Title: Lecture - Classical Physics, PHYS 776

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<u>Maxwell</u> for $F_{\mu\nu} = F_{[\mu\nu]}$

$$\begin{cases} \nabla_{[\mu} F_{\nu\rho]} = 0 & (1) \\ \nabla_\mu F^{\mu\nu} = -4\pi J^\nu & (2) \end{cases}$$

(1) $\Rightarrow$ $\exists A_\mu$ s.t. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ so that (1) is automatically satisfied.

But A_μ <u>not</u> unique, freedom $A_\mu \mapsto A_\mu + \nabla_\mu \lambda$ (GAUGE)
$(F_{\mu\nu} \mapsto F_{\mu\nu})$

Plugging $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$ in (2)

$$\left(\Box := \nabla_\mu \nabla^\mu \equiv \eta^{\mu\nu}\nabla_\mu\nabla_\nu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta \right)$$

$$\rightarrow \Box A_\nu - \nabla_\nu(\nabla_\mu A^\mu) = -4\pi J_\nu$$

Idea: use gauge freedom to get rid of $\nabla_\mu A^\mu$

Claim: For each A_μ, $\exists\ \lambda(A)$ such that $A'_\mu = A_\mu + \nabla_\mu \lambda(A)$ is s.t. $\nabla_\mu A'^\mu = 0$

"Pf": $0 = \nabla_\mu A'^\mu = \nabla^\mu A_\mu + \Box\lambda$ iff $\Box\lambda = -\nabla_\mu A^\mu$ can be solved for λ.

AUGE

(P

(Pf postponed)

$\rightsquigarrow$ this choice of $\lambda = \lambda(A)$ defines the Lorentz gauge, $\nabla_\mu A^{(L)\mu} = 0$

$\rightarrow$ In Lorentz gauge, Maxwell equations reduce to

$$\boxed{\Box A^{(L)}_\mu = -4\pi J_\mu}$$

<u>Check</u>: $\nabla^\mu J_\mu = \Box \nabla^\mu A^{(L)}_\mu = 0$ ✓

v.s. $\nabla^\mu J_\mu = \nabla^\mu \nabla^\nu F_{\mu\nu} \underset{\uparrow}{=} 0$

by symmetry

$= 0$

can be

λ

WAVE EQ

$(\mu = 0)$

$$\Box \phi = -4\pi \rho$$

$$\Box = -\partial_t^2 + \Delta$$

- Homogeneous eq $\quad \Box\phi = 0$

1. in 1+1 d : $\quad \Box = -\partial_t^2 + \partial_x^2$

light cone coordinates

$$\begin{cases} u = t - x \\ v = t + x \end{cases}$$

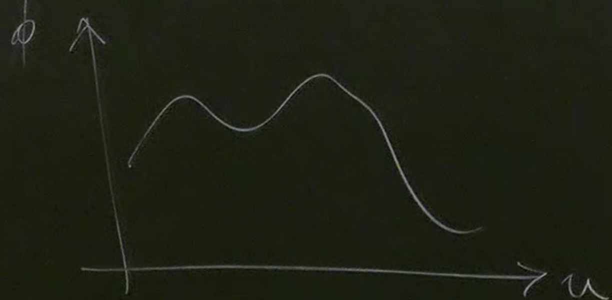
$\rightsquigarrow \Box = -\partial_u \partial_v$

$0 = \Box \phi = -\partial_u \partial_v \phi$

$\rightarrow \partial_v \phi = 0$ i.e. any $\phi = \phi(u)$ is a sol.

$= \phi(ct - x)$

a profile moving "unchanged" towards the right at speed c



$\rightsquigarrow$ "right mover"

$\rightarrow \partial_u \phi = 0$ i.e. any $\phi = \phi(v) = \phi(ct + x)$ is a sol

$\rightsquigarrow$ "left movers"

② in higher dim we can use Fourier analysis.

$$k^\mu = (\omega, \vec{k})$$

$$\phi(x^\mu) = \phi(t, \vec{x}) = \int d^4k \; e^{ik_\mu x^\mu} \hat{\phi}(k)$$

$$= \int d^3\vec{k} \, d\omega \; e^{-i\omega t + i\vec{k}\cdot\vec{x}} \hat{\phi}(\omega, \vec{k})$$

$$\Box e^{ik_\mu x^\mu} = -k^2 e^{ik_\mu x^\mu}$$

$$\Box \phi = \int d^4k \; (-k^2) e^{ik_\mu x^\mu} \hat{\phi}(k) \qquad (k^2 \equiv k^\mu k_\mu)$$

$$= \int d^3\vec{k} \, d\omega \; (\omega^2 - \vec{k}^2) e^{-i\omega t + i\vec{k}\cdot\vec{x}} \hat{\phi}(\omega, \vec{k})$$

$\longrightarrow$ for $\Box\phi = 0$ we need $\hat{\phi}(k)$ be supported "on the light cone"

i.e. on k^μ's s.t. $k^\mu k_\mu = -\omega^2 + \vec{k}^2 = 0$

$$\begin{cases} u = ct - x \\ V = ct + x \end{cases}$$

$\to \partial_u \phi = 0$ i.e. any $\phi = \phi(v)$

That is

$$\hat{\phi}(k^\mu) = \delta(\omega^2 - \vec{k}^2)\, \overbrace{\hat{\varphi}(\vec{k})}^{3d!}$$

$$= \left(\frac{\delta(\omega - |\vec{k}|)}{2\omega} + \frac{\delta(\omega + |\vec{k}|)}{2\omega} \right) \hat{\varphi}(\vec{k})$$

$$\rightsquigarrow \quad \phi(x^\mu) = \int \frac{d^3k}{2\sqrt{\vec{k}^2}} \left(e^{-i|\vec{k}|t + i\vec{k}\vec{x}}\, \hat{\varphi}_+(\vec{k}) + e^{i|\vec{k}|t + i\vec{k}\vec{x}}\, \hat{\varphi}_-(\vec{k}) \right)$$

If ϕ is real, then we get a relation between $\hat{\varphi}_+(\vec{k})$ & $\hat{\varphi}_-(-\vec{k})$

$\phi = \phi(v) = \phi(ct+x)$ is a sol

$\rightsquigarrow$ "left movers"

$\longrightarrow$ for $\Box\phi = 0$ we need $\hat{\phi}(k)$ be supported "on the light cone"

i.e. on k^μ's s.t. $k^\mu k_\mu = -\omega^2 + \vec{k}^2 = 0$

In 1+1d, sanity check:

$$\phi(x^\mu) = \int \frac{dk}{k}\left(e^{-ik(t-x)}\hat{\varphi}_+(k) + e^{ik(t+x)}\hat{\varphi}_-(k)\right)$$

$$= \phi_+(u) + \phi_-(v)$$

SOURCES

Green's function method

heuristics: look first at "point sources" @ y

$$\Box_x G(x,y) = -4\pi\,\delta(x,y) \equiv -4\pi\,\delta^{(4)}(x-y)$$

$$\Delta_x G(x,y) = -4\pi\,\delta(x,y) \equiv -4\pi\,\delta^{(3)}(\vec{x}-\vec{y})$$

Assume we know $G(x,y)$, for now.

Then

$$\phi(x) = \int dy\; G(x,y)\,\rho(y)$$

Pf. $\cdot$ $\Box\phi(x) = \int dy \left(\Box_x G(x,y)\right)\rho(y)$

$$= -4\pi \int dy\; \delta(x,y)\,\rho(y)$$

$$= -4\pi\,\rho(x)$$

$\cdot$ $\Delta\phi = \dots\dots = -4\pi\rho$ ∎

Electrostatics

$\partial_t \equiv 0$

$\Delta \phi(x) = -4\pi \rho(x)$ Laplace/Poisson eq.

$\Delta_x G(x,y) = \delta(x,y)$

THM (Green) Let G be a Green's function of Δ:

$$\phi(x) = \int_R d^3y \; \delta(x-y)\,\phi(y) = -\int_R \left(\Delta_y G(y,x)\right)\phi(y) \overset{\text{i.b.p.}}{=} -\int_R d^3y \; G(y,x)\,\Delta_y \phi(y)$$

Electrostatics

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where $\nabla_{\vec{s}} = \vec{s}\cdot\vec{\nabla}\big|_{\partial R}$ with $\vec{s} =$ outgoing unit normal at ∂R

son eq.

i.b.p. (twice!)

's function of Δ:

$$\int_R \left(\Delta_y G(y,x)\right)\phi(y) = -\int_R d^3y \; G(y,x)\,\Delta_y\phi(y) + \oint_{\partial R}\left(G(y,x)\,\nabla_{\vec{s}}\,\phi(y) - \phi(\vec{y})\,\nabla_{\vec{s}}\,G(y,x)\right)$$

$\vec{s}$ = unit outgoing normal at ∂R

...isson eq.

...een's function of Δ:

i.b.p. (twice!)

$$\int_R \left(\Delta_y G(y,x) \right) \phi(y) \overset{\downarrow}{=} -\int_R d^3y \; G(y,x) \underbrace{\Delta_y \phi(y)}_{\text{will be } -4\pi\rho(y)} + \oint_{\partial R} \Big(\underbrace{G(y,x)\, \nabla_{\vec{s}}\, \phi(y) - \phi(\vec{y}) \nabla_{\vec{s}}\, G(y,x)}_{\text{will allow me to fix/understand boundary conds}} \Big)$$

$\vec{s}$ = outgoing unit normal at ∂R

where $\nabla_{\vec{s}} = s\cdot\nabla\big|_{\partial R}$ with s = unit normal at ∂R

Two main options for boundary cond's for G:

<u>Dirichlet</u>

$$\begin{cases} \Delta_x G_D(x,y) = \delta(x-y) & \text{if } x \in R \\ G_D(x,y) = 0 & \text{if } x \in \partial R \end{cases}$$

<u>Neumann</u>

$$\begin{cases} \Delta_x G_N(x,y) = \delta(x-y) & \text{if } x \in R \\ \vec{s}\cdot\bar{\nabla} G(x,y) = 0 & \text{if } x \in \partial R \end{cases}$$

Plugging into Green's identity (thm)

$$\phi(x) = -\int_R dy \quad G_D(y,x)\, \underbrace{\Delta_y \phi(y)}_{\text{"} -4\pi\rho \text{"}} +$$

...cond's for G:

$\ldots(x-y)$ if $x \in R$

if $x \in \partial R$

$\delta(x-y)$ if $x \in R$

0 if $x \in \partial R$

Plugging into Green's identity (thm)

$$\phi(x) = -\int_R d^2y \; G_D(y,x) \underbrace{\Delta_y \phi(y)}_{\text{"}-4\pi\rho\text{"}} + \oint_{\partial R} d^2y \left(\nabla_{\vec{\xi}} G_D(x,y) \right) \underbrace{\phi(y)}_{\text{D. boundary cond for } \phi}$$

$$\phi(x) = -\int \left(\;\ldots\ldots\; \right) - \oint_{\partial R} d^2y \; G_N(y,x) \underbrace{\nabla_{\vec{\xi}} \phi(y)}_{\text{N. b.c. for } \phi}$$

where $\nabla_{\vec{s}} = s\cdot\nabla\Big|_{\partial R}$ with s = unit normal of ∂R

Two main options for boundary cond's for G:

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Green's function depend on
- choice of diff. operator (Δ)
- — " — bdry conds (D, N)
- ref on R

Plugging into Green's identity (thm)

$$\phi(x) = -\int_R d^3y \; G_D(y,x)\, \underbrace{\Delta_y \phi(y)}_{\text{"}-4\pi\rho\text{"}} + \oint_{\partial R} d^2y \; (\ldots$$

$$\phi(x) = -\int\left(\ldots\ldots \right) - \oint_{\partial R} d^2y \; G_N \ldots$$

Green's functions depend on

- choice of diff. operator (Δ)
- — " — bdry conds (D, N)
- reg on R

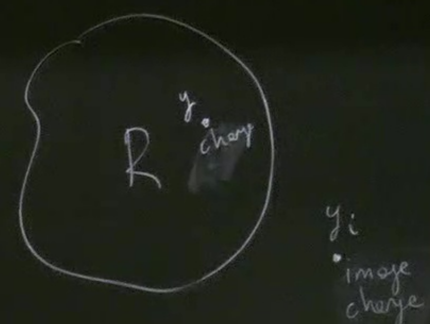
If R is "symmetric enough", we can try the method of images to find solutions.

1) Find a sol. to the Poisson eq without bothering about boundary conditions (nor R)

$$\Delta_x G_0(x,y) = \delta(x,y)$$

2) Note that $G_0(x,y_i)$ with $y_i \notin R$ ("image charge") satisfies the homogeneous (Laplace) eq $\underline{\underline{\text{in}}}$ R

3) The game is to find (y_i) s.t. $G(x,y) = G_0(x,y) + \sum_i G_0(x,y_i)$ satisfies the bdry conds.



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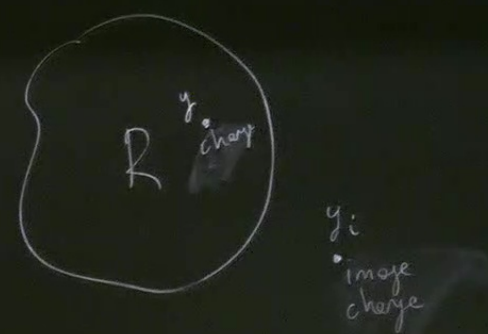
3) The game is to find (y_i) s.t. $G(x,y) = G_0(x,y) + \sum_i q_i G_0(x,y_i)$ satisfies the bdry conds.



i

image charge q_i

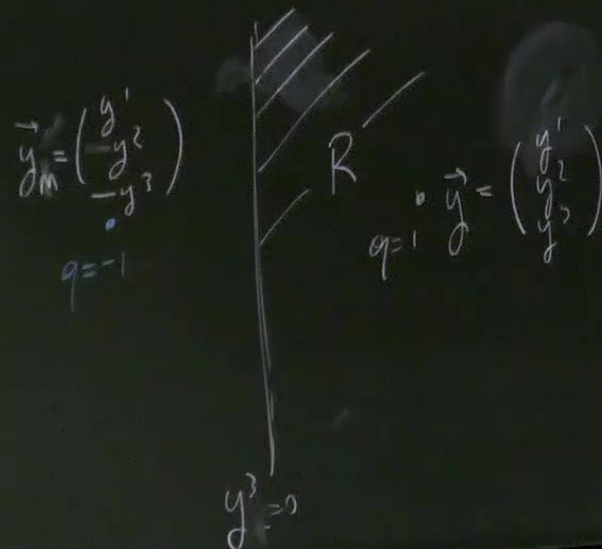
...ing conds.

Ex

$\underline{d=3}$, Dirichlet G.F. for $R = \{z \geq 0\}$ half space.

1) $G_0(\vec{x},\vec{y}) = \dfrac{1}{4\pi}\dfrac{1}{|\vec{x}-\vec{y}|}$

(find this sol playing around with Green's thm & symmetry considerations)



$$G_D(\vec{x},\vec{y}) = \frac{1}{4\pi}\left(\frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}_M|}\right)$$

indeed $G_D(\vec{x},\vec{y})\Big|_{x^3=0} = 0$