Title: Lecture - Classical Physics, PHYS 776

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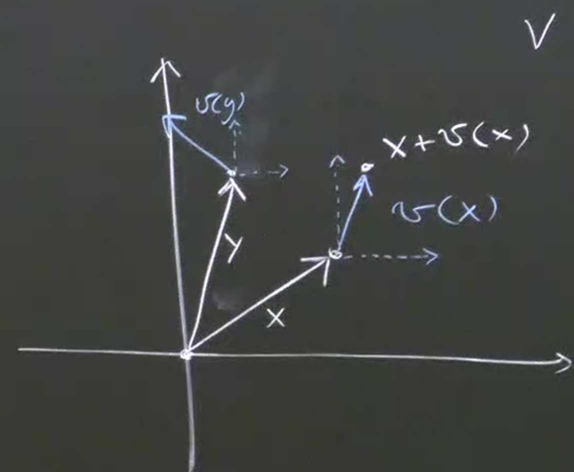
ELECTRODYNAMICS

- vector field over V (Minkowski)

  is an assignment of a vector $\underline{v}(x)$ at every $x \in V$.



- Consider now a function $f: V \to \mathbb{R}$

→ We want to think of the v.f. $V(\underline{x})$ as a derivation.

$$\underline{V}[f](\underline{x}) \equiv \nabla_{\underline{v}} f(\underline{x}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(f(\underline{x}+\epsilon \underline{v}(x)) - f(\underline{x})\right)$$

directional derivative of f in direction v (@x)

Let's consider a basis of V

$$(\underline{e}_\mu)_{\mu=0}^{3}$$

$$\rightsquigarrow \underline{x} = x^\mu \underline{e}_\mu \ , \quad \underline{v}(\underline{x}) = v^\mu(\underline{x})\, \underline{e}_\mu$$

Can think of $f(\underline{x})$ as $F(x^\mu) = f(x^\mu \underline{e}_\mu)$

So F is a function over $\mathbb{R}^n$ $(n=4)$

$$\underbrace{\nabla_{\underline{v}} f(\underline{x})}_{\text{invariant under changes of bases}} = \ldots = \underbrace{v^\mu(x)}_{\text{contravariant vector}} \boxed{\partial_\mu F(x)} \ , \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$\longrightarrow$ COVARIANT VECTOR

$$\partial_\mu F = \nabla_{\underline{e}_\mu} f \equiv \nabla_\mu f$$

Me

sources

ρ = ch. de

$\vec{j}$ = c. cur

Lore

using

the

$\mu)$

$\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$\partial_\mu F = \nabla_{e_\mu} f \equiv \nabla_\mu f$

Maxwell eqs for $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$

sources: ρ = ch. density, $\vec{j}$ = ch. current

$$\vec{\nabla}\cdot\vec{E} = \boxed{4\pi\rho}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

<u>no</u> time derivatives

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$$
$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\left(\boxed{4\pi\vec{j}} + \partial_t\vec{E}\right)$$

w/ time derivatives

Lorentz force

charge

$$\frac{d\vec{p}}{dt} = \vec{F}_L \equiv q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

using that
$$\mathrm{div}(\mathrm{curl}\,\vec{v}) \equiv \vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$$
$$\mathrm{curl}(\mathrm{grad}\,\phi) \equiv \vec{\nabla}\times(\vec{\nabla}\phi) = 0 \qquad \forall \vec{v}, \phi$$

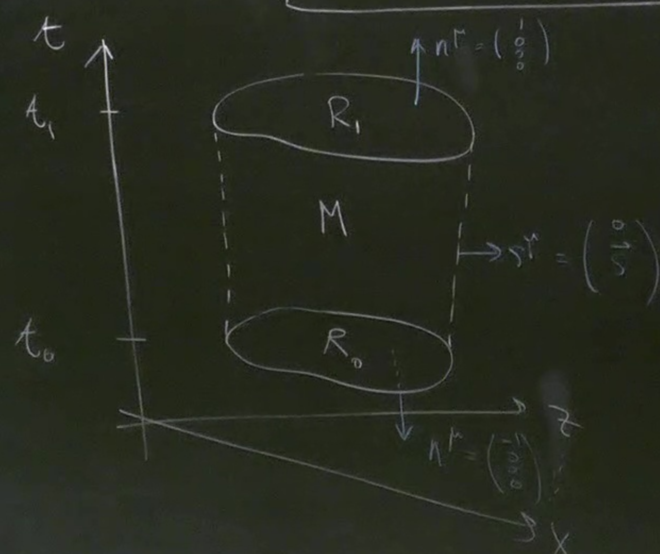
the converse is true as well (e.g. $\vec{\nabla}\times\vec{w} = 0 \Rightarrow \exists\,\phi : \vec{w} = \vec{\nabla}\phi$)

[directional derivative of f in direction v (@x)

CONTINUITY Eq

div (4th) & insert the 1st, we obtain:

$$4\pi \cdot \boxed{\vec{\nabla}\cdot\vec{j} + \partial_t \rho = 0}$$



$$M = R \times [t_0, t_1]$$

$$\partial M = R_0 \cup R_1 \cup (\partial R \times [t_0, t_1])$$

How much ...
of the ...

$Q_R(t)$

on v (@x)

VECTOR

$R \times [t_0, t_1]$

$R_0 \cup R_1 \cup (\partial R \times [t_0, t_1])$

How much charge goes in/out of the box R?

$$Q_R(t) := \int_R \rho(\vec{x}, t)\, d\,\mathrm{Vol}(\vec{x})$$

the charge in the box at time t

$$\frac{d}{dt} Q_R(t) = \int_R \partial_t \rho(\vec{x}, t)\, d^3\mathrm{vol}(\vec{x})$$

continuity eq.

$$= \int_R -\vec{\nabla}\cdot\vec{j}(\vec{x}, t)\, d^3\mathrm{vol}(\vec{x})$$

divergence thm (Stokes)

$$= \oint_{\partial R} -\vec{S}\cdot\vec{j}(\vec{x}, t)\, d^2\mathrm{Area}(\vec{x}) =: \mathrm{Flux}_{\partial R}(\vec{j})$$

If $R = \mathbb{R}^3$, S^3, $\partial R = \emptyset \rightarrow$ total charge is conserved.

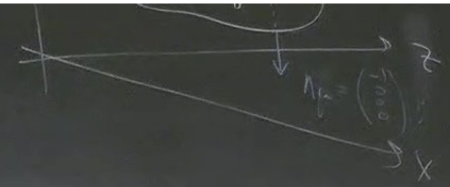
$$Q(t_1) - Q(t_0) = \int_{t_0}^{t_1} \left(\frac{d}{dt} Q_R \right) dt$$

$$= - \underbrace{\int_{t_0}^{t_1} \oint_{\partial R}}_{\int_B \,\leftarrow\, \text{timelike boundary}} \vec{s} \cdot \vec{j} \; d^2Ar(x)\, dt$$

Lorentz covariance?

uity

$$=: \mathrm{Flux}_{\partial R}(\vec{j})$$



(Stokes) $= \oint_{\partial R} -\vec{S}\cdot\vec{j}\,(\vec{x}, t$

Define 4-current

$$J^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$$

Compute its 4-divergence:

$$\boxed{\boxed{\nabla_\mu J^\mu} = \partial_t \rho + \vec{\nabla}\cdot\vec{j} = 0}$$

Note: $0 = \int_M \nabla_\mu J^\mu \, d^4 Vol(x) = \oint n_\mu J^\mu \, d^3 Vol(x)$

$\boxed{\partial M} = R_0 \cup R_1 \cup B$

$$= \int_{R_1} J^0 - \int_{R_0} J^0 + \int_B \vec{S}\cdot\vec{j}$$

$$= Q(t_1) - Q(t_0) + \int_{t_0}^{t_1} Flux(\vec{j})\, dt$$

Rmk charge conserv.

- continuity eq $(3+1,\ 4d)$
- balance law $\left(\frac{dQ}{dt},\ Q(t_1) - Q(t_0)\right)$
- 4-flux = 0

$$= \int_{R_1} J^0 - \int_{R_0} J^0 + \int_{B} \vec{S}\cdot\vec{j}$$

$$= Q(t_1) - Q(t_0) + \int_{t_0}^{t_1} \mathrm{Flux}(\vec{j})\, dt$$

$= 0$

$(x) = \oint n_\mu J^\mu \, d^3\mathrm{Vol}(x)$

$\boxed{\partial M} = R_0 \cup R_1 \cup B$

Rmk charge conserv.

- continuity eq. $(3+1,\ 4d)$
- balance law $\left(\frac{dQ}{dt},\ Q(t_1)-Q(t_0)\right)$
- 4-flux = 0

Change of frame

$\Rightarrow$ mixing of $\vec{E}$ & $\vec{B}$

$\Rightarrow$ in a Lorentz covariant formulation $\vec{E}$ & $\vec{B}$ must be part of 1 tensor:

$$\left(F'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F^{\mu'\nu'} \right)$$

$$\left(J'^{\mu} = \Lambda^{\mu}{}_{\mu'} J^{\mu'} \right)$$

$$F^{\mu\nu} = \begin{cases} E^i & \mu=0, \nu=i \\ -E^i & \mu=i, \nu=0 \\ \epsilon^{ij}{}_k B^k & \mu=i, \nu=j \end{cases} \quad , \quad F^{\mu\nu} = F^{[\mu\nu]} \text{ skew}$$

$$\Rightarrow \boxed{\text{Maxwell: } \begin{cases} \nabla_{[\mu} F_{\nu\rho]} = 0 & \leftarrow \text{no sources eqs} \\ \nabla_\mu F^{\mu\nu} = -4\pi J^\nu & \leftarrow \text{source eqs.} \end{cases}}$$

Continuity eqs :

$$-4\pi \nabla_\mu J^\mu = \underbrace{\nabla_\mu \nabla_\nu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{skew}} \equiv 0$$

Rmk $(\vec{E}, \vec{B})$ Maxwell eqs

"hard" to generalize in higher dim,

but $F^{\mu\nu}$ Maxwell stay unchanged.

$$F^{\mu\nu} = \left(\begin{array}{c|c} & E^i \\ \hline & F_{ij} (\sim B^k) \end{array} \right)$$

$\# E^i$ $=$ $\# F^{ij}$ only in $d=4$

In fact in $\boxed{d=4}$ (without sources)

we have E-M duality $(\vec{E} \leftrightarrow \vec{B})$

$\hookrightarrow$ Lorentz covariantly:

$$F^{\mu\nu} \rightsquigarrow \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Question : prove that $\tilde{F}$ is Lorentz covariant

ϵ^{01}

Continuity eqs :

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$$\epsilon^{0123} = 1, \quad \epsilon^{\mu\nu\rho\sigma} = \begin{cases} 0 \\ \pm 1 \end{cases}$$

d -

$\# E^i$
$\shortparallel$
$\# F^{ij}$
only in $d=4$

Question: prove that $\tilde{F}$ is Lorentz covariant.

Notice what happens if Λ is allowed to be P or T (parity/time reversal)

- In $d=4$, Maxwell eqs (without sources) are also conformally invariant. (stay tuned)

LORENTZ FORCE

Input: $\vec{p} = m\gamma\vec{v} = \vec{P}$

$\downarrow$

$$\frac{dp^i}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)^i = q\left(F^{0i} + \epsilon^i{}_{jk}\, v^j B^k\right)$$

$$= q\left(F^{0i} + F^i{}_j\, v^j\right) \qquad F^{0i} = -F^{i0} = F^i{}_0$$

component-wise

$$\left[U^\mu := \begin{pmatrix}\gamma \\ \gamma\vec{v}\end{pmatrix}\right] \rightsquigarrow \quad = \frac{q}{\gamma} F^i{}_\mu U^\mu$$

$$\left[\gamma \frac{d}{dt} = \frac{d}{d\tau}\right] \Longrightarrow \quad \frac{d}{d\tau} p^i = q F^i{}_\mu U^\mu \ \overset{\text{last eq}}{\rightsquigarrow} \ \boxed{\frac{d}{d\tau} P^\nu = q F^\nu{}_\mu U^\mu}$$

ELECTROMAGNETIC POTENTIAL

Focus on source free eqs:

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & (1) \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 & (2) \end{cases}$$

$$(1) \Rightarrow \exists \vec{A} \text{ s.t. } \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

plug into (2)

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

$$\Rightarrow \exists \phi \text{ s.t. } \boxed{\vec{E} = \vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}}$$

$$q F^{\nu}{}_{\mu} U^{\mu}$$

In covariant terms:

(check)

$$F_{0i} = -E_i = \partial_0 A_i - \partial_i \phi \quad , \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

$$\boxed{\begin{aligned} A_\mu &= (\phi, A_i) \quad \text{electromagnetic potential} \\ F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \end{aligned}}$$

could have been deduced from

$$\begin{cases} F_{\mu\nu} + F_{\nu\mu} = 0 \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases} \Rightarrow \exists A_\mu \text{ s.t. } (\cdots)$$

Rmk

A_μ is not unique!

I have the freedom to redefine

$$A_\mu \longmapsto A_\mu + \nabla_\mu \lambda$$

without affecting $F_{\mu\nu} \mapsto F_{\mu\nu} + 2\cancel{\nabla_{[\mu}\nabla_{\nu]}}\lambda$ $\equiv 0$

$\rightsquigarrow$ GAUGE FREEDOM