Title: Lecture - Classical Physics, PHYS 776

Speakers: Aldo Riello

Collection/Series: Classical Physics (Core), PHYS 776, September 3 - October 4, 2024

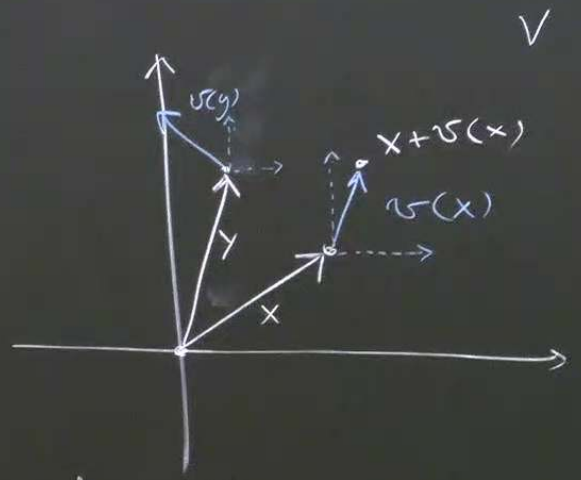
Date: September 24, 2024 - 9:00 AM

URL: https://pirsa.org/24090011

ELECTRODYNAMICS

- vector field over V (Minkowski: $\mathbb{R}^4$)

  is an assignment of a vector $\underline{v}(x)$ at every $x \in V$.



- Consider now a function $f : V \to \mathbb{R}$

→ We want to think of the v.f. $V(\underline{x})$ as a derivation.

$$\underline{V}[f](\underline{x}) \equiv \nabla_{\underline{v}} f(\underline{x}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(f(\underline{x} + \epsilon \underline{V}(x)) - f(\underline{x}) \right)$$

↑ directional derivative of f in direction v (@ x)

Let's consider a basis of V

$$(\underline{e}_\mu)_{\mu=0}^{3}$$

$$\rightsquigarrow \underline{X} = x^\mu \underline{e}_\mu \,, \quad \underline{v}(\underline{x}) = v^\mu(\underline{x})\, \underline{e}_\mu$$

Can think of $f(\underline{x})$ as $F(x^\mu) = f(x^\mu \underline{e}_\mu)$

So F is a function over $\mathbb{R}^n$ $(n=4)$

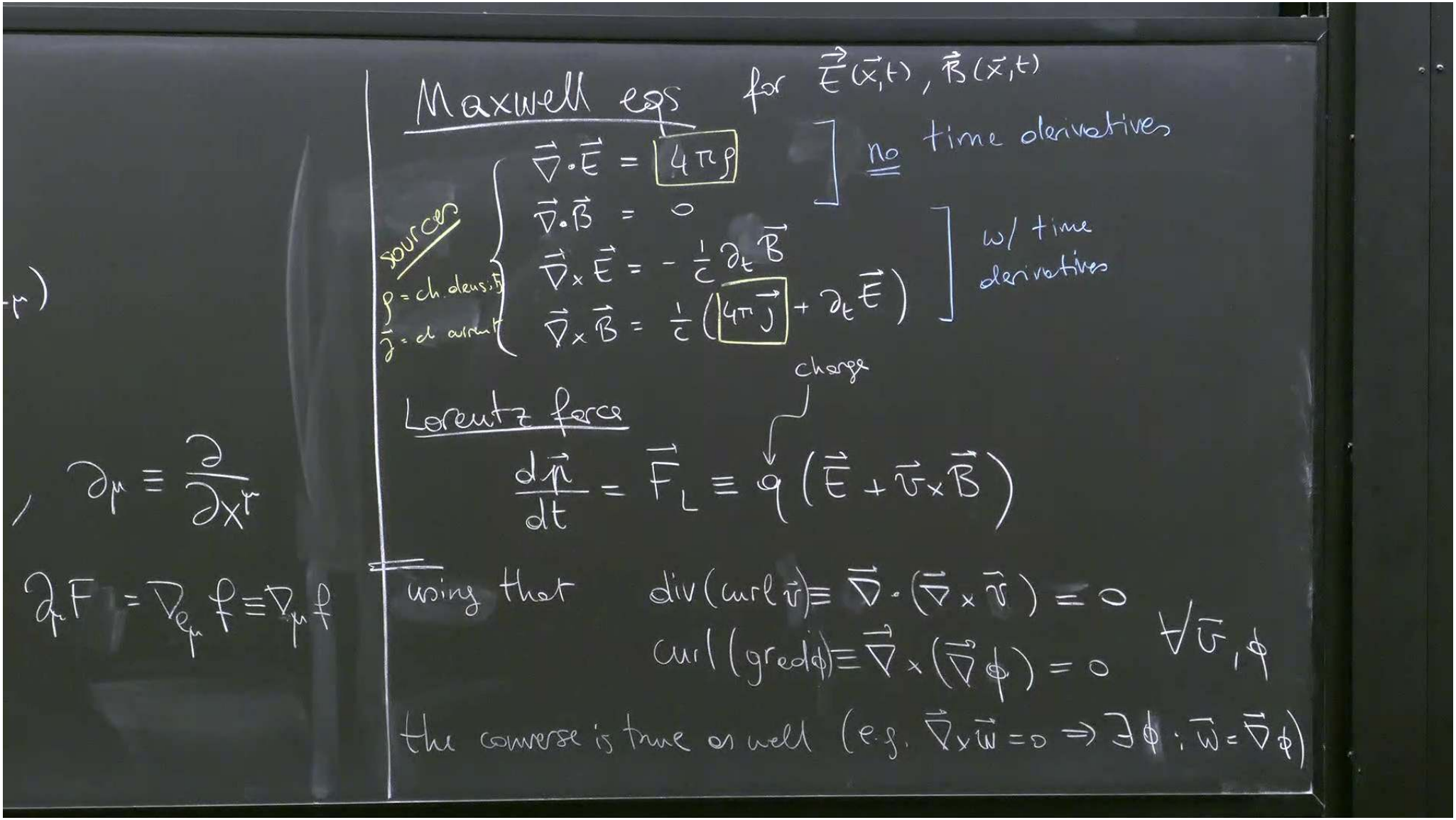
$$\nabla_{\underline{v}} f(\underline{x}) = \dots = v^\mu(x)\, \boxed{\partial_\mu F(x)} \,, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$\nabla_{\underline{v}} f(\underline{x})$: invariant under changes of bases

$v^\mu(x)$: contravariant vector

$\partial_\mu F(x)$: COVARIANT VECTOR

$$\partial_\mu F = \nabla_{\underline{e}_\mu} f \equiv \nabla_\mu f$$

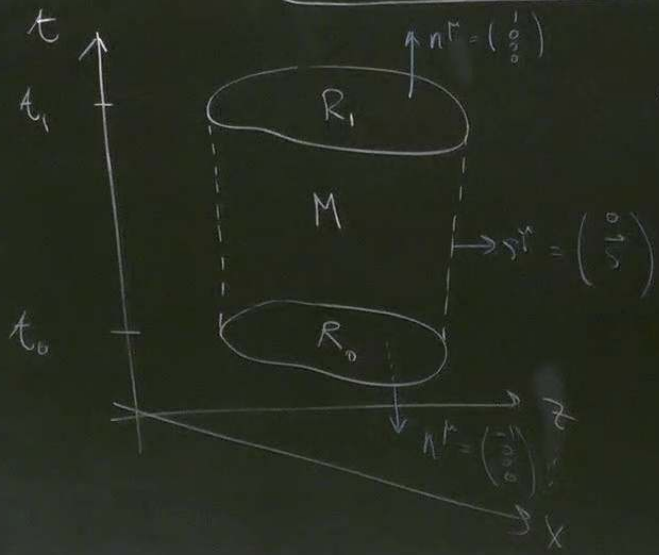

Maxwell eqs for $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$
$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$
$\vec{\nabla}\cdot\vec{B} = 0$
$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$
$\vec{\nabla}\times\vec{B} = \frac{1}{c}\left(4\pi\vec{J} + \partial_t\vec{E}\right)$
no time derivatives
w/ time derivatives
sources
ρ = ch. dens.
$\vec{J}$ = el. current
Lorentz force
charge
$\frac{d\vec{p}}{dt} = \vec{F}_L \equiv q\left(\vec{E} + \vec{v}\times\vec{B}\right)$
using that
$\mathrm{div}(\mathrm{curl}\,\vec{v}) \equiv \vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$
$\mathrm{curl}(\mathrm{grad}\,\phi) \equiv \vec{\nabla}\times(\vec{\nabla}\phi) = 0$
$\forall \vec{v}, \phi$
the converse is true as well (e.g. $\vec{\nabla}\times\vec{w} = 0 \Rightarrow \exists\phi : \vec{w} = \vec{\nabla}\phi$)
$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$
$\partial_\mu F = \partial_\mu f \equiv \partial_\mu f$

directional derivative of f in direction v (∂_x)

CONTINUITY Eq

div (4th) & insert the 1st, we obtain:

$$4\pi \cdot \boxed{\vec{\nabla} \cdot \vec{j} + \partial_t \rho = 0}$$



$M = R \times [t_0, t_1]$

$\partial M = R_0 \cup R_1 \cup (\partial R \times [t_0, t_1])$

How much
of the

$Q_R(t)$

on $\mathcal{V}$ $(@x)$

VECTOR

$R \times [t_0, t_1]$

$R_0 \cup R_1 \cup (\partial R \times [t_0, t_1])$

How much charge goes in/out of the box R?

$$Q_R(t) := \int_R \rho(\vec{x}, t)\, d\mathrm{Vol}(\vec{x})$$

the charge in the box at time t

$$\frac{d}{dt} Q_R(t) = \int_R \partial_t \rho(\vec{x}, t)\, d^3\mathrm{vol}(\vec{x})$$

continuity eq.

$$= \int_R -\vec{\nabla} \cdot \vec{j}(\vec{x}, t)\, d^3\mathrm{vol}(\vec{x})$$

divergence thm (Stokes)

$$= \oint_{\partial R} -\vec{S} \cdot \vec{j}(\vec{x}, t)\, d^2\mathrm{Area}(\vec{x}) =: \mathrm{Flux}_{\partial R}(\vec{j})$$

If $R = \mathbb{R}^3$ or S^3, $\partial R = \emptyset$ → total charge is conserved.

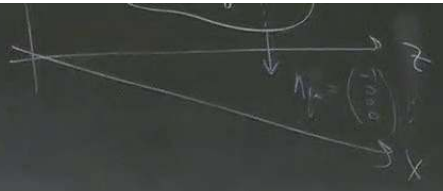
$$Q(t_1) - Q(t_0) = \int_{t_0}^{t_1} \left(\frac{d}{dt} Q_R \right) dt$$

$$= - \underbrace{\int_{t_0}^{t_1} \oint_{\partial R}}_{\int_B} \vec{S} \cdot \vec{j} \; d^3\mathrm{Ar}(x) \, dt$$

$\int_B$ ← timelike boundary

Lorentz covariance?

$=: \mathrm{Flux}_{\partial R}(\vec{j})$



(Stokes) $= \oint_{\partial R} -\vec{S}\cdot\vec{j}\,(\vec{x},t$

Define 4-current

$$J^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$$

Compute its 4-divergence

$$\boxed{\nabla_\mu J^\mu = \partial_t \rho + \vec{\nabla}\vec{j} = 0}$$

Note: $0 = \int_M \nabla_\mu J^\mu \, d^4\mathrm{Vol}(x) = \oint n_\mu J^\mu \, d^3\mathrm{Vol}(x)$

$\boxed{\partial M = R_0 \cup R_1 \cup B}$

$$= \int_{R_1} J^0 - \int_{R_0} J^0 + \int_B \vec{S}\cdot\vec{j}$$

$$= Q(t_1) - Q(t_0) + \int_{t_0}^{t_1} \mathrm{Flux}(\vec{j})\, dt$$

Rmk charge conserv.

- Continuity eq (3+1, 4d)
- balance law $\left(\frac{dQ}{dt}, Q(t_1)-Q(t_0)\right)$
- 4-flux = 0

$$= \int_{R_1} J^0 - \int_{R_0} J^0 + \int_{B} \vec{S}\cdot\vec{j}$$

$$= Q(t_1) - Q(t_0) + \int_{t_0}^{t_1} \mathrm{Flux}(\vec{j})\, dt$$

$\boxtimes$

$$\boxed{= 0}$$

$$x) = \oint n_\mu J^\mu \, d^3 \mathrm{Vol}(x)$$

$$\boxed{\partial M} = R_0 \cup R_1 \cup B$$

Rmk charge conserv.

- continuity eq $(3+1,\ 4d)$
- balance law $\left(\frac{dQ}{dt},\ Q(t_1) - Q(t_0)\right)$
- 4-flux $= 0$

Change of frame
$\Rightarrow$ mixing of $\vec{E}$ & $\vec{B}$
$\Rightarrow$ in a Lorentz covariant formulation $\vec{E}$ & $\vec{B}$ must be part of 1 tensor:

$$F^{\mu\nu} = \begin{cases} E^i & \mu = 0, \nu = i \\ -E^i & \mu = i, \nu = 0 \\ \epsilon^{ij}{}_k B^k & \mu = i, \nu = j \end{cases}$$

$$\left(F'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F^{\mu'\nu'} \right)$$

$$\left(J'^{\mu} = \Lambda^{\mu}{}_{\mu'} J^{\mu'} \right)$$

$$F^{\mu\nu} = F^{[\mu\nu]} \quad \text{skew}$$

$\Rightarrow$ Maxwell:
$\nabla_{[\mu} F_{\nu\rho]} = 0$ $\leftarrow$ no sources eqs
$\nabla_\mu F^{\mu\nu} = -4\pi J^\nu$ $\leftarrow$ source eqs.

Continuity eqs :

$$-4\pi \nabla_\mu J^\nu = \underbrace{\nabla_\mu \nabla_\nu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{skew}} \equiv 0$$

$=$

Rmk $(\vec{E}, \vec{B})$ Maxwell eqs "hard" to generalize in higher dim, but $F^{\mu\nu}$ Maxwell stay unchanged -

$$F^{\mu\nu} = \left(\begin{array}{c|c} & E^i \\ \hline & F_{ij} \; (\sim B^k) \end{array} \right) \quad \begin{array}{l} \# E^i \\ \| \\ \# F^{ij} \\ \text{only in } d=4 \end{array}$$

In fact in $\boxed{d=4}$ (without sources) we have E-M duality $(\vec{E} \leftrightarrow \vec{B})$

↳ Lorentz covariantly

$$F^{\mu\nu} \longleftrightarrow \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Question : prove that $\tilde{F}$ is Lorentz covariant

ϵ^{01}

Continuity eqs:

$$-4\pi \nabla_\mu J^\mu = \underbrace{\nabla_\mu \nabla_\nu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{skew}} \equiv 0$$

Rmk $(\vec{E}, \vec{B})$ Maxwell eqs "hard" to generalize in higher dim, but $F^{\mu\nu}$ Maxwell stay unchanged.

$$F^{\mu\nu} = \left(\begin{array}{c|c} & E^i \\ \hline & F_{ij} \; (\sim B^k) \end{array} \right)$$

$\# E^i = \# F^{ij}$ only in $d=4$

In fact in $\boxed{d=4}$ (without sources) we have E-M duality $(\vec{E} \leftrightarrow \vec{B})$

↳ Lorentz covariantly:

$$F^{\mu\nu} \leftrightsquigarrow \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Question: prove that $\tilde{F}$ is Lorentz covariant

In fact in $d=4$ (without sources)
we have E-M duality ($\vec{E} \leftrightarrow \vec{B}$)

$\epsilon^{0123} = 1$, $\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 0 \\ \pm 1 \end{cases}$

↳ Lorentz covariantly:

$$F^{\mu\nu} \longleftrightarrow \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Question: prove that $\tilde{F}$ is Lorentz covariant.

Notice what happens if Λ is allowed to be P or T (parity/time reversal)

- In $d=4$, Maxwell eqs (without sources) are also conformally invariant. (stay tuned)

$\# E^i$
$=$
$\# F^{ij}$
only in $d=4$

LORENTZ FORCE

Input: $\vec{p} = m\gamma\vec{v} = \vec{P}$

$$\downarrow$$

$$\frac{d\vec{P}^i}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)^i = q\left(F^{0i} + \epsilon^i{}_{jk} v^j B^k\right)$$

$$= q\left(F^{0i} + F^i{}_j v^j\right)$$

$F^{0i} = -F^{i0} = F^i{}_0$ (component-wise)

$$\left[U^\mu = \begin{pmatrix}\gamma \\ \gamma\vec{v}\end{pmatrix}\right] \rightsquigarrow = \frac{q}{\gamma} F^i{}_\mu U^\mu$$

$$\left[\gamma\frac{d}{dt} = \frac{d}{d\tau}\right] \Rightarrow \frac{d}{d\tau} p^i = q F^i{}_\mu U^\mu \overset{\text{last eq.}}{\rightsquigarrow} \boxed{\frac{d}{d\tau} p^\nu = q F^\nu{}_\mu U^\mu}$$

ELECTROMAGNETIC POTENTIAL

Focus on source free eqs:

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & (1) \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 & (2) \end{cases}$$

$(1) \Rightarrow \exists\, \vec{A}$ s.t. $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$

plug into (2)

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

$$\Rightarrow \exists \phi \text{ s.t. } \boxed{\vec{E} = \vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}}$$

In covariant terms:

$$F_{0i} = -E_i = \partial_0 A_i - \partial_i \phi \,, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

(check)

$qF^{\nu}{}_{\mu}U^{\mu}$

$A_\mu = (\phi, A_i)$ electromagnetic potential

$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

could have been deduced from

$$\begin{cases} F_{\mu\nu} + F_{\nu\mu} = 0 \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases} \Rightarrow \exists A_\mu \text{ s.t. } (\cdots)$$

Rmk

A_μ is not unique!

I have the freedom to redefine

$A_\mu \longmapsto A_\mu + \nabla_\mu \lambda$

without affecting $F_{\mu\nu} \mapsto F_{\mu\nu} + 2\nabla_{[\mu}\nabla_{\nu]}\lambda$ ($\equiv 0$)

$\rightsquigarrow$ GAUGE FREEDOM