Title: Lecture - Classical Physics, PHYS 776

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Collection/Series: Classical Physics (Core), PHYS 776, September 3 - October 4, 2024

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Maxwell for $F_{\mu\nu} = F_{[\mu\nu]}$

$$\begin{cases} \nabla_{[\mu} F_{\nu\rho]} = 0 & (1) \\ \nabla_\mu F^{\mu\nu} = -4\pi J^\nu & (2) \end{cases}$$

$(1) \Rightarrow \exists A_\mu$ s.t. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ so that (1) is automatically satisfied.

But A_μ not unique, freedom $A_\mu \mapsto A_\mu + \nabla_\mu \lambda$ (GAUGE)
$(F_{\mu\nu} \mapsto F_{\mu\nu})$

Plugging $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ in (2)

$$\left(\Box := \nabla_\mu \nabla^\mu \equiv \eta^{\mu\nu} \nabla_\mu \nabla_\nu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta \right)$$

$$\rightarrow \Box A_\nu - \nabla_\nu (\nabla_\mu A^\mu) = -4\pi J_\nu$$

Idea: use gauge freedom to get r.d of $\nabla_\mu A^\mu$

Claim: For each A_μ, $\exists\, \lambda(A)$ such that $A'_\mu = A_\mu + \nabla_\mu \lambda(A)$ is s.t. $\nabla_\mu A'^\mu = 0$

"Pf": $0 = \nabla_\mu A'^\mu = \nabla^\mu A_\mu + \Box\lambda$ iff $\Box\lambda = -\nabla_\mu A^\mu$ can be solved for λ.

(Pf postponed)

$\rightsquigarrow$ this choice of $\lambda = \lambda(A)$ defines the Lorentz gauge, $\partial_\mu A^{(L)\mu} = 0$

$\rightsquigarrow$ In Lorentz gauge, Maxwell equations reduce to

$$\boxed{\Box A^{(L)}_\mu = -4\pi J_\mu}$$

Check: $\partial^\mu J_\mu = \Box \partial^\mu A^{(L)}_\mu = 0$ ✓

v.s. $\partial^\mu J_\mu = \partial^\mu \partial^\nu F_{\mu\nu} = 0$ ↑ by symmetry

WAVE EQ

$\Box \phi = -4\pi\rho$ $(\mu = 0)$

$\Box = -\partial_t^2 + \Delta$

- Homogeneous eq $\Box\phi = 0$

(1) in $1+1$ d : $\Box = -\partial_t^2 + \partial_x^2$

light cone coordinates

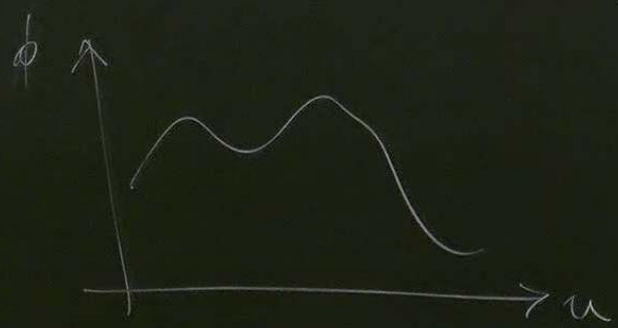
$$\begin{cases} u = t - x \\ v = t + x \end{cases}$$

$\rightsquigarrow \Box = -\partial_u \partial_v$

$0 = \Box\phi = -\partial_u \partial_v \phi$

$\rightarrow \partial_v \phi = 0$ i.e. any $\phi = \phi(u)$ is a sol.

$= \phi(ct - x)$



↑ a profile moving "unchanged" towards the right at speed c

$\rightsquigarrow$ "right mover"

$\rightarrow \partial_u \phi = 0$ i.e. any $\phi = \phi(v) = \phi(ct + x)$ is a sol

$\rightsquigarrow$ "left mover"

(2) in higher dim we can use Fourier analysis

$$\phi(x^\mu) = \phi(t,\vec{x}) = \int d^4k \; e^{ik_\mu x^\mu} \hat{\phi}(k) \qquad k^\mu = (\omega, \vec{k})$$

$$= \int d^3\vec{k}\, d\omega \; e^{-i\omega t + i\vec{k}\cdot\vec{x}} \hat{\phi}(\omega,\vec{k})$$

$$\Box e^{ik_\mu x^\mu} = -k^2 e^{ik_\mu x^\mu} = \int d^3\vec{k}\, d\omega \; e$$

$$\Box\phi = \int d^4k \; (-k^2) e^{ik_\mu x^\mu} \hat{\phi}(k) \qquad (k^2 \equiv k^\mu k_\mu)$$

$$= \int d^3\vec{k}\, d\omega \; (\omega^2 - \vec{k}^2) e^{-i\omega t + i\vec{k}\cdot\vec{x}} \hat{\phi}(\omega,\vec{k})$$

$\Rightarrow$ for $\Box\phi = 0$ we need $\hat{\phi}(k)$ to be supported "on the light cone"

i.e. on k^μ's s.t. $k^\mu k_\mu = -\omega^2 + \vec{k}^2 = 0$

$$\begin{cases} u = t - x \\ v = t + x \end{cases}$$

$\rightarrow \partial_u \phi = 0$ i.e. any $\phi = \phi(v)$

That is

$$\hat{\phi}(k^\mu) = \delta(\omega^2 - \vec{k}^2)\, \hat{\varphi}(\vec{k})$$

(3d!)

$$= \left(\frac{\delta(\omega - |\vec{k}|)}{2\omega} + \frac{\delta(\omega + |\vec{k}|)}{2\omega} \right) \hat{\varphi}(\vec{k})$$

$$\rightsquigarrow \phi(x^\mu) = \int \frac{d^3k}{2\sqrt{\vec{k}^2}} \left(e^{-i|\vec{k}|t + i\vec{k}\cdot\vec{x}}\, \hat{\varphi}_+(\vec{k}) + e^{i|\vec{k}|t + i\vec{k}\vec{x}}\, \hat{\varphi}_-(\vec{k}) \right)$$

If ϕ is real, then we get a relation between $\hat{\varphi}_+(\vec{k})$ & $\hat{\varphi}_-(-\vec{k})$

$\phi = \phi(u) = \phi(ct+x)$ is a sol

$\rightsquigarrow$ "left movers"

$\rightarrow$ for $\Box\phi = 0$ we need $\hat{\phi}(k)$ to be supported "on the light cone"

i.e. on k^μ's s.t. $k^\mu k_\mu = -\omega^2 + \vec{k}^2 = 0$

In 1+1d, sanity check:

$$\phi(x^\mu) = \int \frac{dk}{k} \left(e^{-ik(t-x)} \hat{\varphi}_+(k) + e^{ik(t+x)} \hat{\varphi}_-(k) \right)$$

$$= \phi_+(u) + \phi_-(v)$$

SOURCES

Green's function method

heuristics: look first at "point sources" @ y

$$\Box_x G(x,y) = -4\pi \delta(x,y) \equiv -4\pi \delta^{(4)}(x-y)$$

$$\Delta_x G(x,y) = -4\pi \delta(x,y) \equiv -4\pi \delta^{(3)}(\vec{x}-\vec{y})$$

Assume we know $G(x,y)$, for now.

Then

$$\phi(x) = \int dy \; G(x,y)\, \rho(y)$$

Pf: • $\Box\phi(x) = \int dy \left(\Box_x G(x,y)\right) \rho(y)$

$$= -4\pi \int dy \; \delta(x,y)\, \rho(y)$$

$$= -4\pi \rho(x)$$

• $\Delta\phi = \ldots = -4\pi\rho$ ∎

Electrostatics

$\partial_t \equiv 0$

$\Delta \phi(x) = -4\pi \rho(x)$ Laplace/Poisson eq.

$\Delta_x G(x,y) = \delta(x,y)$

THM (Green) Let G be a Green's function of Δ: i.b.p.

$$\phi(x) = \int_R d^3y \; \delta(x-y)\,\phi(y) = -\int_R \left(\Delta_y G(y,x)\right)\phi(y) = -\int_R d^3y \; G(y,x)\,\Delta_y \phi(y)$$

Electrostatics

$\partial_t \equiv 0$

$\Delta\phi(x) = -4\pi\rho(x)$ Laplace/Poisson eq.

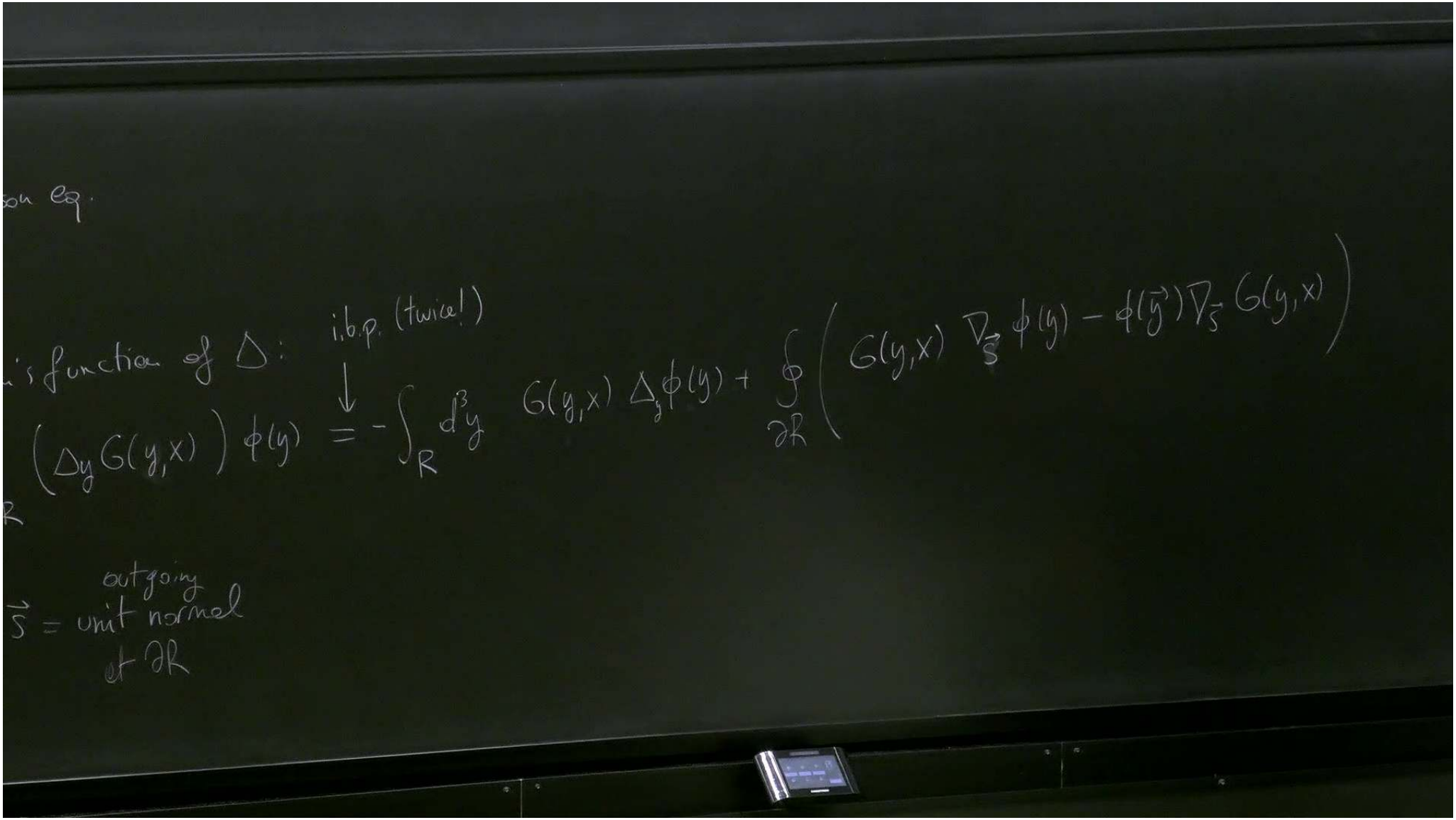
$\downarrow$

$\Delta_x G(x,y) = \delta(x,y)$

THM (Green) Let G be a Green's function of Δ: i.b.p. (twice!)

$$\phi(x) = \int_R d^3y\ \delta(x-y)\,\phi(y) = -\int_R \left(\Delta_y G(y,x)\right)\phi(y) \overset{\downarrow}{=} -\int_R d^3y\ G(y,x)\,\Delta\phi(y)$$

where $\nabla_{\vec{S}} = \vec{S}\cdot\vec{\nabla}\big|_{\partial R}$ with $\vec{S}$ = outgoing unit normal at ∂R


on eq.
is function of Δ:
i.b.p. (twice!)
$(\Delta_y G(y,x))\,\phi(y) = -\int_R d^3y\; G(y,x)\,\Delta_y\phi(y) + \oint_{\partial R}\left(G(y,x)\,\nabla_{\vec{S}}\phi(y) - \phi(\vec{y})\,\nabla_{\vec{S}}G(y,x)\right)$
R
$\vec{S}$ = outgoing unit normal at ∂R

sson eq.

een's function of Δ:

$$\int_R \left(\Delta_y G(y,x)\right)\phi(y) \overset{\text{i.b.p. (twice!)}}{=} -\int_R d^3y\; G(y,x)\,\Delta_y\phi(y) + \oint_{\partial R}\Big(G(y,x)\,\nabla_{\vec{S}}\phi(y) - \phi(\vec{y})\,\nabla_{\vec{S}}G(y,x)\Big)$$

will be $-4\pi\rho(y)$

will allow me to fix/understand boundary conds

$\vec{S}$ = outgoing unit normal at ∂R

where $\nabla_{\vec{S}} = \vec{S}\cdot\nabla\Big|_{\partial R}$ with $\vec{S}$ = unit normal at ∂R

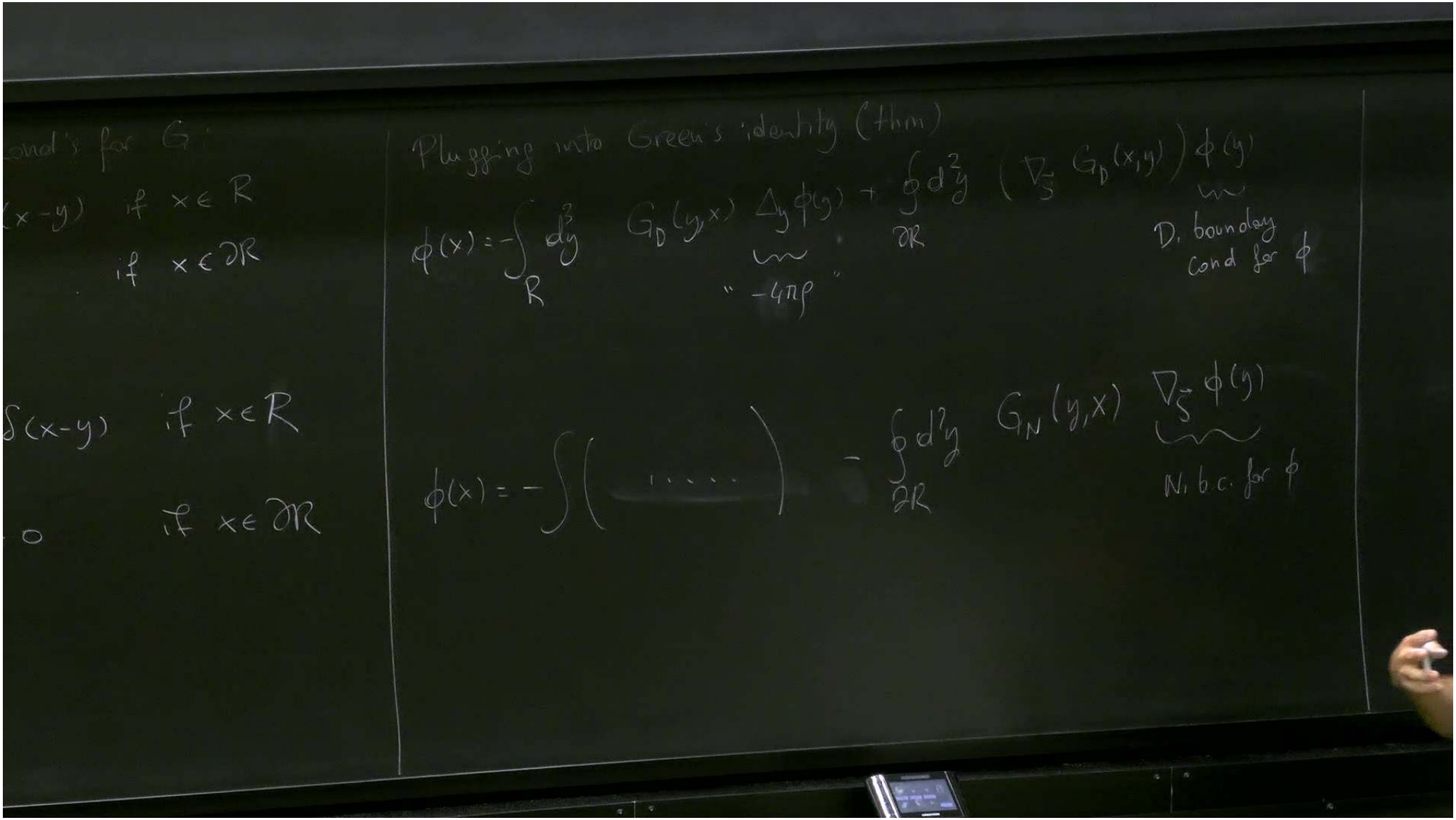
Two main options for boundary cond's for G:

Dirichlet
$$\begin{cases} \Delta_x G_D(x,y) = \delta(x-y) & \text{if } x \in R \\ G_D(x,y) = 0 & \text{if } x \in \partial R \end{cases}$$

Neumann
$$\begin{cases} \Delta_x G_N(x,y) = \delta(x-y) & \text{if } x \in R \\ \vec{S}\cdot\nabla G(x,y) = 0 & \text{if } x \in \partial R \end{cases}$$

Plugging into Green's identity (thm)

$$\phi(x) = -\int_R dy \; G_D(y,x) \underbrace{\Delta_y \phi(y)}_{\text{“}-4\pi\rho\text{”}} +$$


ould's for G
(x-y) if x∈R
if x∈∂R
δ(x-y) if x∈R
0 if x∈∂R
Plugging into Green's identity (thm)
φ(x) = -∫_R d³y G_D(y,x) Δ_y φ(y) + ∮_∂R d²y (∇_S G_D(x,y)) φ(y)
"-4πρ"
D. boundary cond for φ
φ(x) = -∫(.....) - ∮_∂R d²y G_N(y,x) ∇_S φ(y)
N. b.c. for φ

where $\nabla_{\vec{S}} = \vec{S}\cdot\nabla\Big|_{\partial R}$ with $\vec{S}$ = unit normal at ∂R

Two main options for boundary cond's for G:

Dirichlet
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Green's function depend on
- choice of diff. operator (Δ)
- — " — bdry conds (D, N)
- region R

Plugging into Green's identity (thm)

$$\phi(x) = -\int_R d^3y \; G_D(y,x) \underbrace{\Delta_y \phi(y)}_{\text{"} -4\pi\rho \text{"}} + \oint_{\partial R} d^2y \; (\dots$$

$$\phi(x) = -\int(\dots) - \oint_{\partial R} d^2y \; G_N$$

Green's functions depend on
- choice of diff. operator (Δ)
- — " — bdry conds (D, N)
- region R

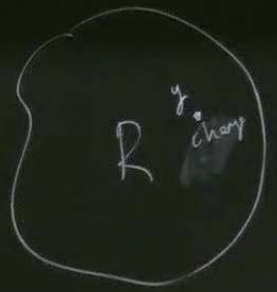
If R is "symmetric enough", we can try the method of images to find solutions

1) Find a sol. to the Poisson eq without bothering about boundary conditions (nor R)

$$\Delta_x G_0(x,y) = \delta(x,y)$$

2) Note that $G_0(x,y)$ with $y_i \notin R$ ("image charges")
satisfies the homogeneous (Laplace) eq in R

3) The game is to find (y_i) s.t. $G(x,y) = G_0(x,y) + \sum_i G_0(x,y_i)$ satisfies the bdry conds.



y_i
image charge

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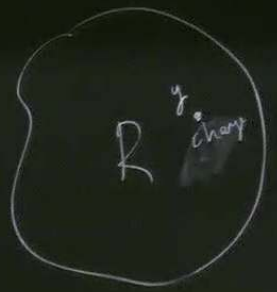
3) The game is to find (y_i) s.t. $G(x,y) = G_0(x,y) + \sum_i q_i G_0(x,y_i)$ satisfies the bdry conds.



y_i: image charge

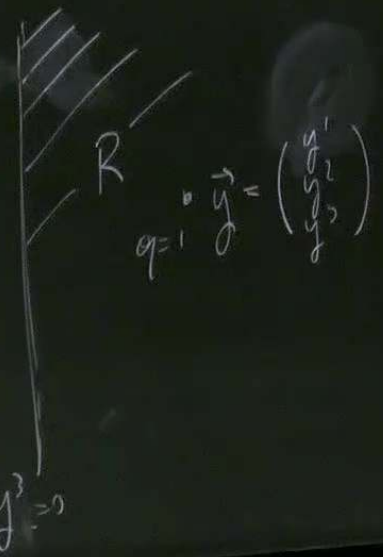
Ex

d = 3, Dirichlet G.f. for $R = \{z \geq 0\}$ half space.

1) $G_0(\vec{x}, \vec{y}) = \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}$

(find this out playing around with Green's thm & symmetry considerations)

$\vec{y}_m = \begin{pmatrix} y^1 \\ y^2 \\ -y^3 \end{pmatrix}$ $q = -1$



$G_D(\vec{x}, \vec{y}) = \frac{1}{4\pi} \left(\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x} - \vec{y}_m|} \right)$

indeed $G_D(\vec{x}, \vec{y})\Big|_{x^3 = 0} = 0$

i

image charge q_i

ary conds.