Title: Tutorial: Causal Inference Meets Quantum Physics

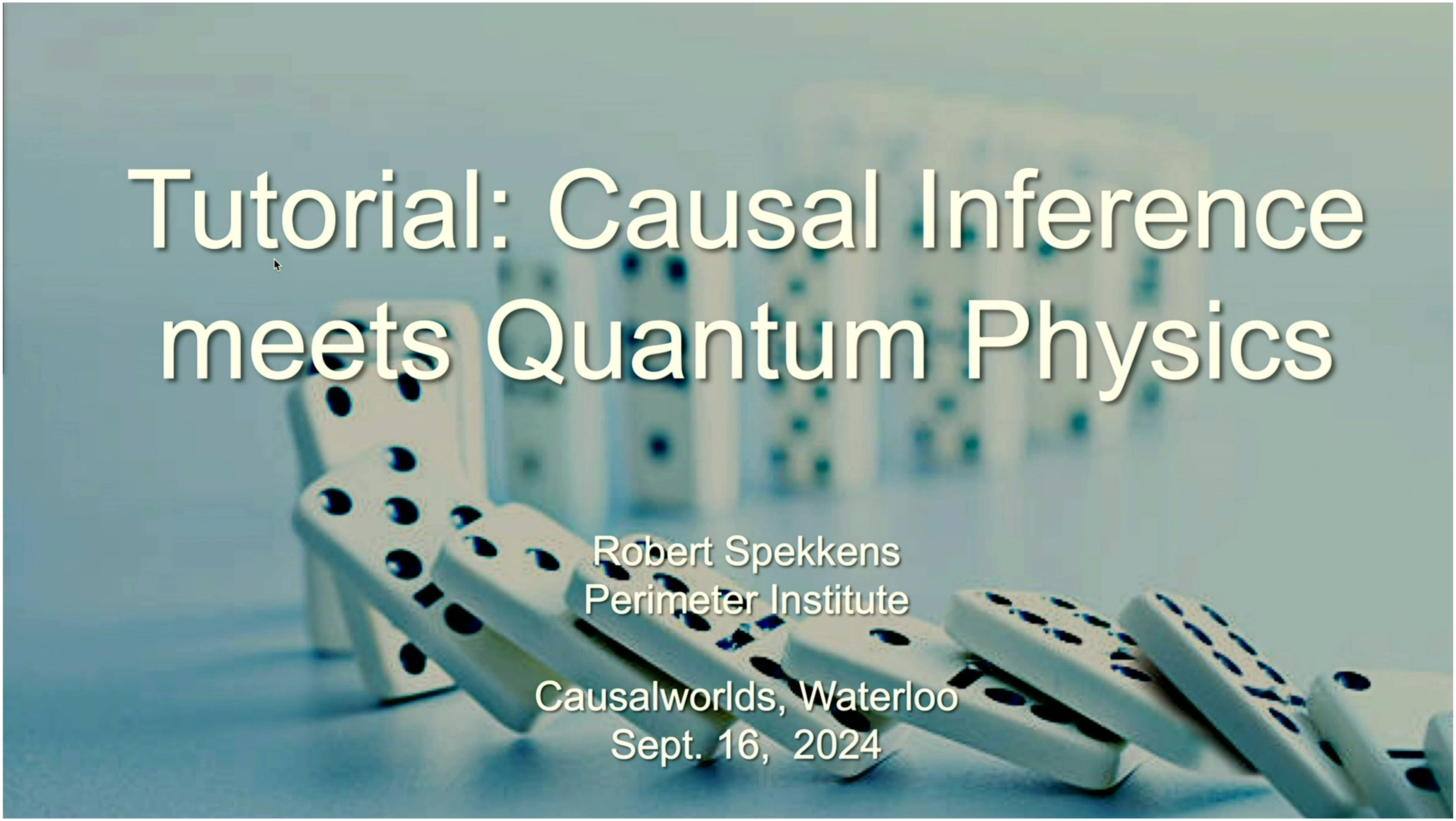
Speakers: Robert Spekkens

Series: Quantum Foundations, Quantum Information

Date: September 16, 2024 - 9:20 AM

URL: https://pirsa.org/24090083

Abstract: Can the effectiveness of a medical treatment be determined without the expense of a randomized controlled trial? Can the impact of a new policy be disentangled from other factors that happen to vary at the same time? Questions such as these are the purview of the field of causal inference, a general-purpose science of cause and effect, applicable in domains ranging from epidemiology to economics. Researchers in this field seek in particular to find techniques for extracting causal conclusions from statistical data. Meanwhile, one of the most significant results in the foundations of quantum theoryâ€”Bellâ€™s theoremâ€”can also be understood as an attempt to disentangle correlation and causation. Recently, it has been recognized that Bellâ€™s result is an early foray into the field of causal inference and that the insights derived from almost 60 years of research on his theorem can supplement and improve upon state-of-the-art causal inference techniques. In the other direction, the conceptual framework developed by causal inference researchers provides a fruitful new perspective on what could possibly count as a satisfactory causal explanation of the quantum correlations observed in Bell experiments. Efforts to elaborate upon these connections have led to an exciting flow of techniques and insights across the disciplinary divide. This tutorial will highlight some of what is happening at the intersection of these two fields.


Tutorial: Causal Inference meets Quantum Physics
Robert Spekkens
Perimeter Institute
Causalworlds, Waterloo
Sept. 16, 2024

15 hours of lectures
Available online at **https://pirsa.org/c23016**

Causal Inference: Classical and Quantum

PHYS 777-007
Lecturer: Robert Spekkens
TA: Marina Ansanelli

March 6, 2023

Causarum Investigatio
"Investigate the causes"

The field of Causal Inference

⇄

The field of Quantum Foundations

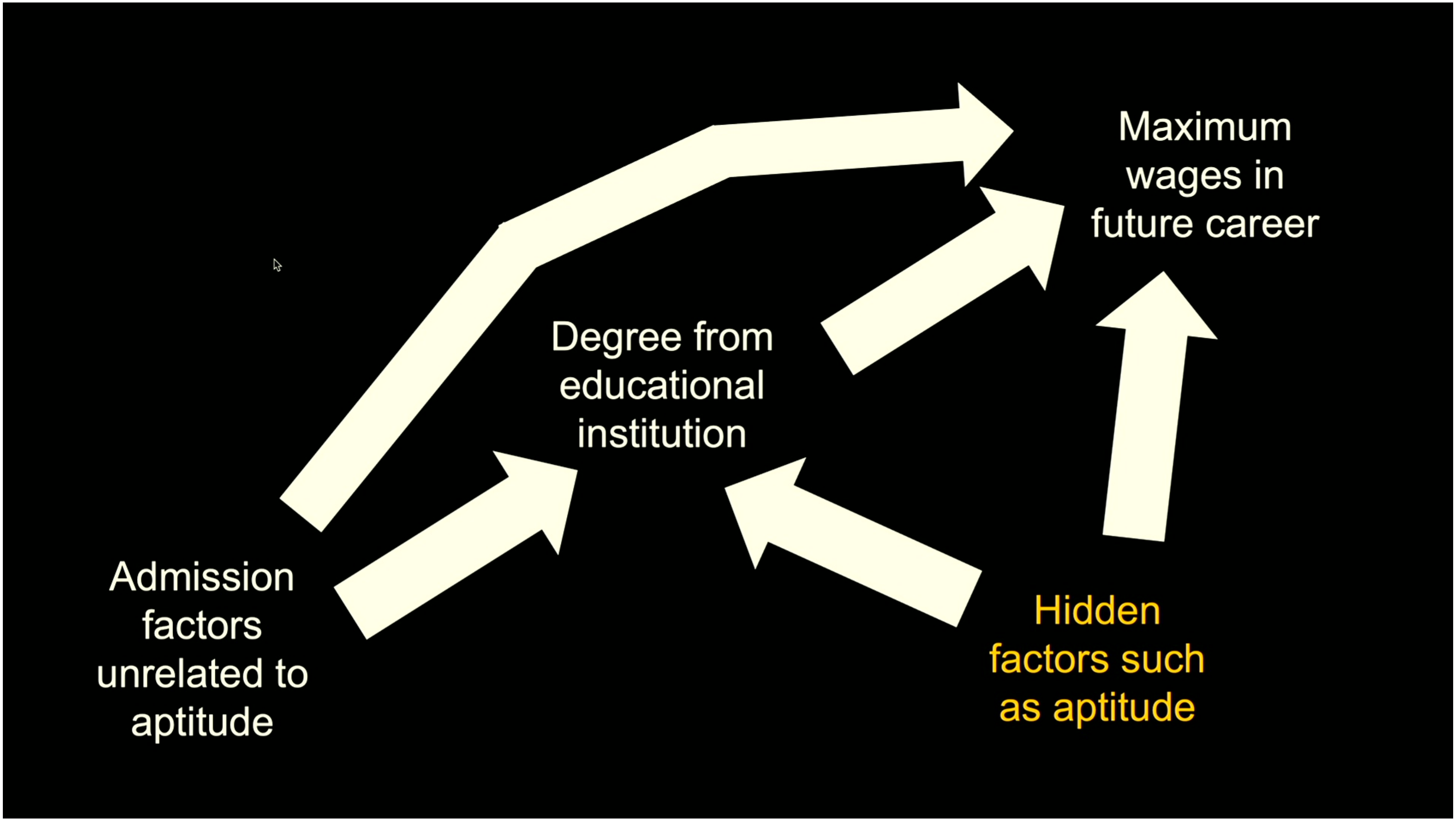
Instrumental inequalities

Causal Inference in the presence of hidden variables

Maximum wages in future career above some threshold?

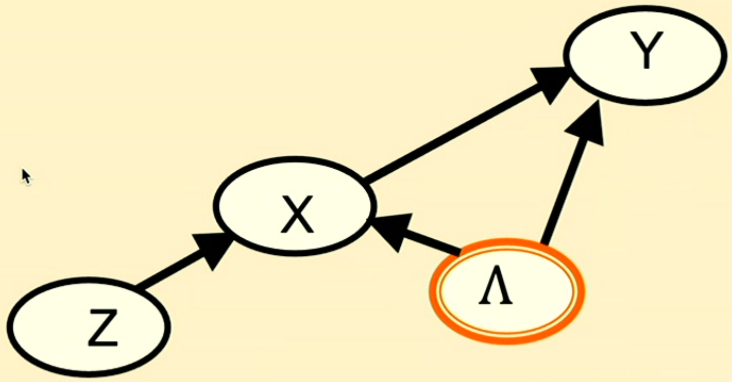
Degree from educational institution?

	Yes	No
Yes	79%	21%
No	43%	57%


Maximum wages in future career
Degree from educational institution
Admission factors unrelated to aptitude
Hidden factors such as aptitude

Causal structure

Parameters



$P_{X|\Lambda Z}$

$P_{Y|\Lambda X}$

P_{Λ}

$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

Example of causal compatibility constraint:

$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$

Pearl, 1993

The evidence

Z=0

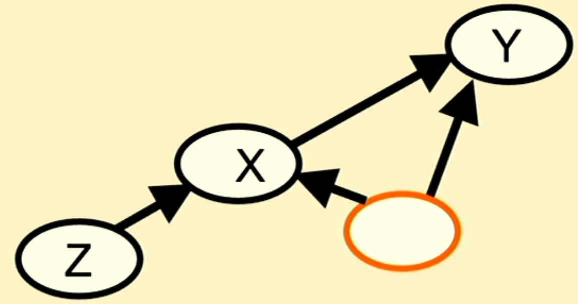
	Y=0	Y=1
X=0	0.79	0.21
X=1	0.43	0.57

Z=1

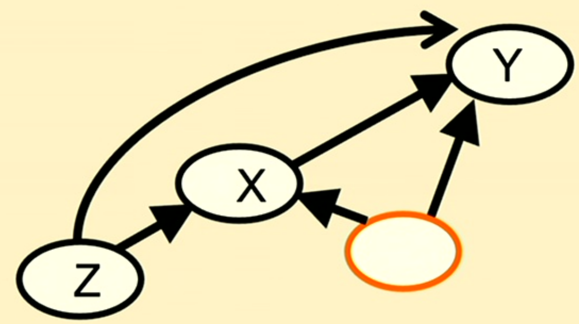
	Y=0	Y=1
X=0	0.59	0.41
X=1	0.39	0.61

Violates Pearl's instrumental inequality!

The hypotheses



Implies a constraint:
Pearls's instrumental inequality

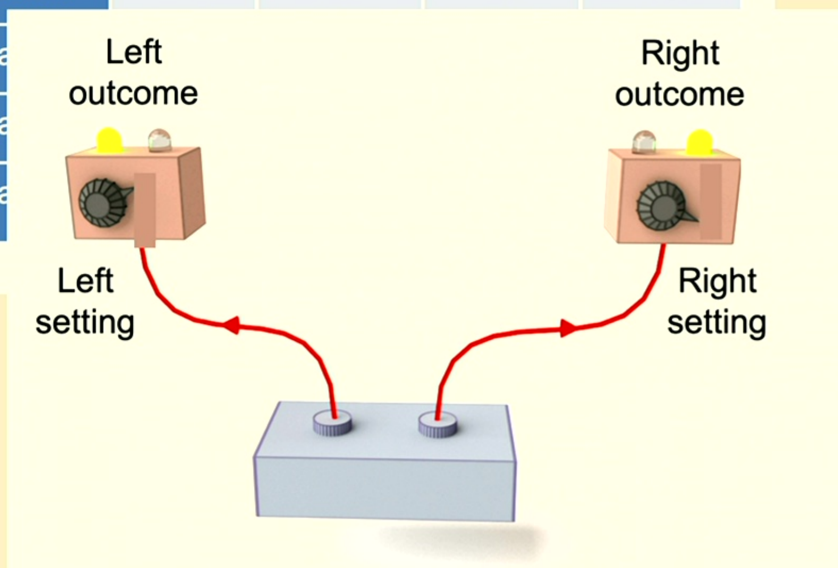


Bell inequalities

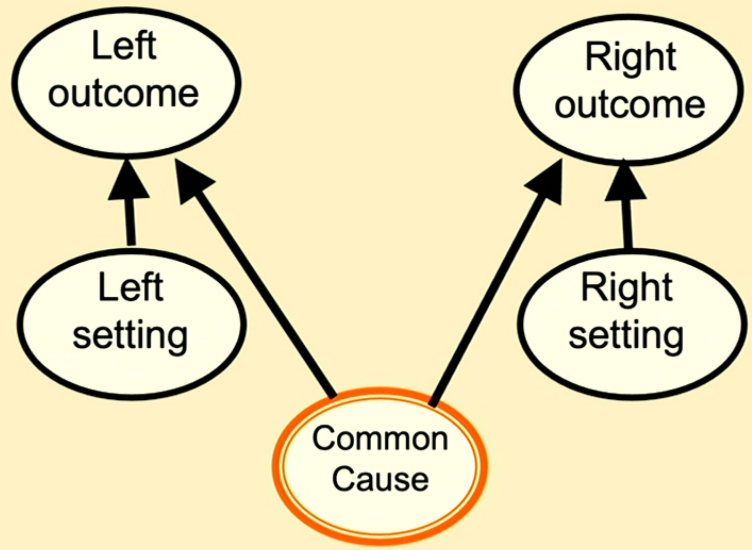
Left setting and Right setting \ Left outcome and Right outcome	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	43%	7%	7%	43%
0 and 1	43%	7%	7%	43%
1 and 0	43%	7%	7%	43%
1 and 1	7%	43%	43%	7%

The evidence

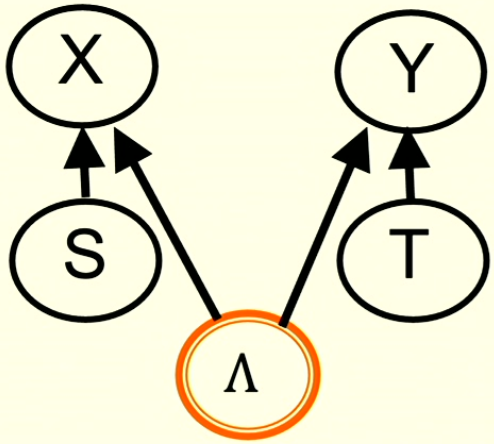
Left setting and Right setting		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
	0 and 0	43%	7%	7%	43%
	0 a				
	1 a				
	1 a				



The natural hypothesis



Causal structure Parameters



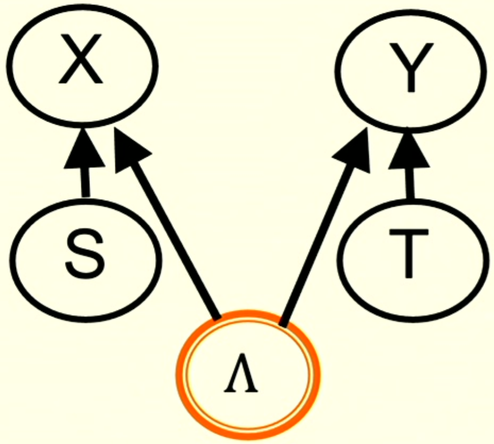
$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Examples of causal compatiblity constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

Causal structure Parameters



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$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Examples of causal compatiblity constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Clauser, Horne, Shimony and Holte, Phys. Rev. Lett.23, 880 (1967)

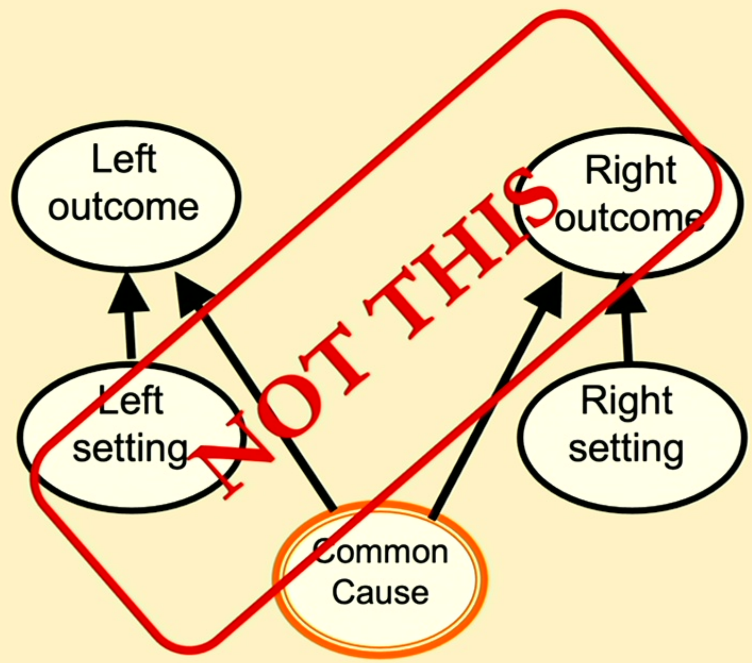
The evidence

Left outcome and Right outcome

Left setting and Right setting	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	43%	7%	7%	43%
0 and 1	43%	7%	7%	43%
1 and 0	43%	7%	7%	43%
1 and 1	7%	43%	43%	7%

Violates the
Bell Inequalities!

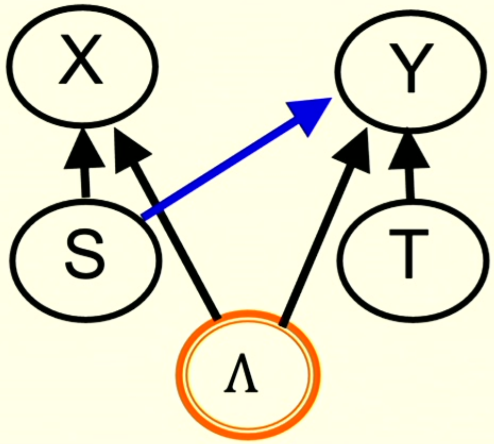
The natural hypothesis



Implies a constraint:
Bell Inequalities

Incompatible

Causal structure Parameters



$P_{X|S\Lambda}$

$P_{Y|ST\Lambda}$

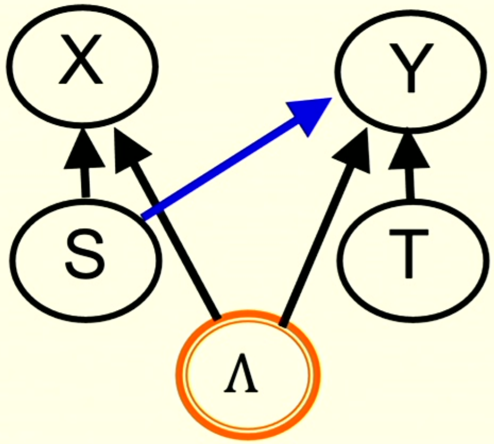
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|ST\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Causal compatibility constraints:

$P_{X|ST} = P_{X|S}$ But the data *also* satisfies $P_{Y|ST} = P_{Y|T}$

Causal structure



Parameters

$P_{X|S\Lambda}$

$P_{Y|ST\Lambda}$

P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|ST\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Causal compatibility constraints:

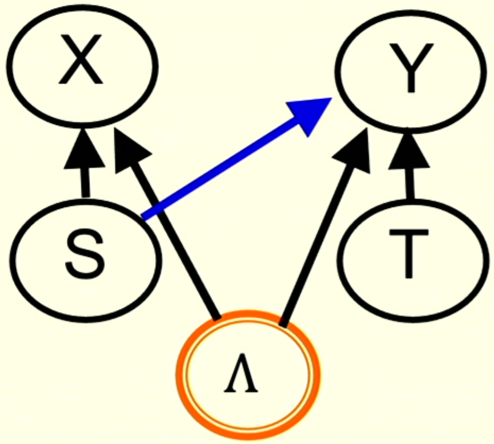
$P_{X|ST} = P_{X|S}$

But the data *also* satisfies $P_{Y|ST} = P_{Y|T}$

Reproducing this requires fine-tuning

Wood and RWS, New J. Phys. 17, 033002 (2015)

Causal structure | Parameters



$P_{X|S\Lambda}$
$P_{Y|ST\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|ST\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Causal compatibility constraints:

$P_{X|ST} = P_{X|S}$ But the data *also* satisfies $P_{Y|ST} = P_{Y|T}$

Reproducing this requires violation of Leibniz's methodological principle

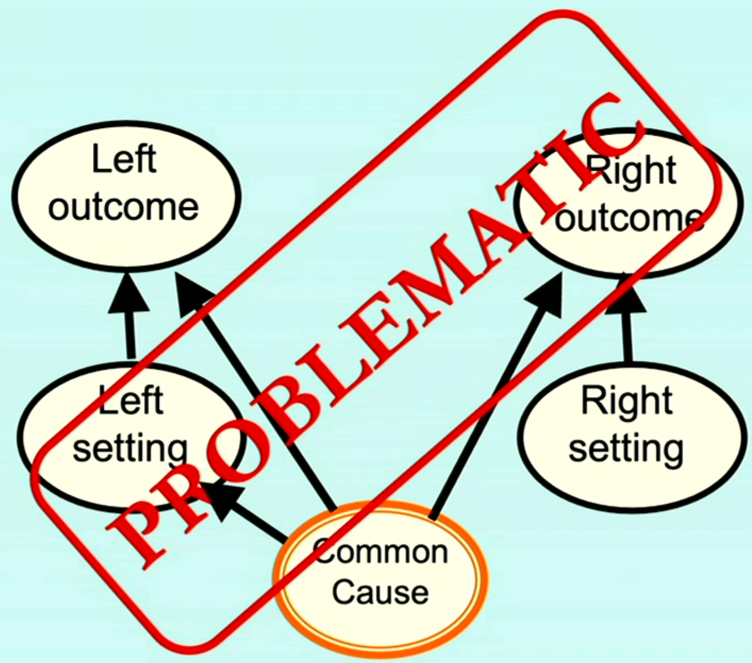
Schmid, Selby, Spekkens, to appear


QUANTUM
THEORY
Diego Delso / CC BY-SA
RELATIVITY

The evidence

		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
Left setting and Right setting	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

Superdeterminism

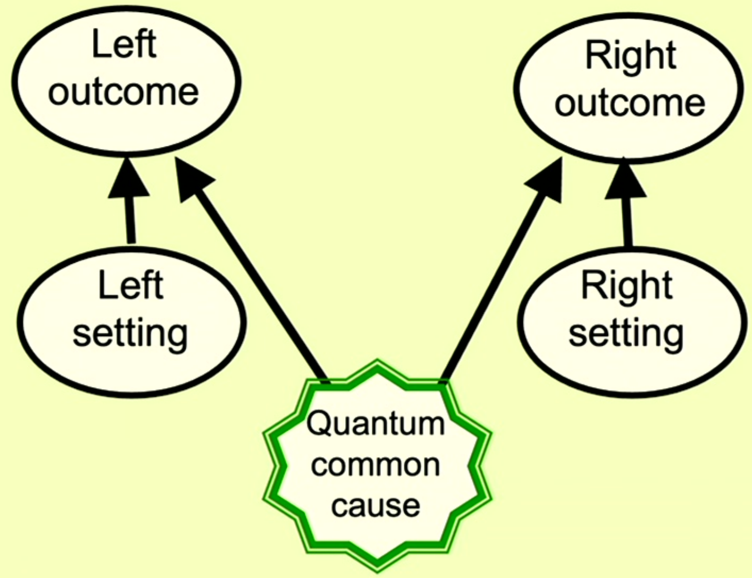


The evidence

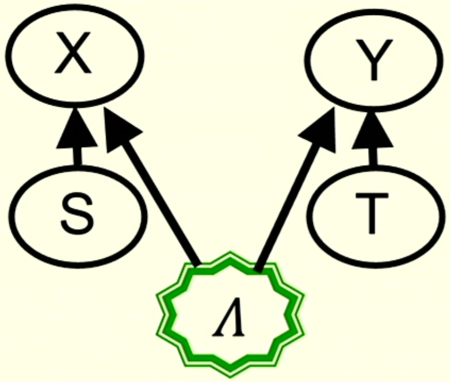
Left outcome and Right outcome

Left setting and Right setting		0 and 0	0 and 1	1 and 0	1 and 1
	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

A new possibility



Quantum causal model



$\rho_{X|S\Lambda}$

$\rho_{Y|T\Lambda}$

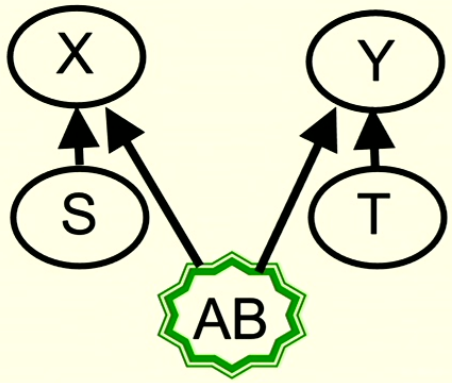
ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Leifer and RWS, PRA 88, 052130 (2013)
Henson, Lal & Pusey NJP 16, 113043 (2014)
Costa, Shrapnel NJP 18(6) (2016)
Allen, Barrett, Horsman, Lee & RWS, PRX 7, 031021 (2017)
Barrett, Lorenz, Oreshkov, arXiv:1906.10726

Quantum causal model



$\rho_{X|SA}$
$\rho_{Y|TB}$
ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Leifer and RWS, PRA 88, 052130 (2013)
Henson, Lal & Pusey NJP 16, 113043 (2014)
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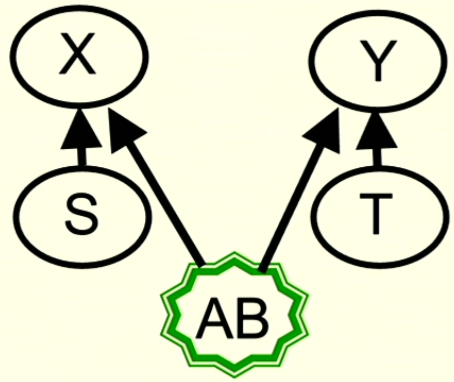
Quantum causal model



$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

ρ_{AB}

$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$

Leifer and RWS, PRA 88, 052130 (2013)
Henson, Lal & Pusey NJP 16, 113043 (2014)
Costa, Shrapnel NJP 18(6) (2016)
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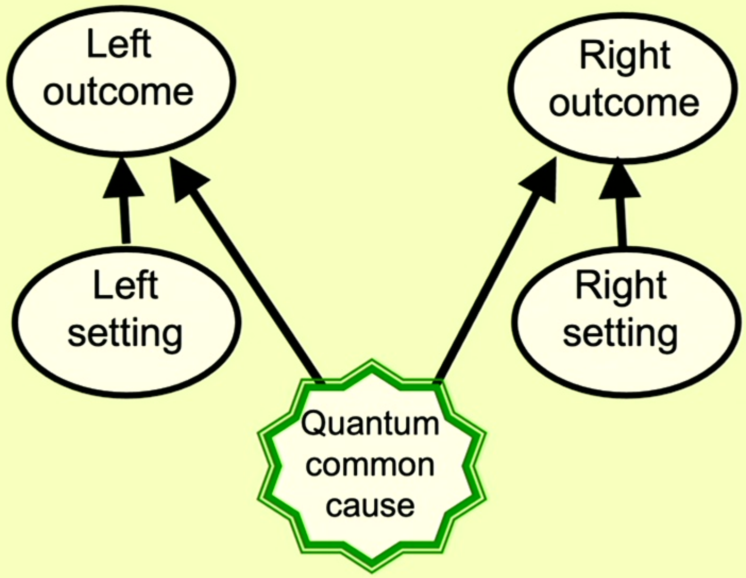
The evidence

A new possibility

Left setting and Right setting	Left outcome and Right outcome: 0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	43%	7%	7%	43%
0 and 1	43%	7%	7%	43%
1 and 0	43%	7%	7%	43%
1 and 1	7%	43%	43%	7%

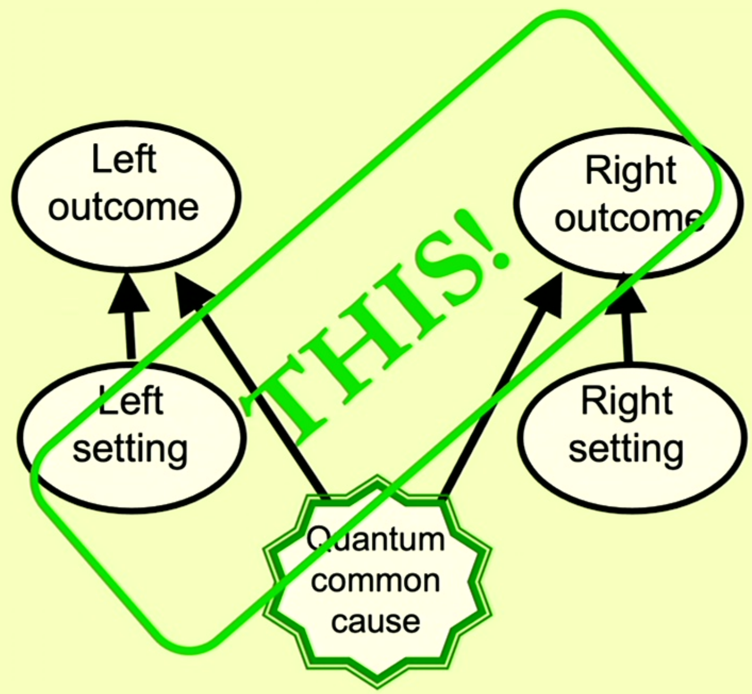


Compatible

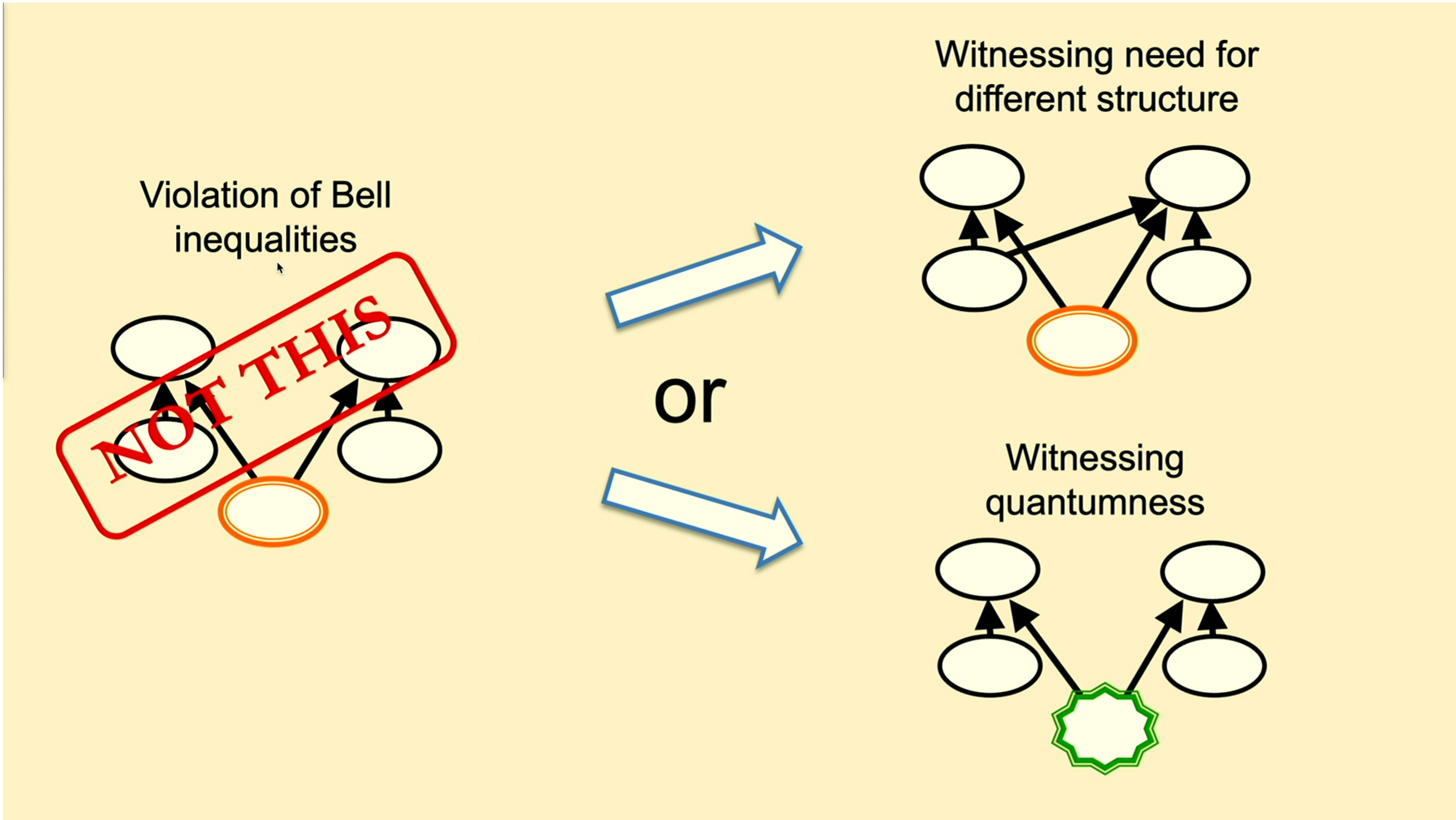
The evidence

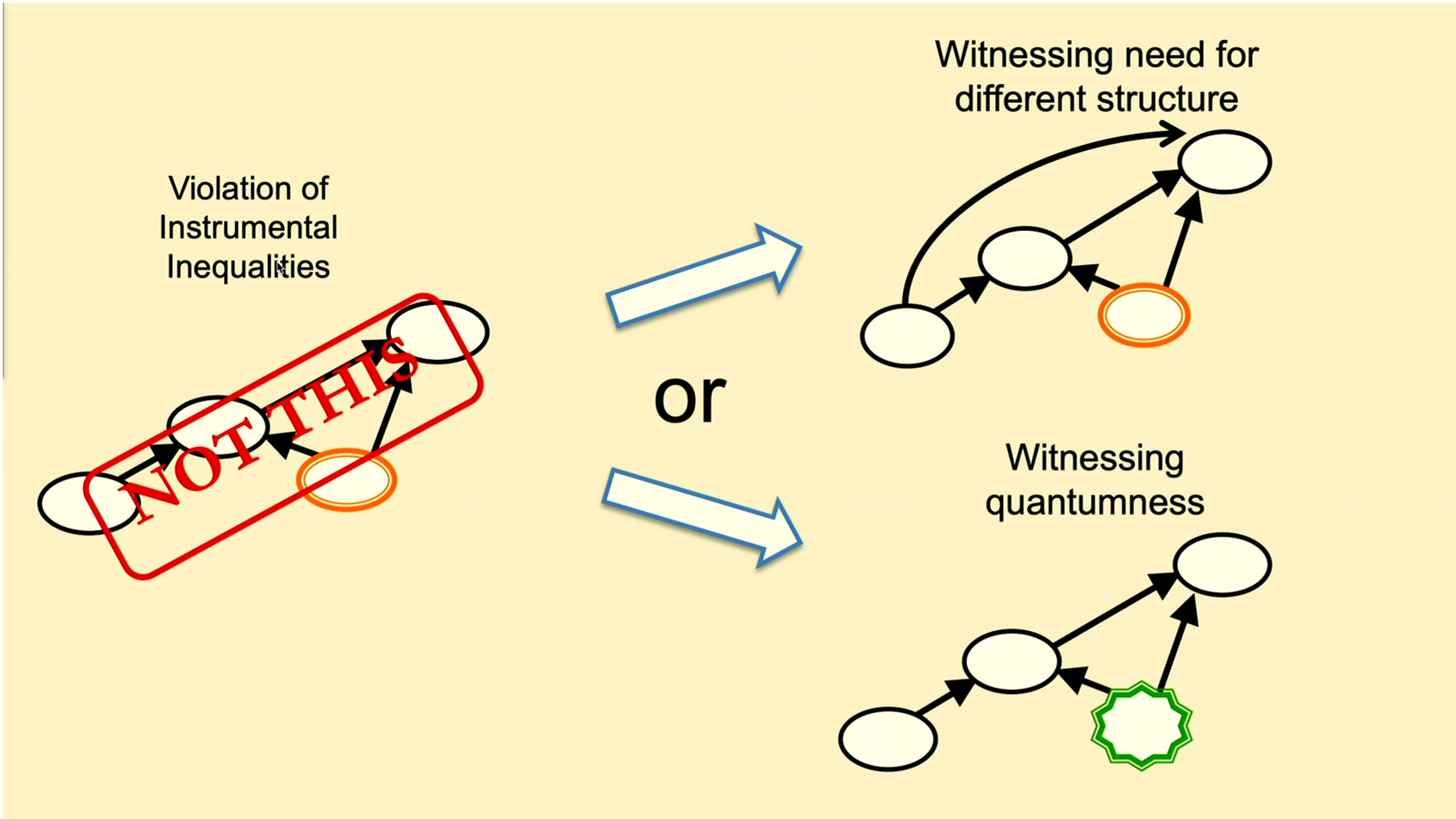
		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
Left setting and Right setting	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

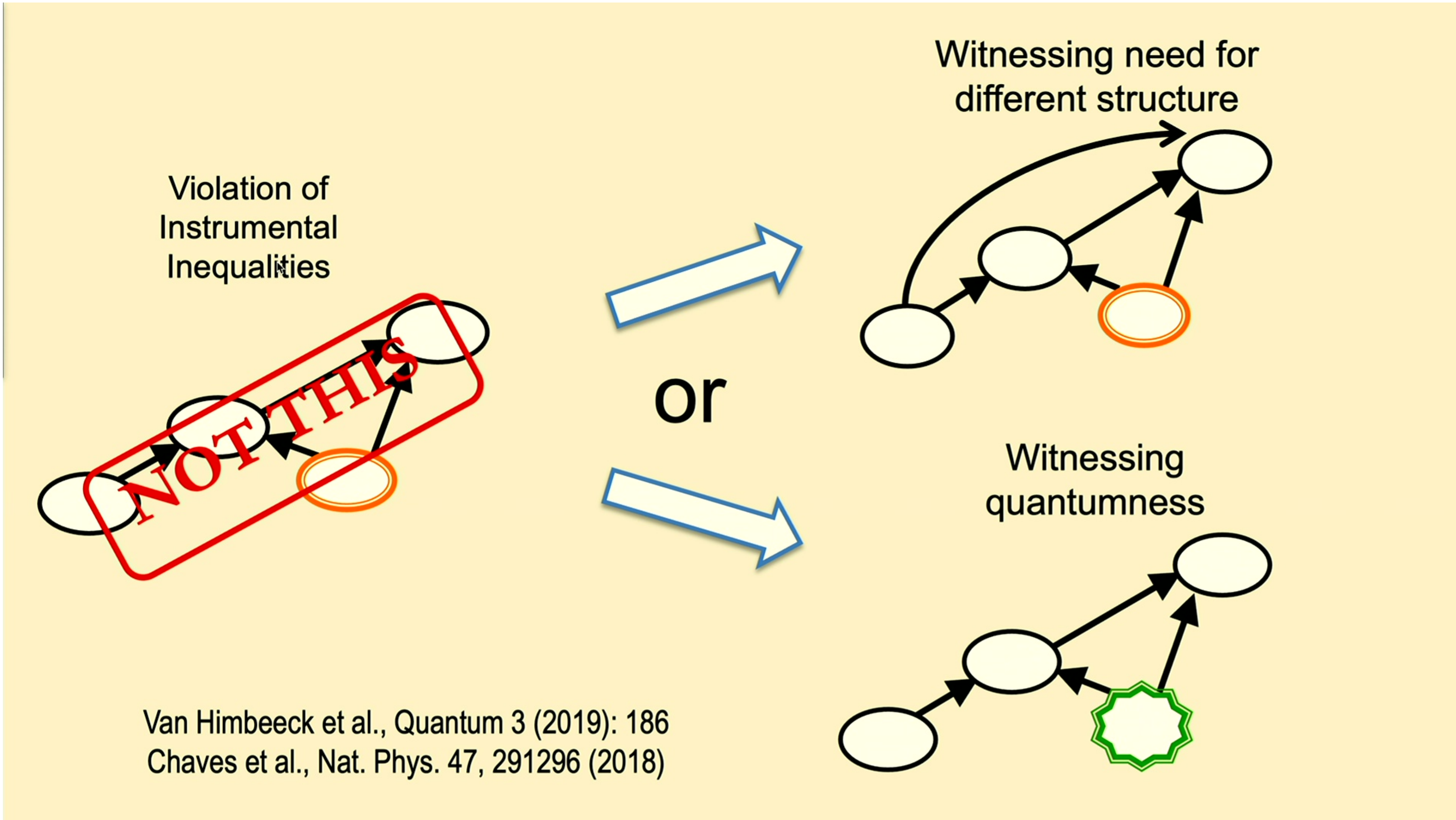
A new possibility



Compatible


Violation of Bell inequalities
NOT THIS
or
Witnessing need for different structure
Witnessing quantumness

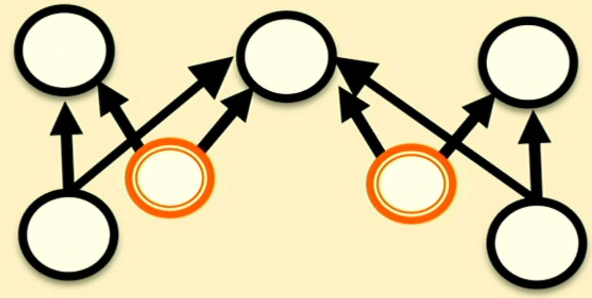

Violation of
Instrumental
Inequalities
NOT THIS
or
Witnessing need for
different structure
Witnessing
quantumness


Violation of
Instrumental
Inequalities
NOT THIS
or
Witnessing need for
different structure
Witnessing
quantumness
Van Himbeeck et al., Quantum 3 (2019): 186
Chaves et al., Nat. Phys. 47, 291296 (2018)

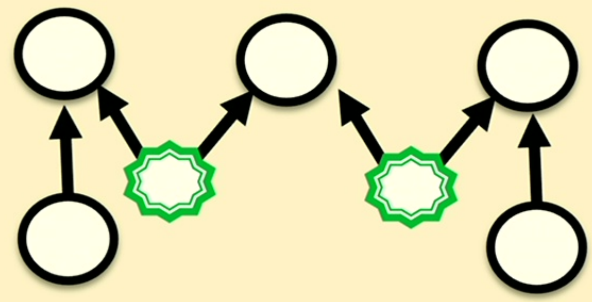
Violation of certain causal compatibility inequalities

or

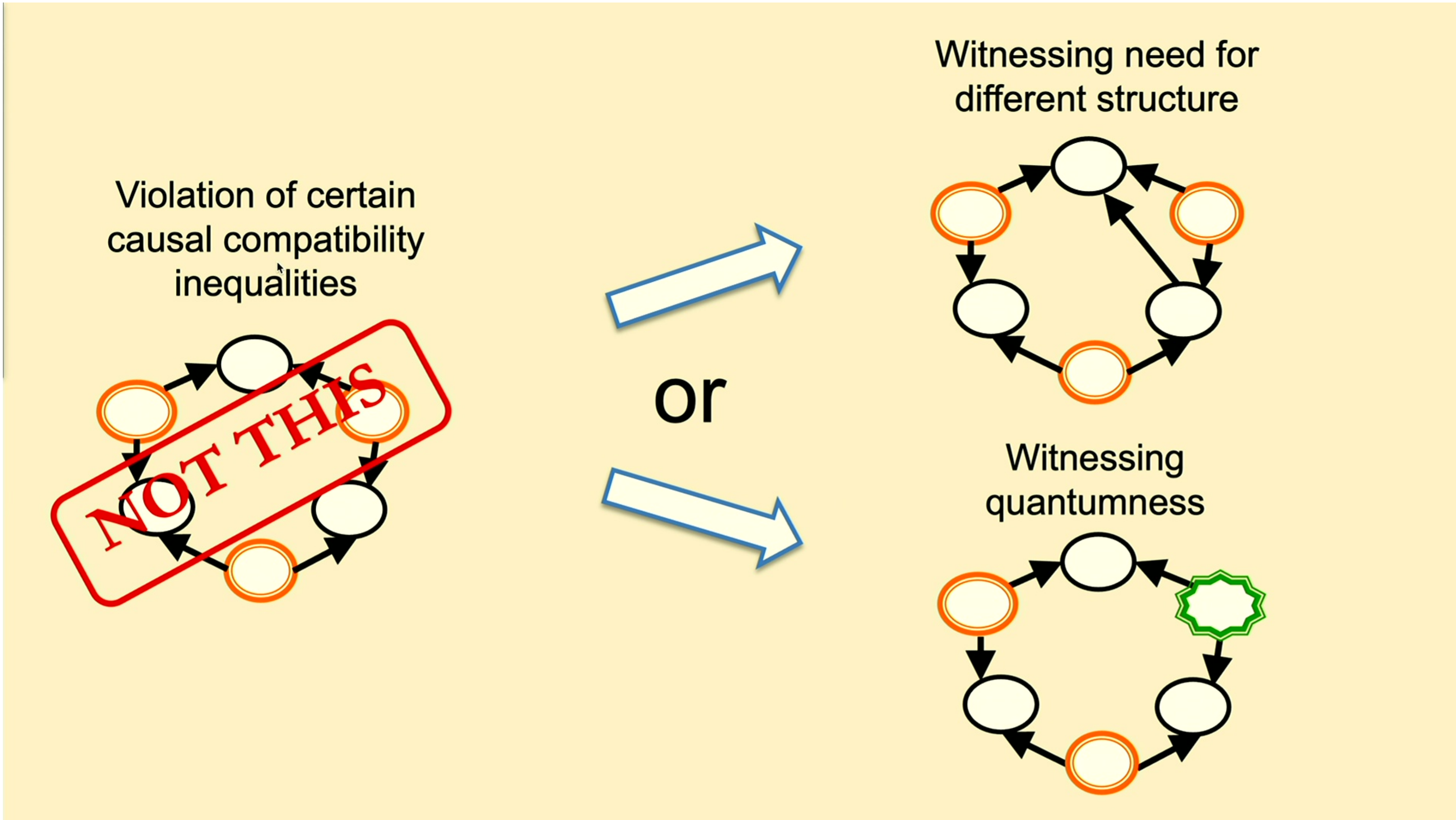
Witnessing need for different structure



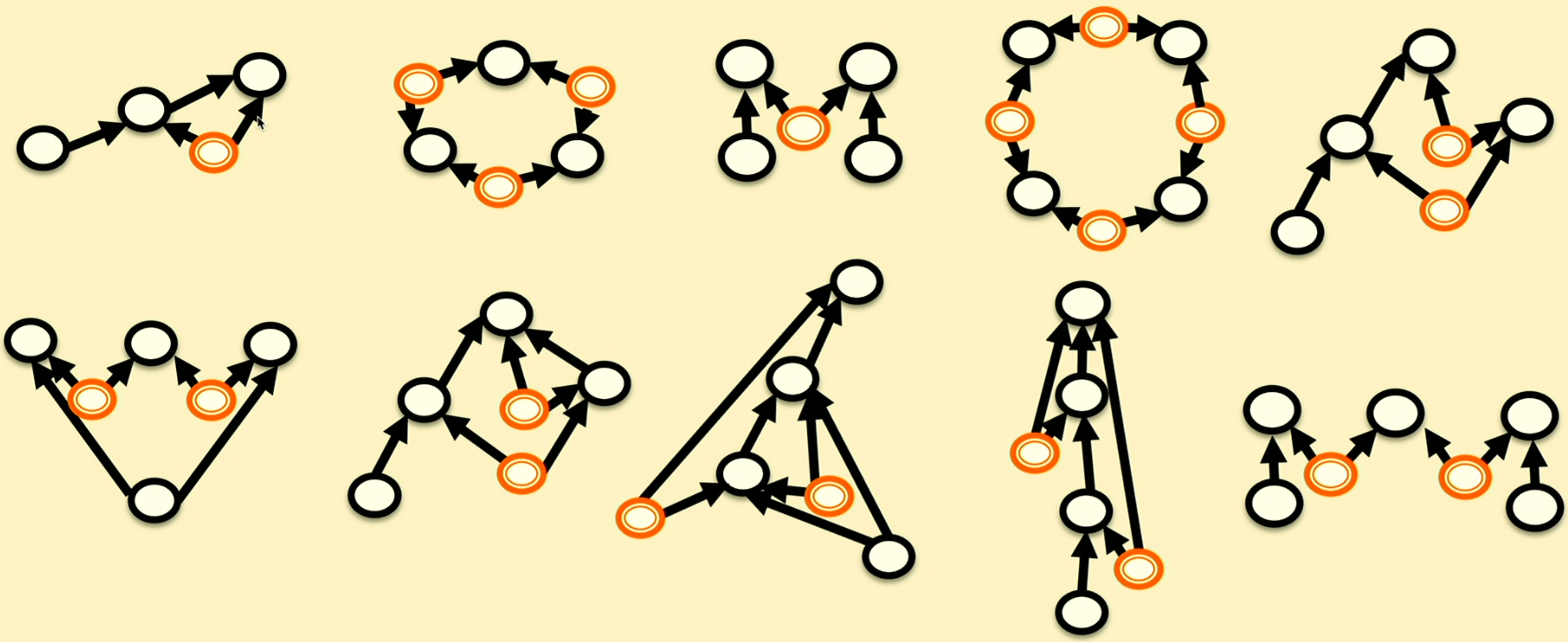
Witnessing quantumness

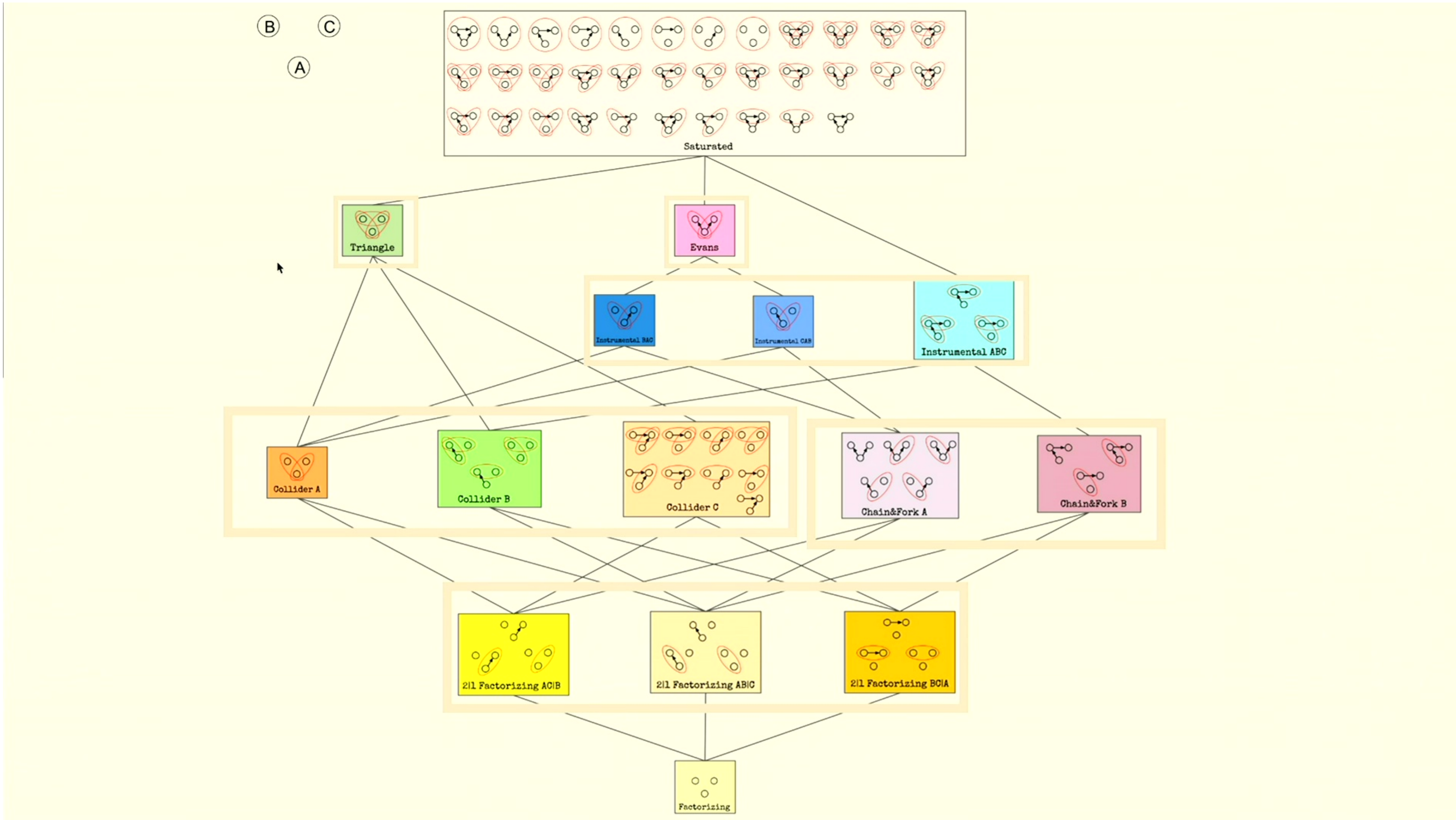


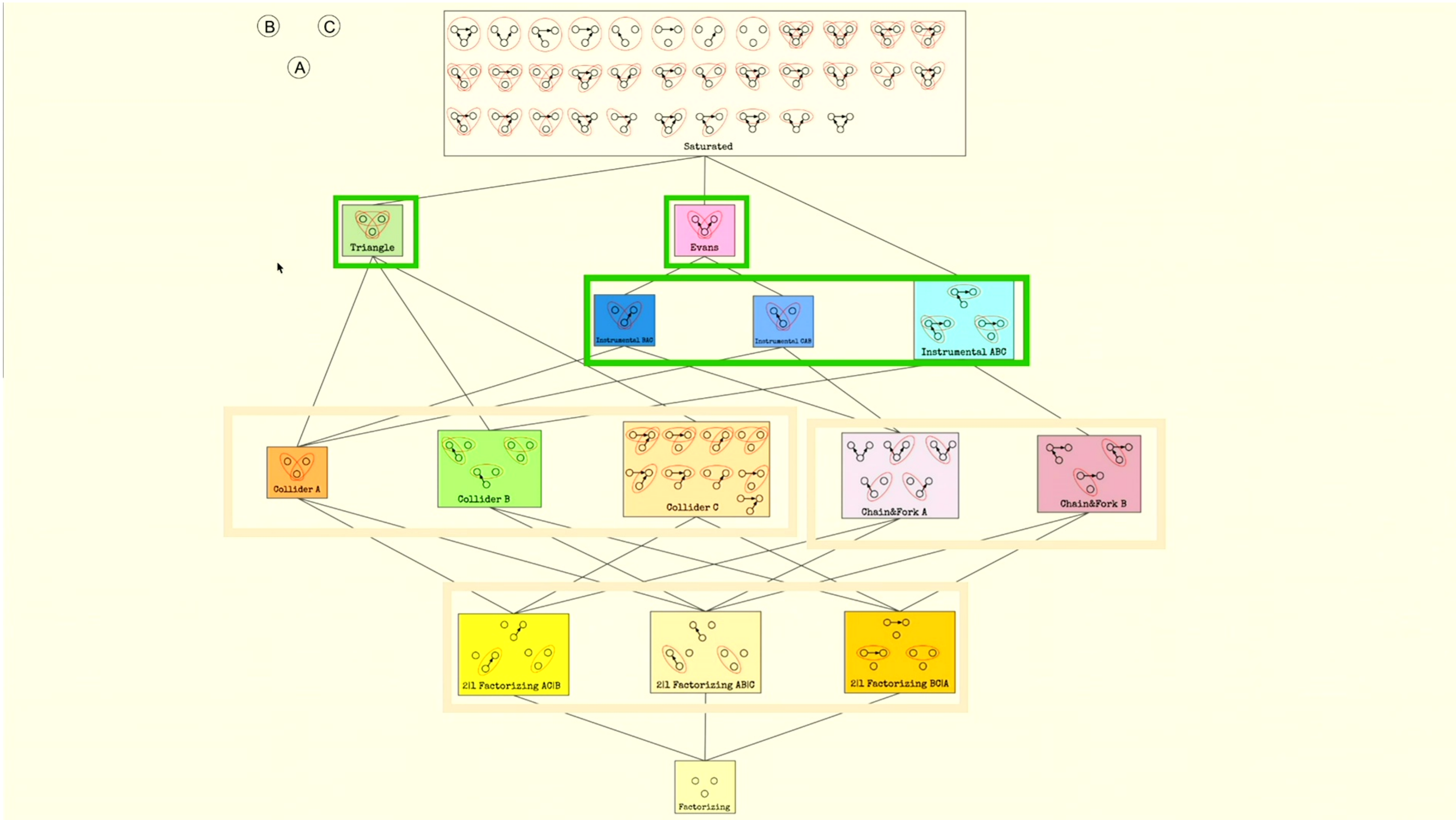
Branciard et al., Phys. Rev. A 85, 032119 (2012)


Violation of certain causal compatibility inequalities
NOT THIS
or
Witnessing need for different structure
Witnessing quantumness

Some causal structures that admit of quantum-classical gaps:




B
C
A
Saturated
Triangle
Evans
Instrumental BAC
Instrumental CAB
Instrumental ABC
Collider A
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A
Factorizing


B
C
A
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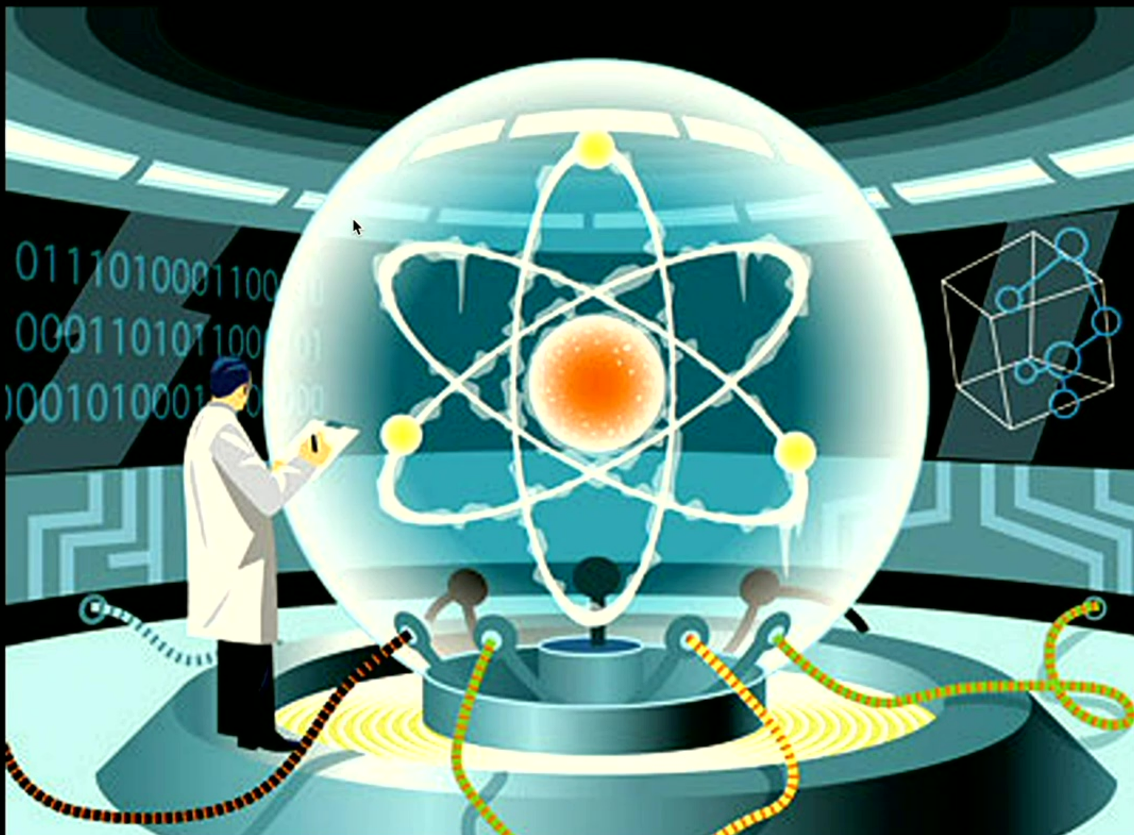
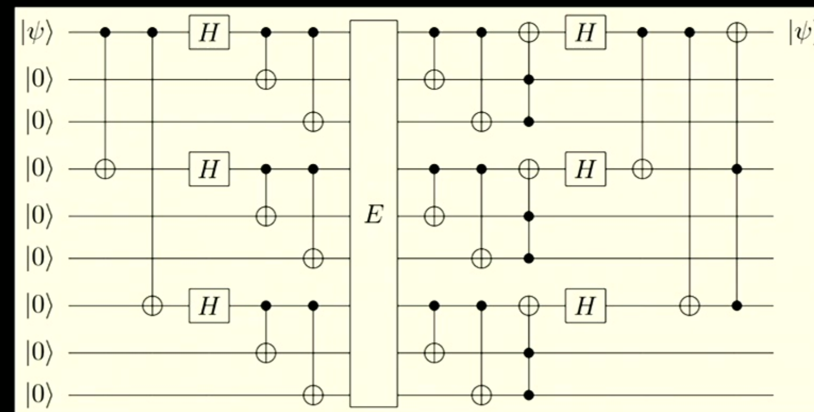
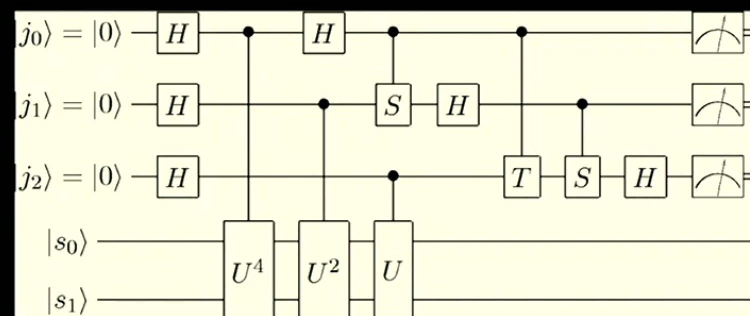
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A
Factorizing

Applications for Quantum Technology





$|\psi\rangle$
$|0\rangle$
H
E
$|\psi\rangle$



$|j_0\rangle = |0\rangle$
$|j_1\rangle = |0\rangle$
$|j_2\rangle = |0\rangle$
$|s_0\rangle$
$|s_1\rangle$
H
S
T
U^4
U^2
U

The formalism and conceptual scheme of causal inference resolved various puzzles of statistics (e.g., Simpson’s paradox, Berkson’s paradox)

The lesson:

We must unscramble the omelette of inference and causation both conceptually and in the formalism

But what hope do we have of succeeding in the quantum context if we do not understand how to do so in the classical context?

Quantum
Causation and Inference

Classical
Causation and Inference

≈

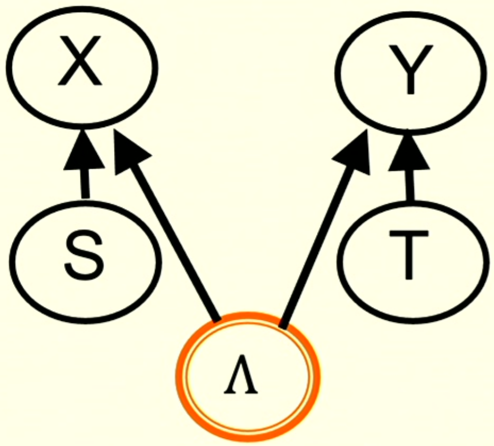
Relativistic
Notions of Space
and Time

PreRelativistic
Notions of Space
and Time


QUANTUM
THEORY
RELATIVITY

Deriving causal compatibility inequalities

Bell scenario



$P_{X|S\Lambda}$

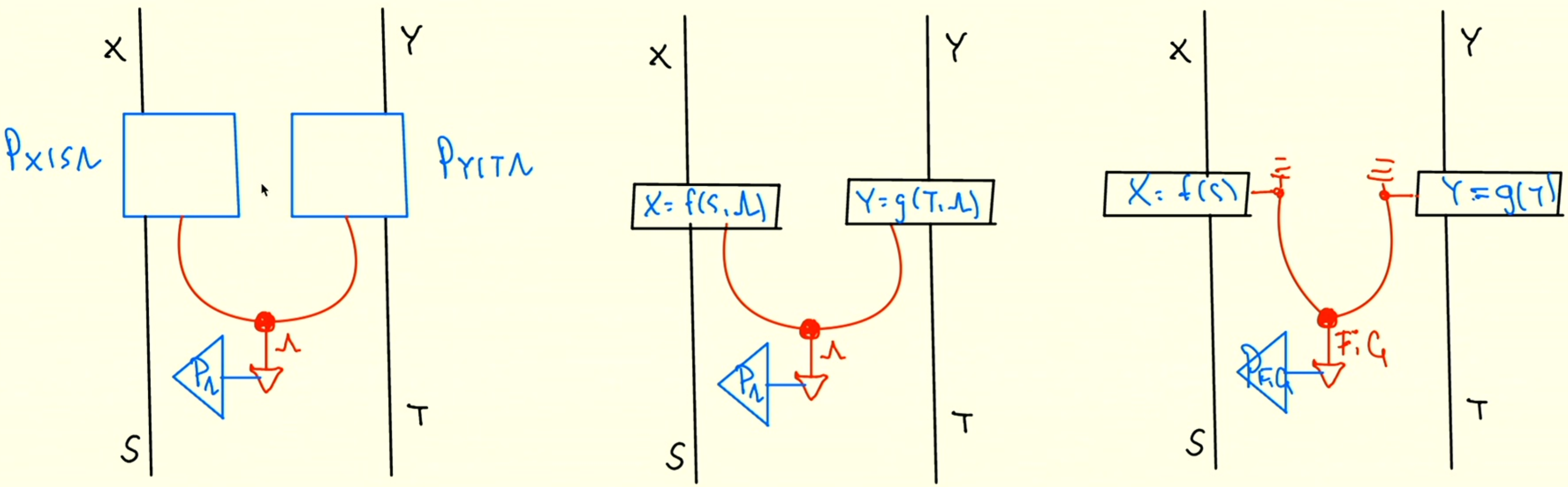
$P_{Y|T\Lambda}$

P_{Λ}

P_S

P_T

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$


X
Y
Pxisλ
Pyitλ
S
T
X = f(S, λ)
Y = g(T, λ)
X = f(S)
Y = g(T)
F, G

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

If X,Y,S,T are binary, Λ can have cardinality 16

$$p_{xy|st} := P_{XY|ST}(xy|st) \qquad q_{fg} := P_{F,G}(f,g)$$

$$x, y, s, t \in \{0,1\} \qquad f, g \in \{\mathrm{id}, \mathrm{fp}, \mathrm{r}_0, \mathrm{r}_1\}$$

$$p_{00|00} = q_{\mathrm{r}_0,\mathrm{r}_0} + q_{\mathrm{r}_0,\mathrm{id}} + q_{\mathrm{id},\mathrm{r}_0} + q_{\mathrm{id},\mathrm{id}}$$

$$p_{00|01} = q_{\mathrm{r}_0,\mathrm{r}_1} + q_{\mathrm{r}_0,\mathrm{fp}} + q_{\mathrm{id},\mathrm{r}_1} + q_{\mathrm{id},\mathrm{fp}}$$

$$0 \leq q_{fg} \leq 1 \; \forall f, g$$

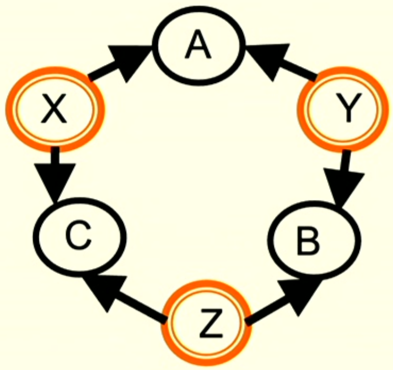
$\vdots$

16 linear equalities + inequalities
One must implement linear quantifier elimination to implicitize the 16 q's.

Techniques for determining upper bounds on cardinalities of the latent variables in more general causal structures

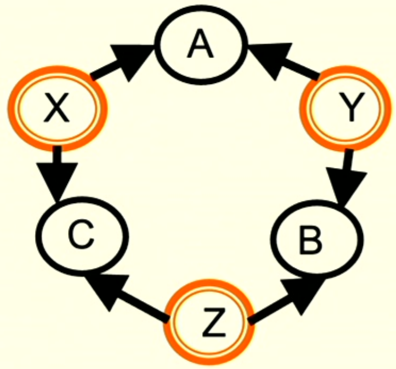
R. Evans, Annals of Statistics, **46**, 2623 (2018)

D. Rosset, N. Gisin, and E. Wolfe. Quantum Inf. & Comp. **18,** 0910 (2018)



A, B, C binary → Sufficient for X, Y, Z to be 6-valued

Triangle scenario



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_X
P_Y
P_Z

$$P_{ABC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_{C|ZX}\mathrm{P}_X\mathrm{P}_Y\mathrm{P}_Z$$

With more than one latent variable,
we require **nonlinear** quantifier elimination
which scales badly

Other techniques implicitizing parameters referring to hidden variables

Entropy cone techniques:

R. Chaves and T. Fritz, Phys. Rev. A 85 (2012)
T. Fritz, New J. Phys. 14 103001 (2012)
R. Chaves, L. Luft, D. Gross, New J. Phys. 16, 043001 (2014)
R. Chaves, L. Luft, T. O. Maciel, D. Gross, D. Janzing, B. Schölkopf, Proceedings of UAI 2014
M. Weilenmann and R. Colbeck,Proc. Roy. Soc. A 473.2207 (2017): 20170483.

Covariance matrix techniques:

A. Kela, K. von Prillwitz, J. Aberg, R. Chaves, and D. Gross, arXiv:1701.00652 (2017).

Strategy 2: Implicitization of parameters referring to counterfactual possibilities

Example 1:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \textstyle\sum_Z P_{XYZ}$$
$$P_{YZ} := \textstyle\sum_X P_{XYZ}$$
$$P_{XZ} := \textstyle\sum_Y P_{XYZ}$$

?

Example 2:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11] = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Example 2:

$$P_{XY}^{\text{target}} = \cancel{\tfrac{1}{2}[00] + \tfrac{1}{2}[11]} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that $P_{XY} = P_{XY}^{\text{target}}$, $P_{YZ} = P_{YZ}^{\text{target}}$, $P_{XZ} = P_{XZ}^{\text{target}}$ where $P_{XY} := \sum_Z P_{XYZ}$, $P_{YZ} := \sum_X P_{XYZ}$, $P_{XZ} := \sum_Y P_{XYZ}$?

Answer: no!

consider $[000], [001], [010], [011], [100], [101], [110], [111]$

Consider binary X, Y and Z

$$\exists P_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\Downarrow$$

Marginal compatibility constraints

$$P_X = \sum_Y P_{XY} = \sum_Z P_{XZ}$$

$$P_Y = \sum_X P_{XY} = \sum_Z P_{YZ}$$

$$P_Z = \sum_X P_{XZ} = \sum_Y P_{YZ}$$

$$P_X + P_Y + P_Z - P_{XY} - P_{YZ} - P_{XZ} \leq 1$$

How this is proven:

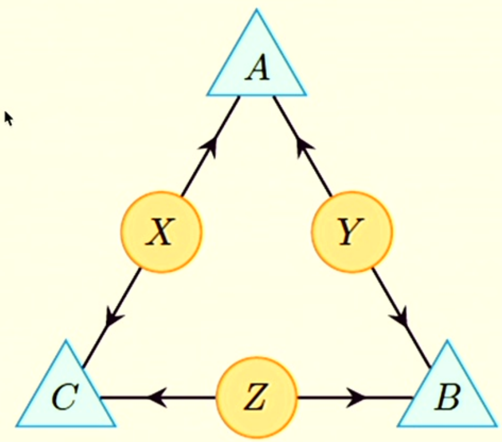
$$\exists Q_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

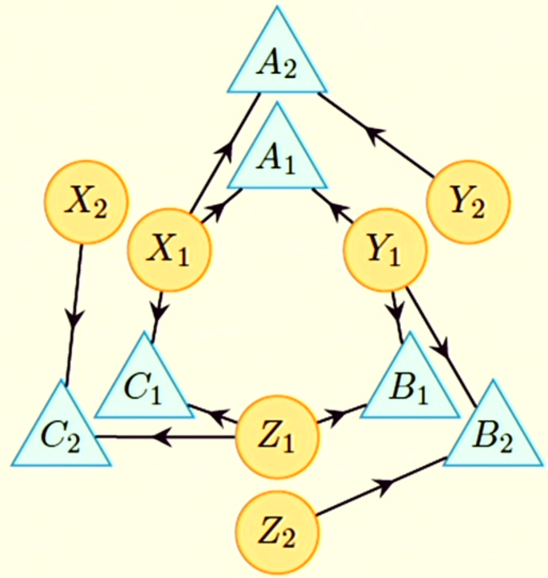
$$\forall x,y : P_{XY}(xy) = \textstyle\sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \textstyle\sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \textstyle\sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \textstyle\sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \textstyle\sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \textstyle\sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

Linear quantifier elimination

$$\forall x : P_X(x) = \textstyle\sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$
$$\forall y : P_Y(y) = \textstyle\sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$
$$\forall z : P_Z(z) = \textstyle\sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$
$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

Inflation DAGs


A
X
Y
C
Z
B

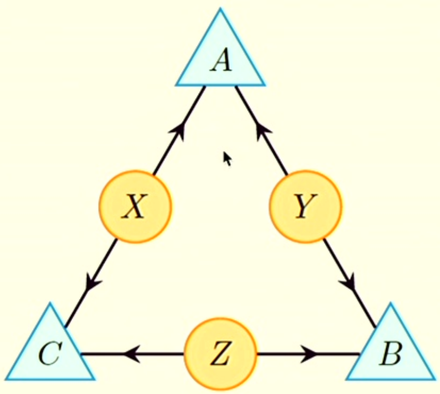

A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2

$$G' \in \text{inflations}(G)$$

if and only if

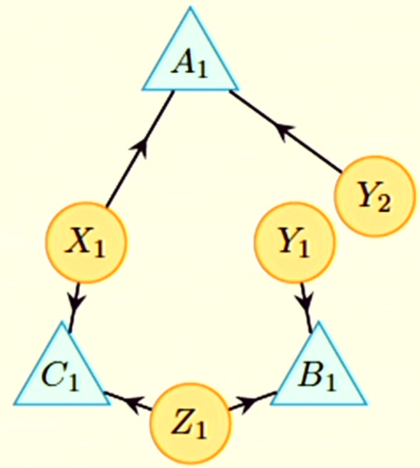
$$\forall A_i \in \text{nodes}(G') : \ \text{ansubgraph}_{G'}(A_i) \sim \text{ansubgraph}_{G}(A).$$

model M on DAG G



$P_{A|XY}$
$P_{B|YZ}$
$P_{C|XZ}$
P_X
P_Y
P_Z

M’ = G $\rightarrow$ G’ Inflation of M

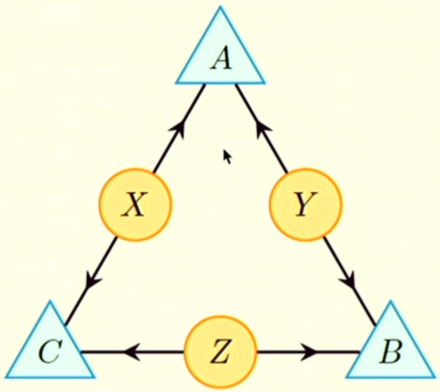


$P_{A_1|X_1Y_2}$
$P_{B_1|Y_1Z_1}$
$P_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

with symmetry constraint:

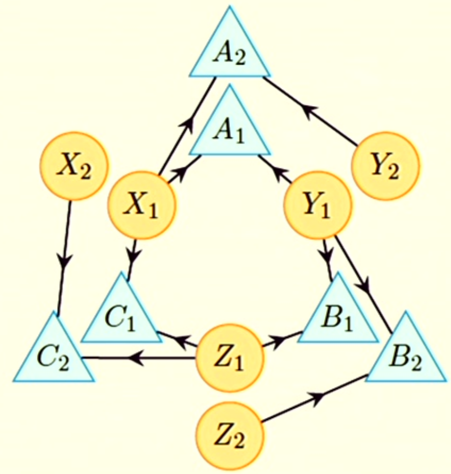
$$P_{Y_1} = P_{Y_2}$$

model M on DAG G



$$\mathrm{P}_{A|XY}$$
$$\mathrm{P}_{B|YZ}$$
$$\mathrm{P}_{C|XZ}$$
$$\mathrm{P}_{X}$$
$$\mathrm{P}_{Y}$$
$$\mathrm{P}_{Z}$$

M' = G → G' Inflation of M

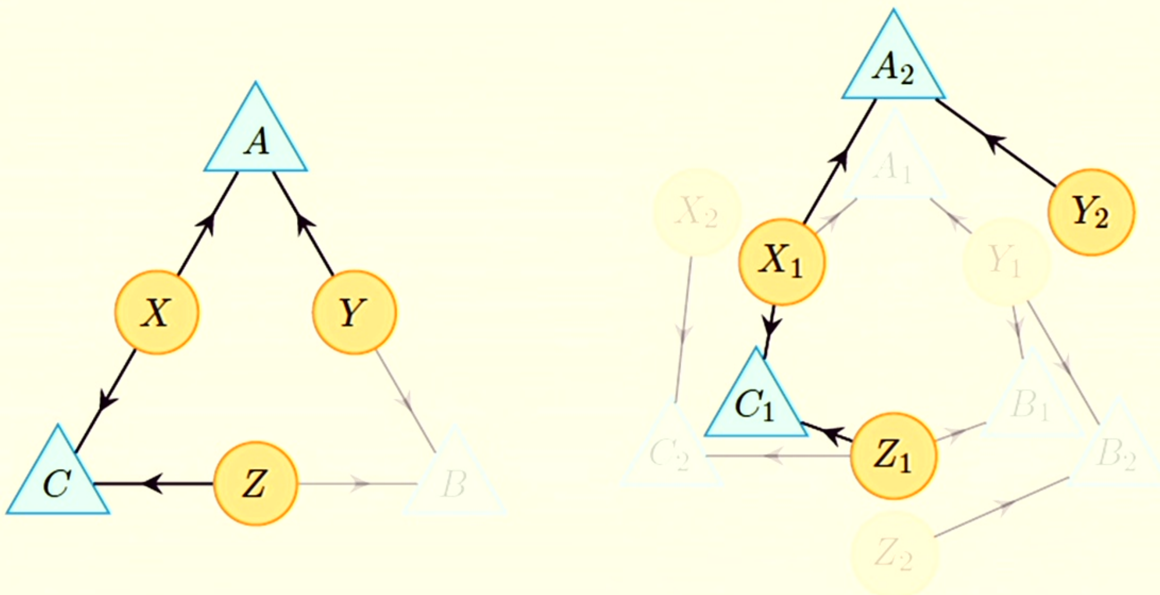


$\mathrm{P}_{A_1\|X_1Y_1}$	$\mathrm{P}_{A_2\|X_1Y_2}$
$\mathrm{P}_{B_1\|Y_1Z_1}$	$\mathrm{P}_{B_2\|Y_1Z_2}$
$\mathrm{P}_{C_1\|X_1Z_1}$	$\mathrm{P}_{C_2\|X_2Z_1}$
P_{X_1}	P_{X_2}
P_{Y_1}	P_{Y_2}
P_{Z_1}	P_{Z_2}

with symmetry constraints:

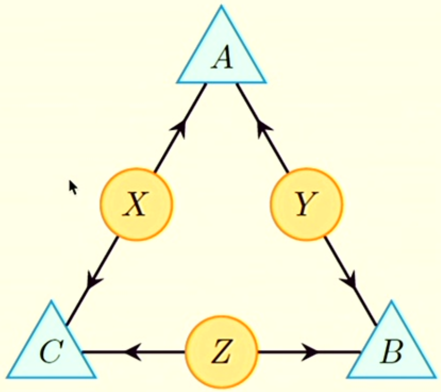
$$\mathrm{P}_{A_1|X_1Y_1} = \mathrm{P}_{A_2|X_1Y_2}$$
$$\mathrm{P}_{B_1|Y_1Z_1} = \mathrm{P}_{B_2|Y_1Z_2}$$
$$\mathrm{P}_{C_1|X_1Z_1} = \mathrm{P}_{C_2|X_2Z_1}$$
$$\mathrm{P}_{X_1} = \mathrm{P}_{X_2}$$
$$\mathrm{P}_{Y_1} = \mathrm{P}_{Y_2}$$
$$\mathrm{P}_{Z_1} = \mathrm{P}_{Z_2}$$

Injectable sets of observed variables
in the inflation DAG



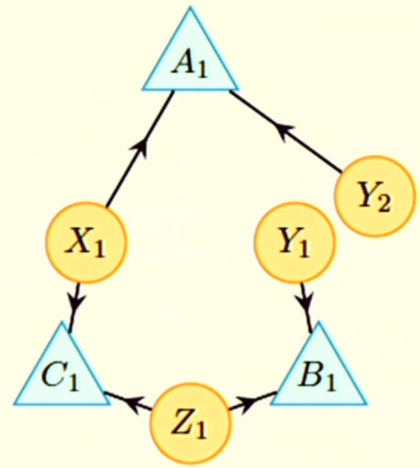
$\{A_2 C_1\}$ is an injectable set

model M on DAG G



$P_{A|XY}$
$P_{B|YZ}$
$P_{C|XZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M

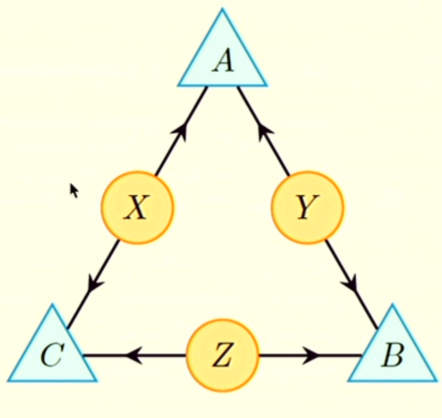


$P_{A_1|X_1Y_2}$
$P_{B_1|Y_1Z_1}$
$P_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$\{A_1 C_1\}$ is an injectable set

model M on DAG G

M' = G → G' Inflation of M



$\mathrm{P}_{A|XY}$

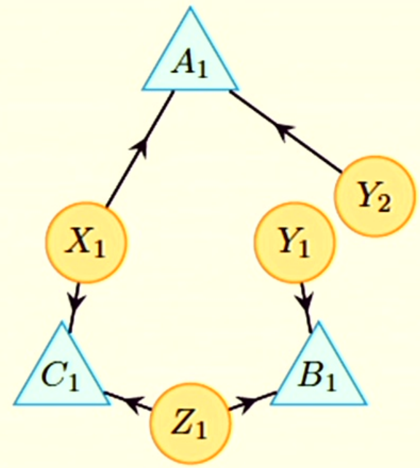
$\mathrm{P}_{B|YZ}$

$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

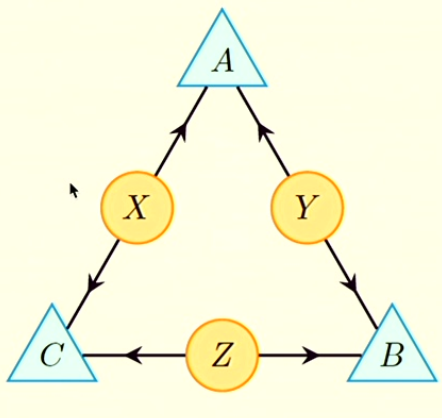
P_{Z_1}

$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

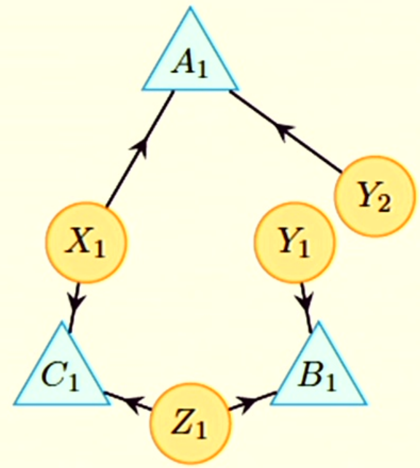
$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M' = G $\rightarrow$ G' Inflation of M



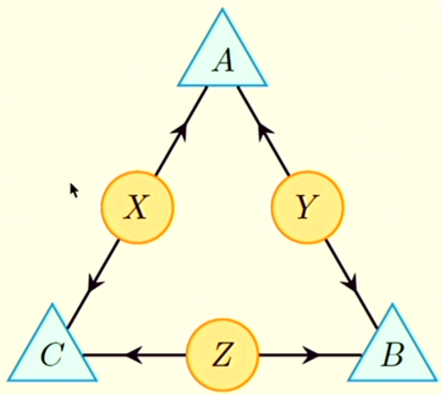
$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$
$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

$$P_{AC} \text{ compatible with } M \quad \Longrightarrow \quad P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

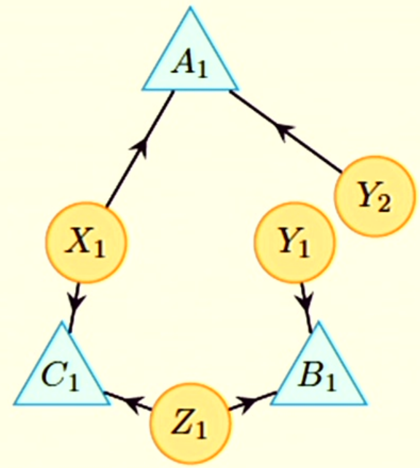
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

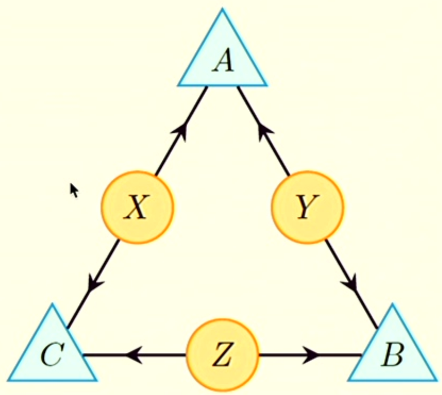
$\{A_1B_1\}$ is *not* an injectable set

$$\mathrm{P}_{A_1B_1} = \left(\sum_{X_1Y_2} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{Y_2}\mathrm{P}_{X_1}\right)\left(\sum_{Z_1Y_1} \mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{Y_1}\mathrm{P}_{Z_1}\right)$$

$$\mathrm{P}_{AB} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

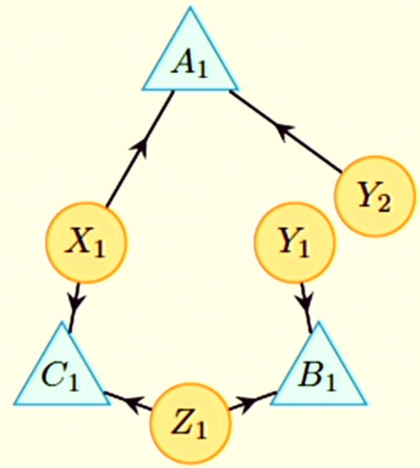
P_{AB} compatible with M $\quad\not\Longrightarrow\quad$ $P_{A_1B_1} = P_{AB}$ compatible with M'

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



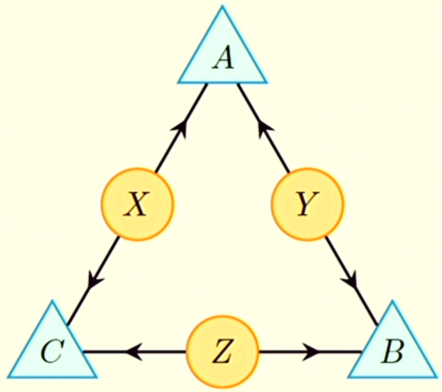
$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

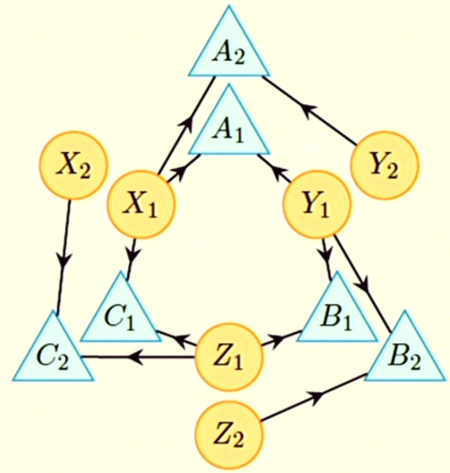
$(P_A, P_B, P_C, P_{AC}, P_{BC})$ compatible with M $\Longrightarrow$ $(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$ compatible with M'

where $\mathrm{P}_{A_1} = \mathrm{P}_A$, $\mathrm{P}_{B_1} = \mathrm{P}_B$, $\mathrm{P}_{C_1} = \mathrm{P}_C$, $\mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$, $\mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

model M on DAG G



M' = G → G' Inflation of M



Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_2\}, \{B_2\}, \{C_2\},$
$\{A_1B_1\}, \{A_1, B_2\}, \{B_1C_1\}, \{B_1, C_2\}, \{C_1, A_1\}, \{C_1, A_2\},$
$\{A_1B_1C_1\}$

M' = G → G' Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is not compatible with M

$\Longleftarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is not compatible with M'

M’ = G → G’ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Deriving
causal compatibility inequalities
by the inflation technique

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

$\Downarrow$

$I_{\mathcal{S}'}$ is **satisfied** for
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

$\Longleftarrow$

$I_{\mathcal{S}}$ is **satisfied** for
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

M' = G → G' Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

is compatible with M

$$\Downarrow$$

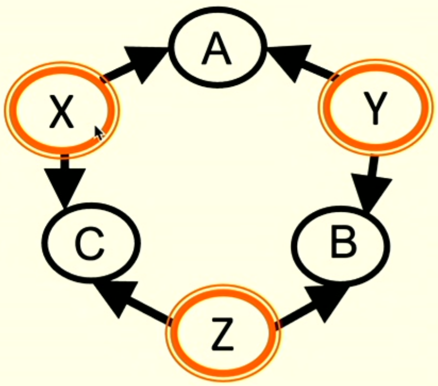
$I_{\mathcal{S}}$ is **satisfied** for

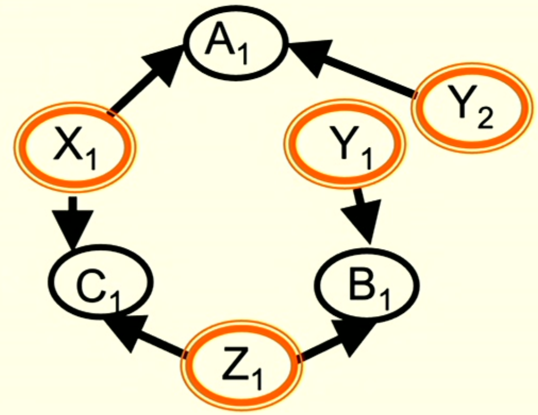
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

M’ = G $\rightarrow$ G’ Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M $\Longleftarrow$ $I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'



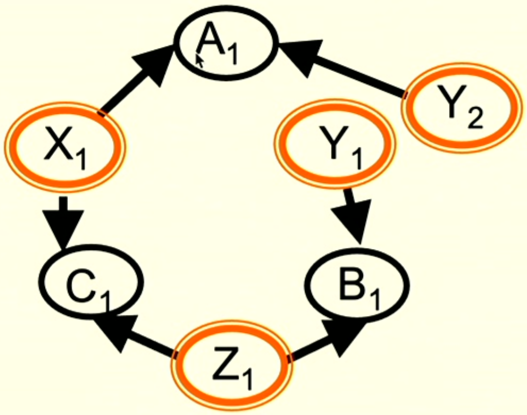


$$P_A+P_B+P_C-P_AP_B-P_{BC}-P_{AC} \leq 1$$
is a causal compatibility inequality for M

$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$
is a causal compatibility inequality for M'

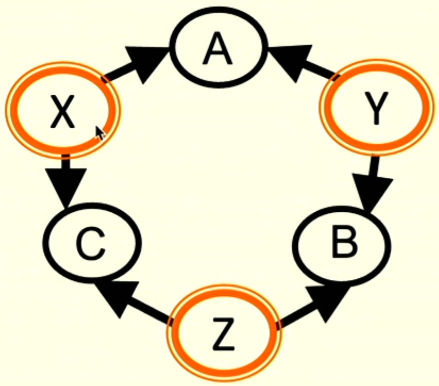
$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

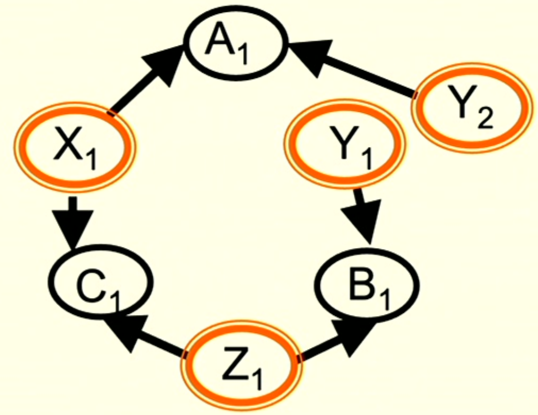
Linear quantifier elimination



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

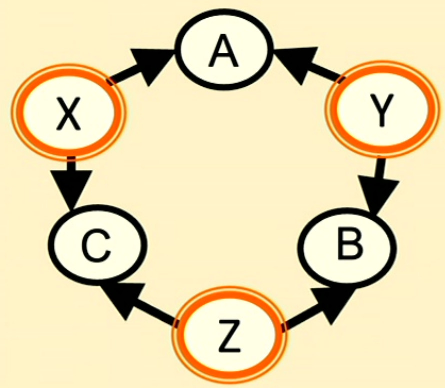




$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$ is a causal compatibility inequality for M

⇐

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$ is a causal compatibility inequality for M'

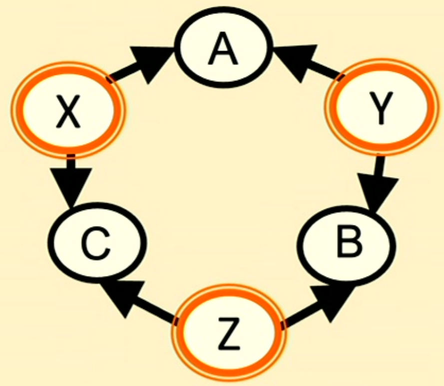


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

causal compatibility inequality

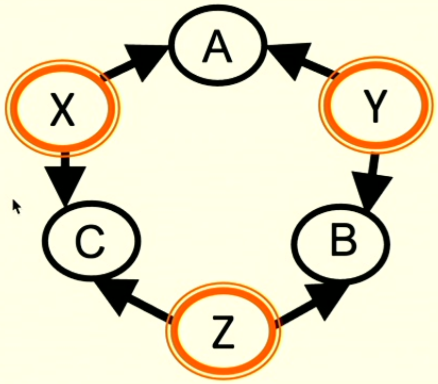
rules out

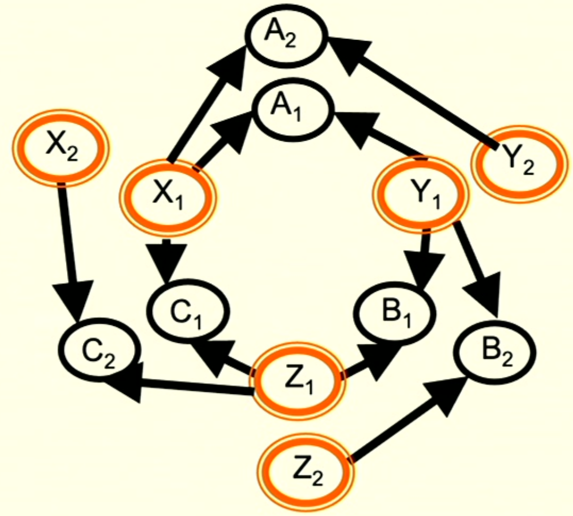
$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$



$$
\begin{aligned}
&P_A(1)P_B(1)P_C(1) \\
&\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) \\
&+ P_{AC}(11)P_B(1) + P_{ABC}(000)
\end{aligned}
$$

causal compatibility inequality





$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

is a causal compatibility inequality for M

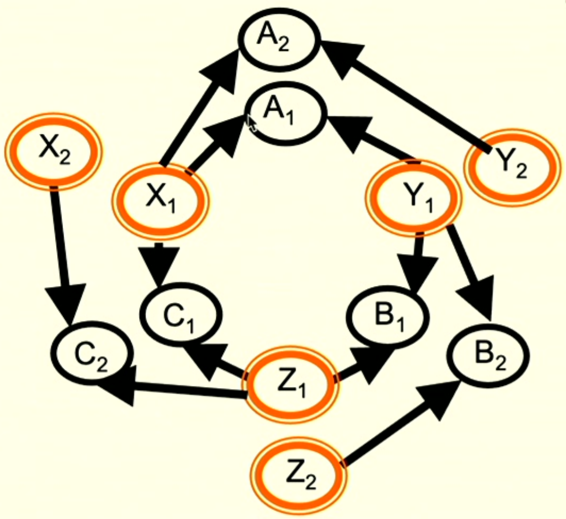
$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

is a causal compatibility inequality for M'

$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$
is a valid set of marginals $\Longrightarrow$

$$P_{A_2B_2C_2}(111) \leq P_{A_1B_2C_2}(111) + P_{B_1C_2A_2}(111) + P_{A_2C_1B_2}(111) + P_{A_1B_1C_1}(000)$$

Linear quantifier elimination



$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$
is compatible with M'

$$A_1B_2 \perp_d C_2 \Longrightarrow P_{A_1B_2C_2} = P_{A_1B_2}P_{C_2},$$
$$B_1C_2 \perp_d A_2 \Longrightarrow P_{B_1C_2A_2} = P_{B_1C_2}P_{A_2},$$
$$A_2C_1 \perp_d B_2 \Longrightarrow P_{A_2C_1B_2} = P_{A_2C_1}P_{B_2},$$
$$A_2 \perp_d B_2 \perp_d C_2 \Longrightarrow P_{A_2B_2C_2} = P_{A_2}P_{B_2}P_{C_2}$$

$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$
is compatible with M' $\Longrightarrow$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

Linear inequality constraints from marginal compatibility (from linear quantifier elimination)

Polynomial inequality constraints for causal compatibility with the original DAG ⇐

+

Polynomial equality constraints from causal compatibility with the inflated DAG (e.g., from d-separation relations)

The technique defines an algorithm for deriving causal compatibility inequalities and for testing compatibility

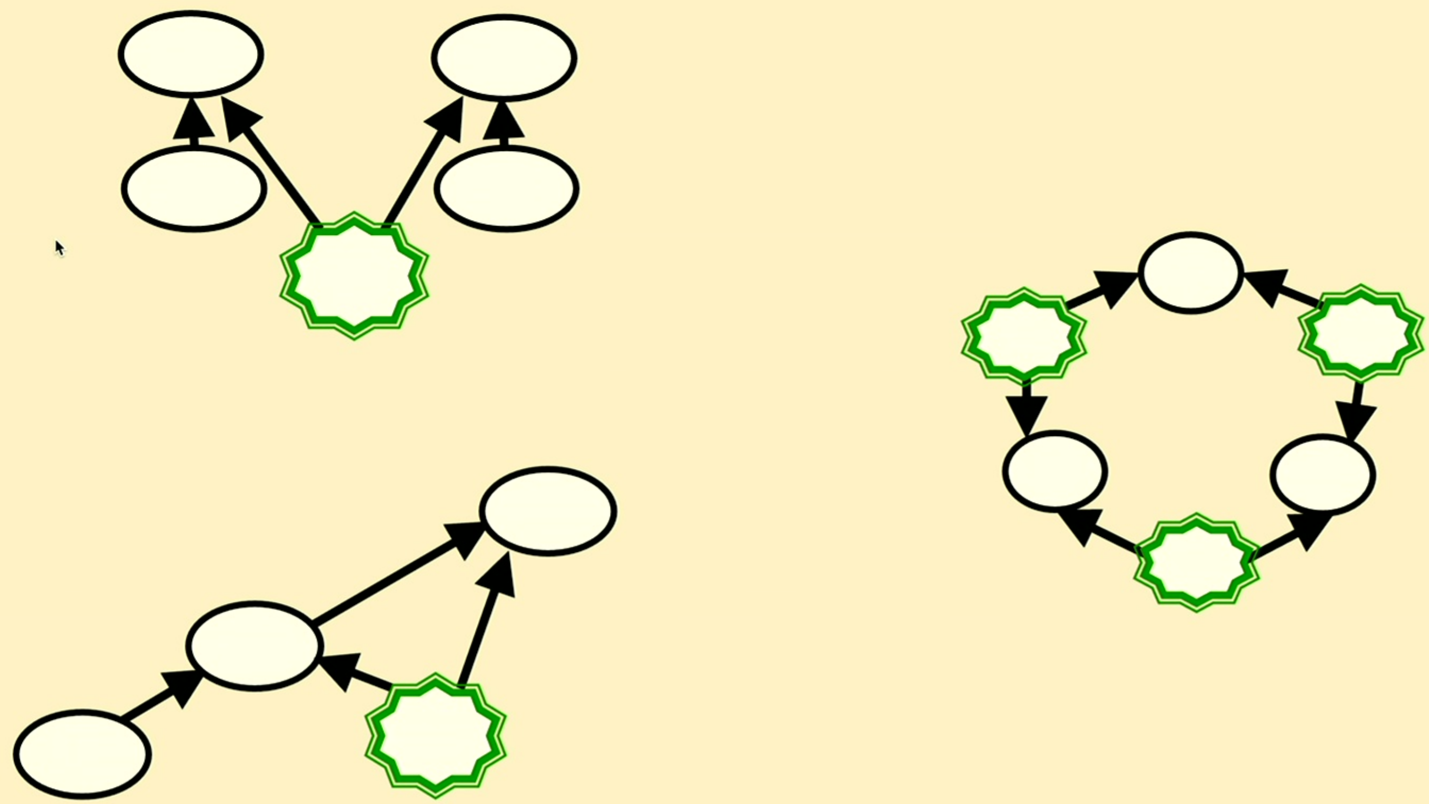
Proof that this provides a convergent heirarchy of tests:
Navascués & Wolfe, J. Causal Inf. 8(1) 70 (2020)

Approaches to Bell arguments that follow essentially
the logic of the inflation technique:

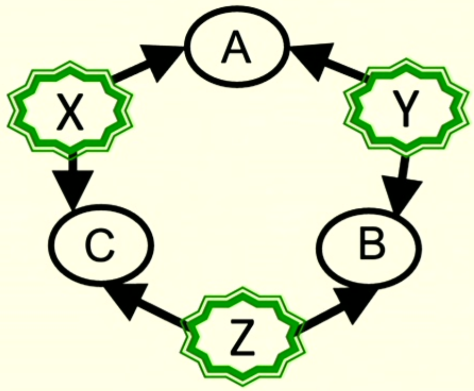
Fine's proof of CHSH inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
...

What probability distributions over classical variables are compatible with a given causal structure when the latent systems can be quantum?

The analogue of finding the Tsirelson bound for the Bell scenario



Triangle scenario



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

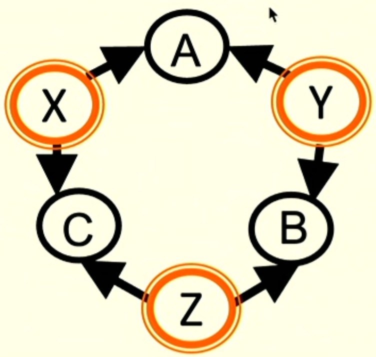
ρ_X

ρ_Y

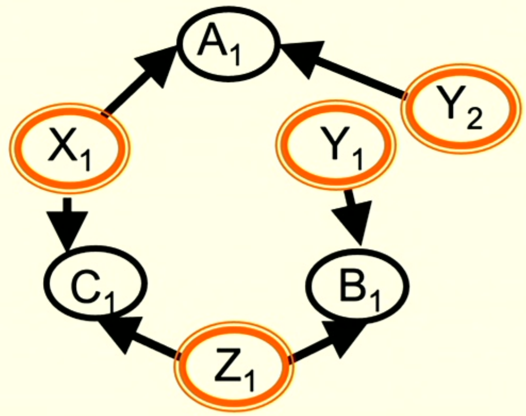
ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

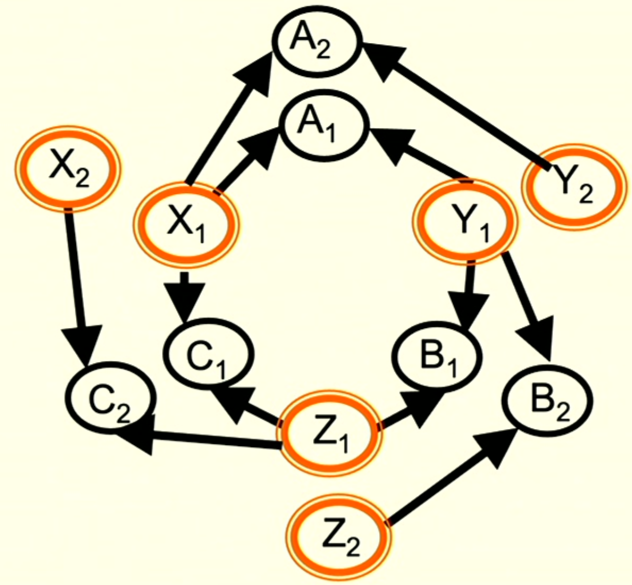


Triangle



Cut inflation of Triangle

Non-fan-out

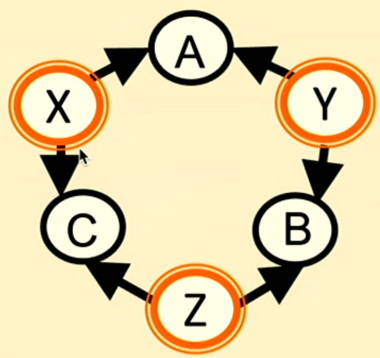


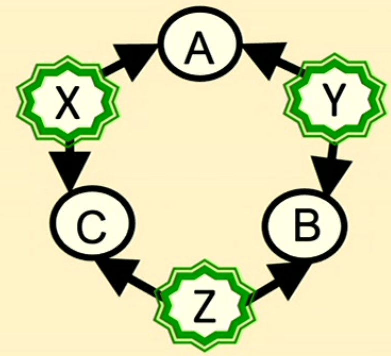
Spiral inflation of Triangle

Fan-out

Classical latents

Quantum latents



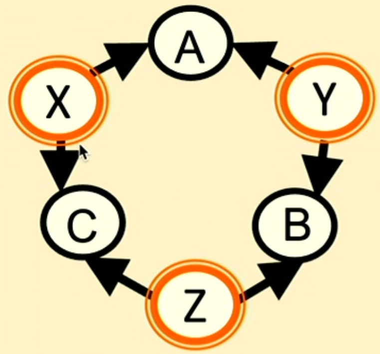


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

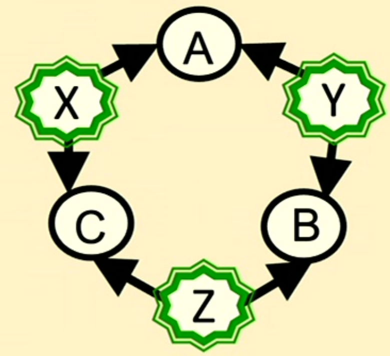
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

This inequality was obtained from a non-fanout inflation and therefore holds for both quantum and classical latents

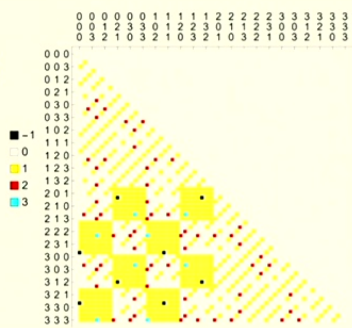
Classical latents



Quantum latents



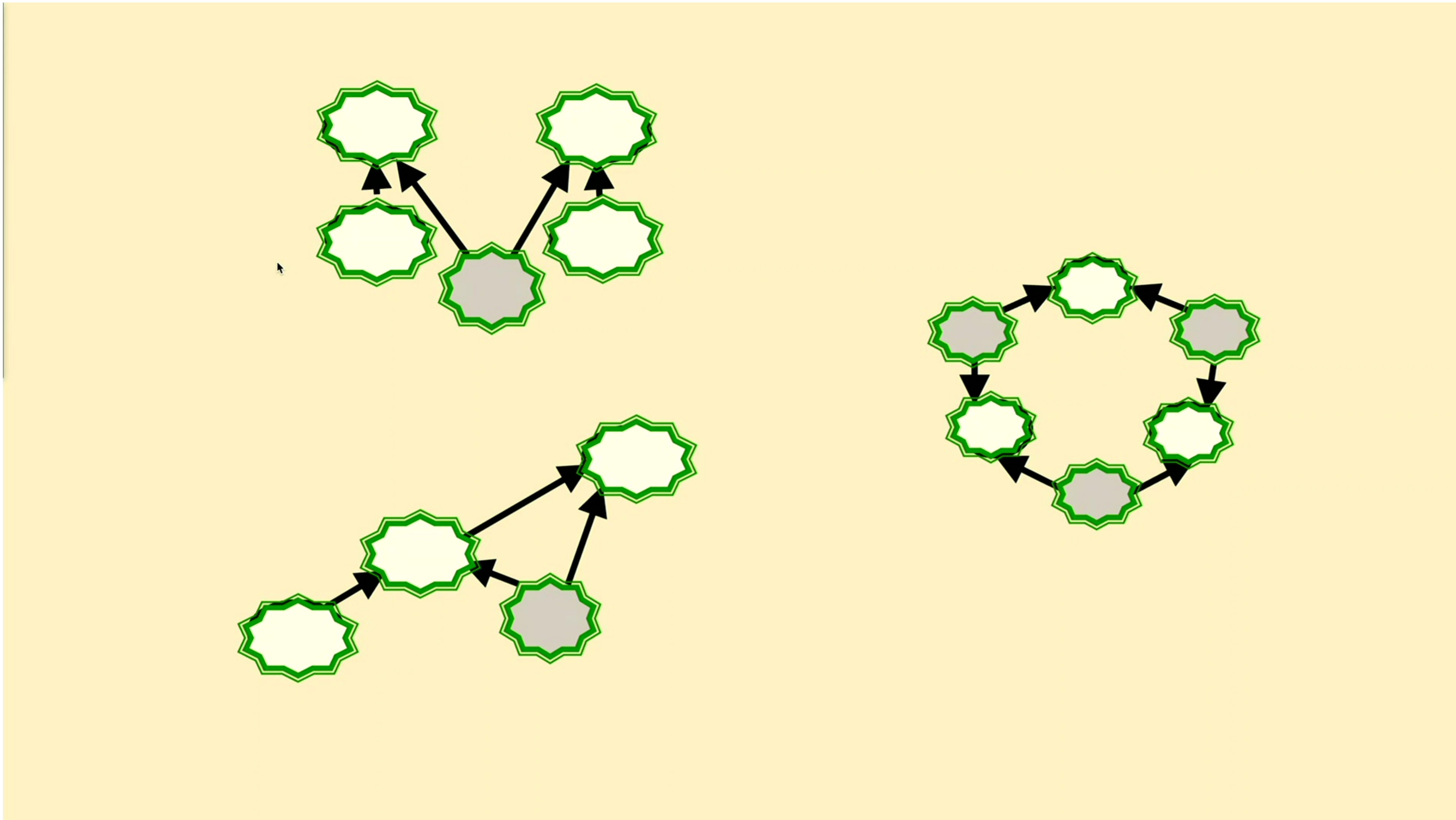
$$\sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$



(b) The coefficients of a quadratic inequality

This inequality can be quantumly violated

Polino et al., Nat. Comm. 14, 909 (2023)



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Causal Inference: Classical and Quantum

PHYS 777-007
Lecturer: Robert Spekkens
TA: Marina Ansanelli

March 6, 2023

Causarum Investigatio
"Investigate the causes"