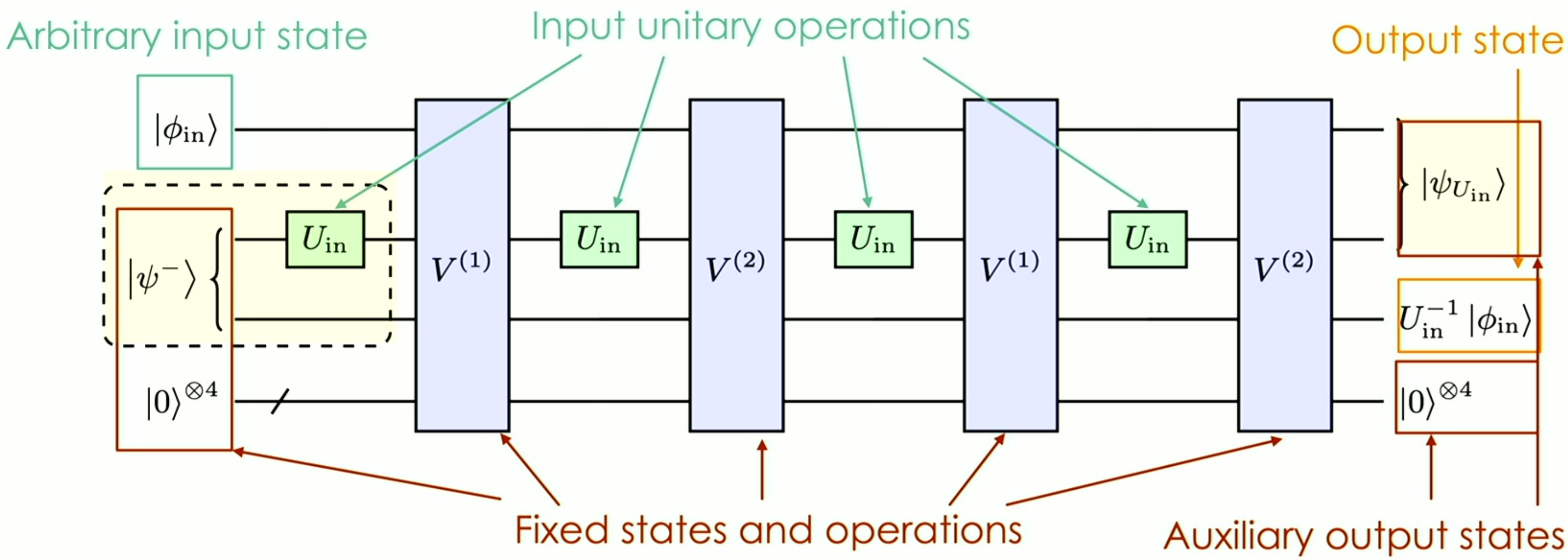
- Initially numerically found by SDP
- SDP is simplified using the symmetry of the problem
- **Catalytic** supermap
- Can be transformed to a clean protocol with one more call

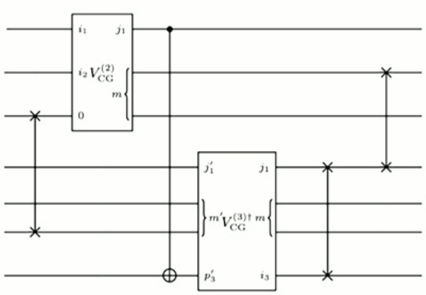
Previous methods required 36 calls (with probabilistic algorithm* + success or draw algorithm**) or 2997 calls (with a parallel algorithm) for 99.5%

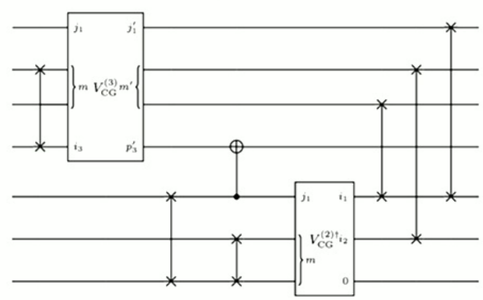
*Quintino, Dong, Shimbo, Soeda and Murao, Phys. Rev. Lett. (2019),
Quintino, Dong, Shimbo, Soeda and Murao, Phys. Rev. A (2019)

**Dong, Quintino, Soeda and Murao, Phys. Rev. Lett. (2021)

Qubit-unitary inversion (close look)



$V^{(1)} =$ 

$V^{(2)} =$ 

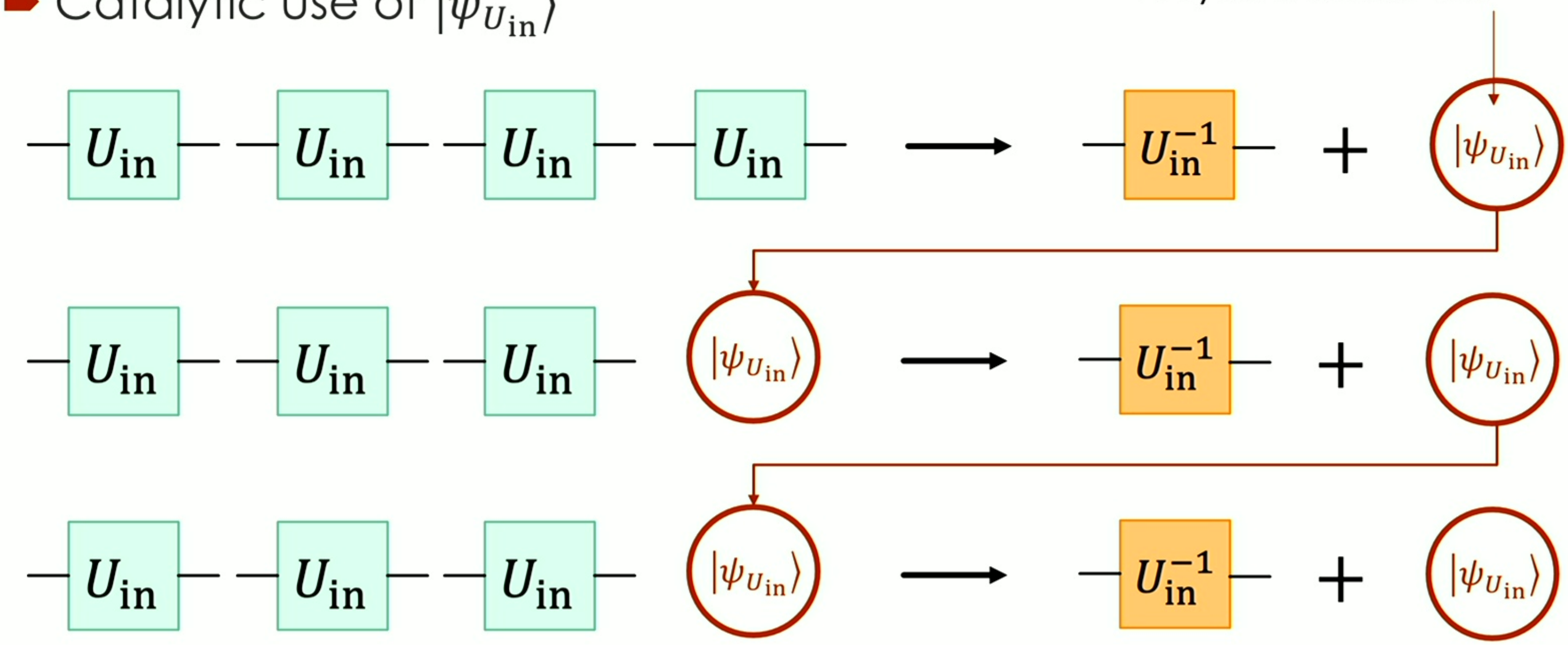
$V_{CG}^{(k)}$: Clebsch-Gordan transforms

D. Bacon, I.L. Chuang, A.W. Harrow, PRL (2006)
+ recent developments (efficient circuits)

Characteristics of this higher-order quantum algorithm: **Catalytic**

- Catalytic use of $|\psi_{U_{\mathrm{in}}}\rangle$

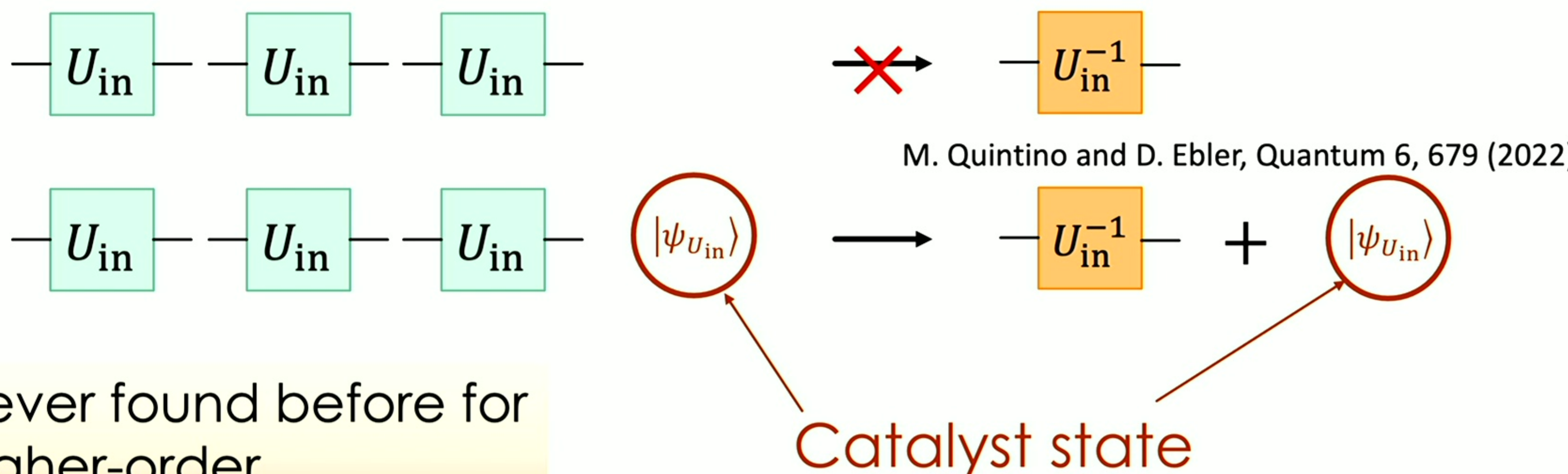
a part of learned quantum data is saved in a quantum state and recycled in later use



So, from the second run, the actual cost of calls is only 3 instead of 4!

Catalytic higher-order quantum algorithm

- Catalytic use of $|\psi_{U_{\text{in}}}\rangle$

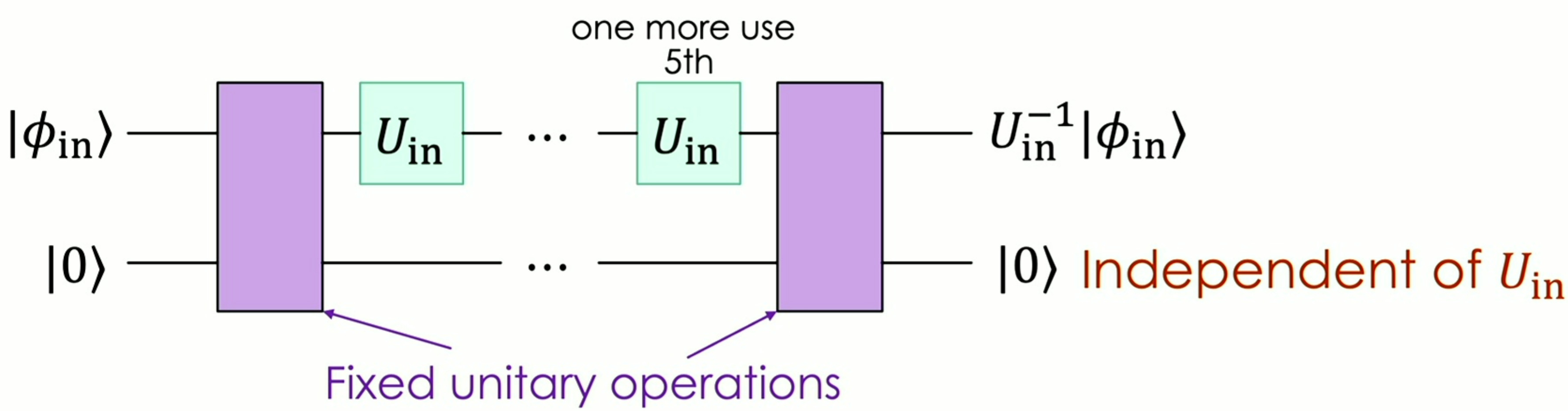


Never found before for Higher-order quantum algorithms!

Originally defined for state transformations
D. Jonathan, D and M. Plenio, M. B, PRL, 83, 3566 (1999).

Characteristics of this protocol: **Clean protocol**

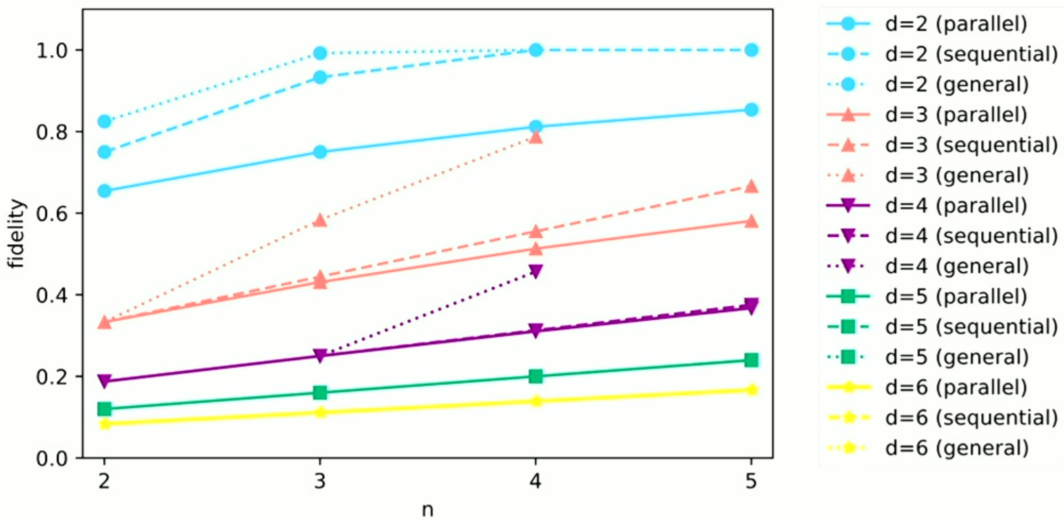
- Clean protocol: With **one more use** of U_{in}, the catalysis state is "cleaned" by the property of the singlet state $U_{\mathrm{in}} \otimes I \left| \psi_{U_{\mathrm{in}}} \right\rangle = U_{\mathrm{in}} \otimes U_{\mathrm{in}} \left| \Psi^- \right\rangle = \left| \Psi^- \right\rangle$ singlet

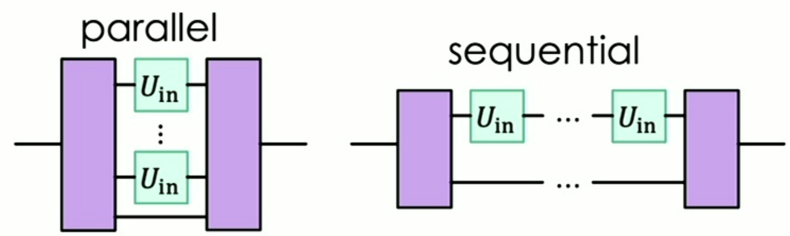


- No initialization cost is required for the auxiliary system (environment)

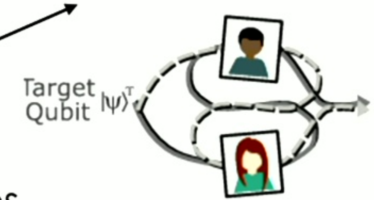
Z. Gavorová et al. arXiv:2011.10031 (2020).

For d >2…
Numerical calculation of SDP





general (indefinite causal order)



beyond circuit model
but preserves CPTP maps

Sci. Adv., 3: e1602589 (2017)

parallel $\leq$ sequential $\leq$ general, but
$k \leq d-1$ → parallel = sequential = general

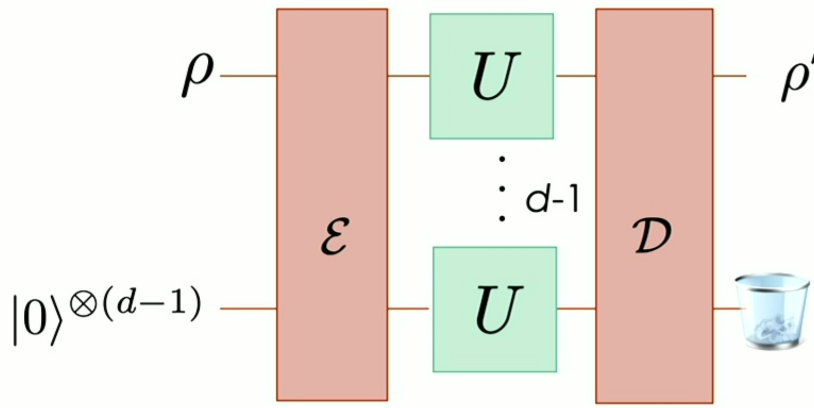
Recently, for general d, a **sequential** algorithm was found with $\approx (\pi/2)d^2$!
Y.A Chen, Y. Mo, Y Liu, L. Zhang, X. Wang, arXiv:2403.04704

Recently, we found **a lower bound**: d^2 for **sequential** algorithms
T. Odake, S. Yoshida and M. Murao, arXiv:2405.07625

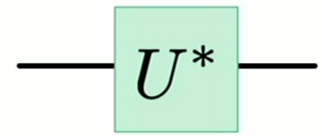
J. Miyazaki, A. Soeda and M. Murao, Phys. Rev. Research 2019

A universal algorithm for $U \to U^*$ (deterministic exact)

We found a universal implementation algorithm with *d-1* uses of *U*!



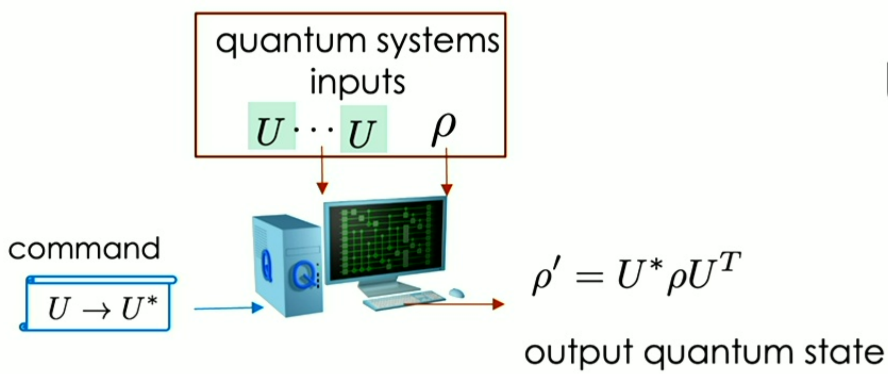
$\rho' = U^* \rho U^T$ $\cong$



$$\mathcal{E}(\rho) = A\rho A^\dagger,\ \mathcal{D}(\rho) = A^\dagger \rho A$$

$$A = \frac{1}{\sqrt{(d-1)!}} \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) |\sigma_2\rangle \otimes \cdots \otimes |\sigma_d\rangle\langle\sigma_1|$$

A: isometry to the (*d-1*)-dim antisymmetric subspace



Using (d-1)× U Deterministic & exact

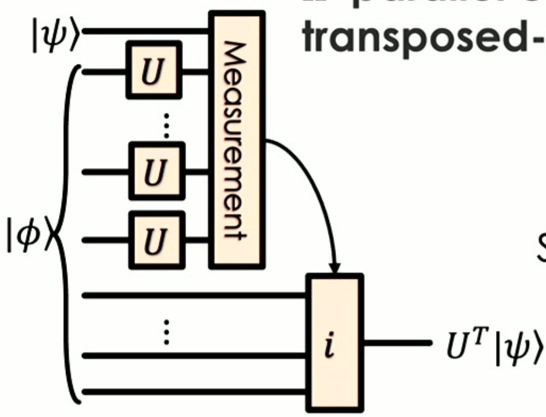
Recovers the d=2 case as well

Also proved: With *d*-2 uses, not possible even probabilistically

Quintino, Dong, Shimbo, Soeda and Murao, Phys. Rev. A, 100, 062339 (2019)

Unitary transposition $U \to U^T$: Probabilistic algorithms

- k **parallel** calls – optimal success probability can be achieved **by port-based transposed-gate teleportation**



Port-based transposed-gate teleportation

Port-based teleportation:

S. Ishizaka and T. Hiroshima, Phys. Rev. Lett., 101, 240501 (2008)
S. Ishizaka and T. Hiroshima, Phys. Rev. A, 79, 042306 (2009)

Success prob. $p = 1 - \frac{d^2 - 1}{k + d^2 - 1}$ $\left(1 - \frac{3}{k+3} \text{ for } d = 2\right)$

Input state can be inserted **after** the uses of blackboxes ("learning type")

The optimality was proven by showing that the probability matches to the probability of the probabilistic storing and retrieving of a unitary channel
Michal Sedlak, Alessandro Bisio and Mario Ziman, PRL 2019

- k **sequential** calls – A repeat until success protocol gives

Transposed-gate teleportation → $U^T X^i Z^j |\phi\rangle$ → apply conjugation → reset to the initial state $|\phi\rangle$

$U^* U^T = \mathbb{I}$ by applying $Z^j X^i$

Repeat until success!

$i, j = 0$ → $U^T |\phi\rangle$

Success prob. $p = 1 - (1 - 1/d^2)^{\lceil k/d \rceil}$ Exponential improvement! for $k \geq d^2$

Numerical results of SDP for obtaining optimal fidelities of unitary transposition and inversion

D. Grinko and M. Ozols, arXiv:2207.05713

$f(U)$	d \ n	1	2	3	4
U^{T}	2	0.500000	0.750000	0.933013	1.000000
	3	0.222222	0.407407	0.626597	0.799250
U^{-1}	2	0.500000	0.750000	0.933013	1.000000
	3	0.222222	0.333333	0.444444	0.555556

TABLE 2. Optimal values of the SDP (173). The column $f(U)$ indicates the task, for which we want to find a deterministic sequential superchannel $\mathcal{C}$. In both tasks we reproduce the results of [QE22] for $d = 2$, $n \leqslant 3$ and $d = 3$, $n \leqslant 2$. We also reproduce the results for $f(U) = U^{-1}$ [YSM22] when $d = 2$, $n = 4$ and $d = 3$, $n = 3$ and $d = 3$, $n = 4$. Finally, we obtain new results for the unitary transposition task for $d = 2$, $n = 4$ and $d = 3$, $n = 3$ and $d = 3$, $n = 4$.

Optimal average fidelity for unitary transposition: $U^{\otimes k} \mapsto U^T$

		Parallel		Sequential		Indefinite causal order
$d = 2$	$k = 2$	$\cos^2\left(\frac{\pi}{5}\right) \approx 0.6545$	<	0.7500	<	0.8249
$d = 2$	$k = 3$	$\frac{3}{4} = 0.75$	<	0.9330	<	0.9921
$d = 3$	$k = 2$	0.3333	<	0.4074	<	0.4349

Optimal average fidelity for unitary inversion: $U^{\otimes k} \mapsto U^{-1}$

		Parallel		Sequential		Indefinite causal order
$d = 2$	$k = 2$	$\cos^2\left(\frac{\pi}{5}\right) \approx 0.6545$	<	0.7500	<	0.8249
$d = 2$	$k = 3$	$\frac{3}{4} = 0.75$	<	0.9330	<	0.9921
$d = 3$	$k = 2$	0.3333	=	0.3333	=	0.3333

Figure 8: Optimal average fidelity for deterministic protocols transforming k uses of U into its transpose (upper

M.T. Quintino and D. Ebler, Quantum (2022)

Unitary transposition requires transformation in the Gelfand–Tsetlin basis

T. Odake, S. Yoshida and M. Murao, arXiv:2405.07625 (2024)

Analytical lower bound with definite causal order

	Lower bound		Minimum known	
	previous methods	our method	$d=2$	$d \geq 3$
$f(U) = U^{-1}$	4^* $(d=2$ [1]), 6^* $(3 \leq d \leq 7$ [1]), $d-1$ $(d \geq 8$ [2])	$\boldsymbol{d^2}$	4 [1]	$\sim (\pi/2)d^2$ [3]
$f(U) = U^T$	4^* $(d = 2$[4]), 5^* $(d \geq 3)$ [4]	$\mathbf{4}$ $(d=2)$, $\boldsymbol{d+3}$ $(d \geq 3)$	4^* [4]	$\sim (\pi/2)d^2$ [3]
$f(U) = U^*$	$d-1$ [2]	$d-1$	$d-1$ [5]	

(*numerical) analytical By construction

[1] S. Yoshida, A. Soeda and M. Murao, PRL 131, 120602 (2023)
[2] M.T. Quintino, Q. Dong, A. Shimbo, A. Soeda and M. Murao, PRA 100, 062339 (2019)
[3] Y.-A. Chen, Y. Mo, Y. Liu, L. Zhang, and X. Wang, arXiv:2403.04704
[4] D. Grinko and M. Ozols, arXiv:2207.05713
[5] J. Miyazaki, A. Soeda, and M. Murao, PRR 1, 013007 (2019)

Query complexity of higher-order unitary transformations:

$$U \to U^* \; < \; U \to U^T \; < \; U \to U^\dagger \quad ???$$

Sequential and Indefinite causal order does not help*

Indefinite causal order helps

Indefinite causal order Helps if $k \geq d$

D. Ebler et al. IEEE Transactions on Information Theory 69, 8 (2023)

Thank you for
your attention!