Title: Lecture - Classical Physics, PHYS 776

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RELATIVISTIC MECHANICS

1) principle of inertia ✓

2) $\frac{d\vec{p}}{dt} = \vec{F}$ ✗

3) Newton's third law ✓
(conservation of momentum)

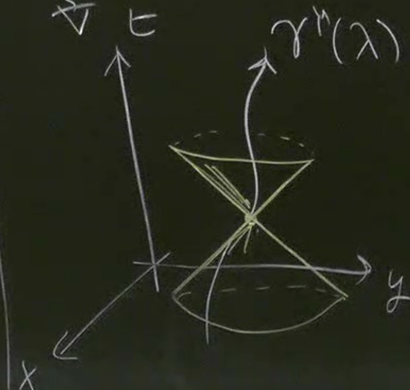
(2) is invariant under Galilean boosts

$$\begin{cases} t \mapsto t \\ \vec{x} \mapsto \vec{x} - \vec{v}t \end{cases} \Rightarrow \vec{a} \mapsto \vec{a}$$

(m is const) $\Rightarrow$ $\vec{F} \mapsto \vec{F}$

2) needs revision in SR!

$\vec{x}(t)$ not a good parametriz in SR
where t is frame dep
& 3-vectors are not
"natural" objects.



$$\gamma^\mu : \mathbb{R} \longrightarrow \mathbb{R}^{1,3}$$
$$\lambda \mapsto x^\mu = \gamma^\mu($$

"lousy clock" parameter

$$\begin{pmatrix} t \\ \vec{x} \end{pmatrix} = ($$

particle's world line

$R \longrightarrow \mathbb{R}^{1,3}$

$\lambda \longmapsto x^\mu = \gamma^\mu(\lambda)$

clock" $\begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} \gamma^0(\lambda) \\ \vec{\gamma}(\lambda) \end{pmatrix}$

eter

's world line

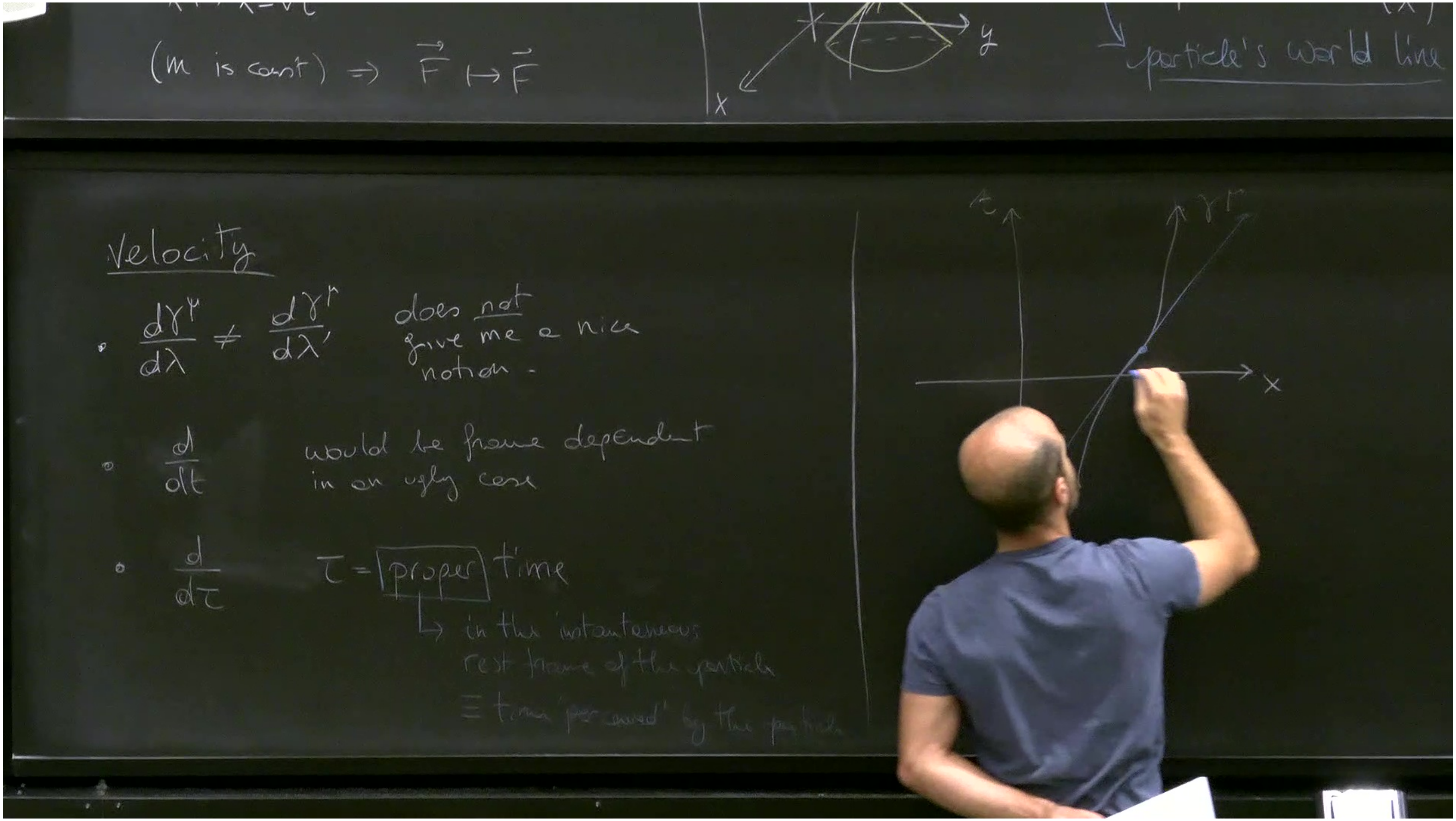
$$\frac{d\gamma^0}{d\lambda} > 0$$ particle moves forward in time

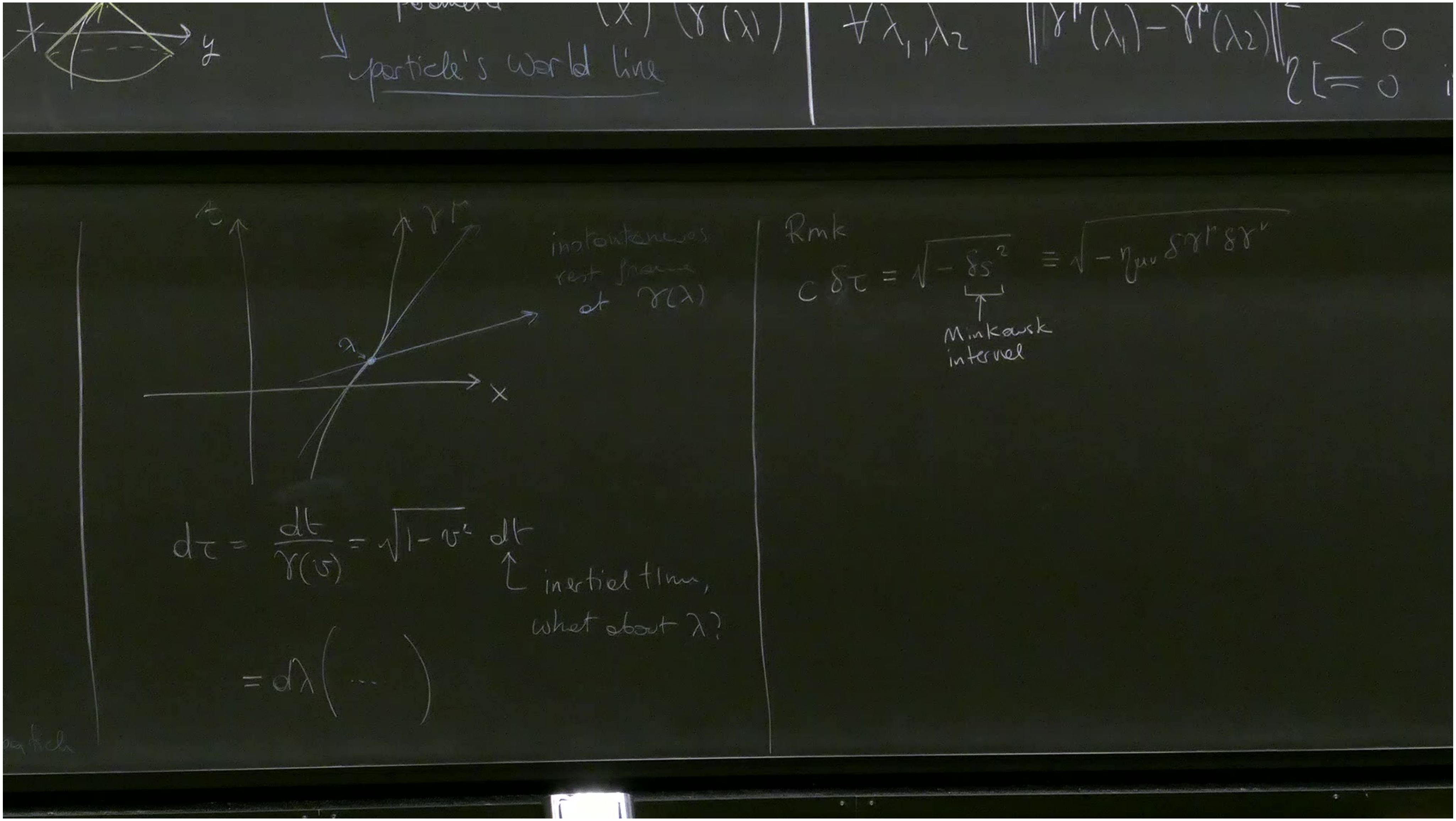
↳ if I wanted I could invert $\lambda \mapsto t = \gamma^0(\lambda)$
and parametrize γ^μ wrt t.

More generally, λ can be arbitrarily redefined $\lambda \mapsto \lambda' = f(\lambda)$
$\frac{df}{d\lambda} > 0$.

$\Longrightarrow$ All expression must be invariant under $\lambda \mapsto \lambda'$.

$$\forall \lambda_1, \lambda_2 \quad \|\gamma^\mu(\lambda_1) - \gamma^\mu(\lambda_2)\|^2_\eta < 0 \qquad [= 0 \text{ iff massless}] \longrightarrow \boxed{\left|\frac{d\gamma^\mu}{d\lambda}\right|^2_\eta < 0}$$


(m is const) ⇒ $\vec{F} \mapsto \vec{F}$
y
x
particle's world line
Velocity
• $\frac{d\gamma^\mu}{d\lambda} \neq \frac{d\gamma^\mu}{d\lambda'}$ does not give me a nice notion.
• $\frac{d}{dt}$ would be frame dependent in an ugly case
• $\frac{d}{d\tau}$ τ = proper time
in the 'instantaneous' rest frame of the particle
≡ time 'perceived' by the particle
t
γ^μ
x


particle's world line
$\forall \lambda_1, \lambda_2 \quad \|\gamma^\mu(\lambda_1) - \gamma^\mu(\lambda_2)\|^2_\eta < 0$
instantaneous rest frame at $\gamma(\lambda)$
Rmk
$c\,\delta\tau = \sqrt{-\delta s^2} \equiv \sqrt{-\eta_{\mu\nu}\,\delta\gamma^\mu\,\delta\gamma^\nu}$
Minkowsk interval
$d\tau = \frac{dt}{\gamma(v)} = \sqrt{1-v^2}\,dt$
inertial time, what about λ?
$= d\lambda(\ \dots\ )$

$(\gamma(\lambda))$ $\quad \forall \lambda_1, \lambda_2 \quad \|\gamma^\mu(\lambda_1) - \gamma^\mu(\lambda_2)\|^2_\eta < 0 \longrightarrow \left|\frac{d\gamma}{d\lambda}\right|_\eta < 0$

[$= 0$ iff massless] CAUSALITY

Rmk

$$c\,\delta\tau = \sqrt{-\underbrace{\delta s^2}_{\text{Minkowsk interval}}} \equiv \sqrt{-\eta_{\mu\nu}\,\delta\gamma^\mu\,\delta\gamma^\nu}$$

$$\delta\gamma^\mu(\lambda) = \gamma^\mu(\lambda+\delta\lambda) - \gamma^\mu(\lambda)$$

$$= \frac{d\gamma^\mu}{d\lambda}\,\delta\lambda + \mathcal{O}(\delta\lambda^2)$$

$$\rightarrow c\,\delta\tau = \sqrt{-\delta s^2} = \sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}\;\delta\lambda$$

check: $\lambda = t \rightarrow \; = \sqrt{\left(\frac{d\gamma^0}{dt}\right)^2 - \left(\frac{d\vec\gamma}{dt}\right)^2}\;\delta t = \sqrt{1-\vec{v}^2}\;\delta t$ ✓

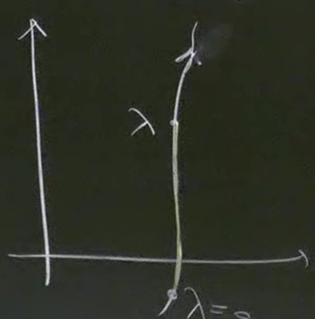
rest frame of the particle
$\equiv$ time "perceived" by the particle

$= d\lambda\left(\cdots \right)$

$$d\tau = \sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}\, d\lambda$$

Ex: check that $d\tau$ is invariant under $\lambda \mapsto \lambda' = f(\lambda)$

$$\tau(\lambda) = \int_0^\lambda d\tau$$



Using τ we can define a nice notion of velocity!

$$\boxed{u^\mu := \frac{d\gamma^\mu}{d\tau}} = \frac{d\gamma^\mu/d\lambda}{d\tau/d\lambda} \underset{\lambda = t}{\overset{\uparrow}{=}} \gamma(v)\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

Rmk.

1) indep of $\lambda \mapsto \lambda'$
2) "Lorentz covariant"
   i.e. is a contravariant vector wrt action of $SO^+(1,3)$

b.c. $d\gamma^\mu$ = covariant, $d\tau$ = invariant what about λ?

Check. $\lambda = t \rightarrow = \sqrt{\left(\frac{d\gamma^0}{dt}\right)^2 - \left(\frac{d\vec{\gamma}}{dt}\right)^2}\, \delta t = \sqrt{1-\vec{v}^2}\, \delta t$ ✓

$\left(\begin{matrix} 1 \\ \vec{v} \end{matrix} \right)$

Rmk

u^μ " $= \frac{d\gamma^\mu/d\lambda}{|d\gamma^\mu/d\lambda|_\eta}$ " $\longrightarrow \boxed{u^\mu u_\mu = -1}$

rmk this had to be so, bc I can compute $u^\mu u_\mu$ in any (inertial) frame and in particular in the instant rest frame

$\rightarrow u^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

4-momentum

$$p^\mu = m u^\mu$$

↑ Lorentz invariant/scalar called <u>mass</u>

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \mathcal{O}(v^4)$$

- $u^\mu u_\mu = -1 \implies p_\mu p^\mu = -m^2$
- in a given inertial frame:

$$p^\mu = \left(\gamma(v) m, \gamma(v) m \vec{v}\right) \overset{v \ll 1}{\approx} \Big(\underbrace{m}_{\boxed{\text{Rest energy}}} + \underbrace{\tfrac{1}{2} m v^2}_{\text{N. Kinetic energy}}, \underbrace{m \vec{v}}_{\text{Newtonian momentum}}\Big) + \mathcal{O}(v^3)$$

$= 1 + \frac{1}{2} v^2 + O(v^4)$

Relativistic 2^{nd} law

$$\frac{dp^\mu}{d\tau} = F^\mu$$

e.g. Lorentz force.

Rmk: if F^μ does not change the mass of the particle, then

$$0 = -\frac{d}{d\tau} m^2 = \frac{d}{d\tau}(p_\mu p^\mu) = p^\mu F_\mu \Rightarrow \boxed{u^\mu F_\mu = 0}$$

$) + O(v^3)$

what about λ?

check. $\lambda = t \to = \sqrt{\left(\frac{d\gamma^0}{dt}\right)^2 - \left(\frac{d\vec{\gamma}}{dt}\right)^2}\, dt = \sqrt{1-\vec{v}^2}\, dt$ ✓

$\left(\frac{1}{\vec{v}}\right)$

Rmk

$u^\mu = \frac{d\gamma^\mu/d\lambda}{|d\gamma^\mu/d\lambda|_\eta} \longrightarrow \boxed{u^\mu u_\mu = -c^2}$

rmk this had to be so, bc I can compute $u^\mu u_\mu$ in any (inertial) frame and in particular in the instant rest frame

$\to u^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

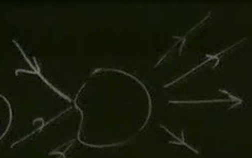
$f(x^\mu)\Big|_{x^\mu = \gamma^\mu(\lambda)}$ I can ask how does it change in time: $\frac{d}{d\tau} f(\gamma) = \frac{d\gamma^\mu}{d\tau} \frac{\partial f}{\partial x^\mu}\Big|_{\gamma^\mu}$

$= u^\mu \partial_\mu f$

$$p^\mu = \left(\gamma(v)\, m,\ \gamma(v)\, m\vec{v}\right) \overset{v\ll 1}{\approx} \left(m + \tfrac{1}{2} m v^2,\ m\vec{v}\right) + \mathcal{O}(v)$$

3rd law

in collisions (in)



(out)

$$\sum_{\alpha=1}^{N} p^\mu_{\alpha,\mathrm{in}} = \sum_{\beta=1}^{M} p^\mu_{\beta,\mathrm{out}}$$

→ rmk: energy & linear mom conservation even for inelastic collisions

⇝ in special relativity heat possesses inertia i.e. it contributes to rest mass!

$$p^\mu = \left(\gamma(v)\, m \,,\, \gamma(v)\, m\vec{v}\right) \overset{v\ll 1}{\approx} \left(m + \tfrac{1}{2} m v^2 \,,\, m\vec{v}\right) + O(v$$

3rd law

in collisions (in)

(out)

$$\sum_{\alpha=1}^{N} p^\mu_{\alpha,in} = \sum_{\beta=1}^{M} p^\mu_{\beta,out}$$

$\boxed{\mu=0}$

→ rmk: energy & linear mom conservation even for inelastic collisions

⇝ in special relativity heat possesses inertia i.e. it contributes to rest mass!

Action principle

$$S[\gamma^\mu] = -m\int d\tau = -m\int d\lambda \sqrt{-\eta_{\mu\nu}}$$

Free particle of mass m.

$\frac{1}{2}mv^2$, $m\vec{v}$ $] + \mathcal{O}(v^3)$

$0 = -\frac{d}{d\tau}$

vation

sions

es inertie

Action principle

$$S[\gamma^\mu] = -m\int_\gamma d\tau = -m\int d\lambda \sqrt{-\eta_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}$$

$\gamma'^\mu \equiv d\gamma^\mu/d\lambda$

[Free particle of mass m]

Euler-Lagrange: $\frac{d}{d\lambda}\frac{\partial L}{\partial \gamma'^\mu} - \underbrace{\frac{\partial L}{\partial \gamma^\mu}}_{\text{free particle} = 0} = 0$

$$0 = \frac{d}{d\lambda}\Pi_\mu \ , \quad \Pi_\mu = \frac{\partial L}{\partial \gamma'^\mu} = \frac{m\gamma'_\mu}{\sqrt{-\gamma'^\nu\gamma'_\nu}} = m u_\mu \equiv p_\mu \quad \checkmark$$

⇝ in special relativity heat possesses inertia
i.e. it contributes to rest mass!

particle in external (EM) potential

$$S(\gamma^\mu) = -\int d\lambda \left(m\sqrt{-\gamma'^\mu\gamma'_\mu} + q\,\underbrace{A_\mu(\gamma(\lambda))\,\gamma'^\mu}\right)$$

$$= -m\int_\gamma d\tau - q\int_\gamma A_\mu\, d\gamma^\mu$$

1) Lorentz invariant if $A_\mu(x)$ is a Lorentz covariant vector field

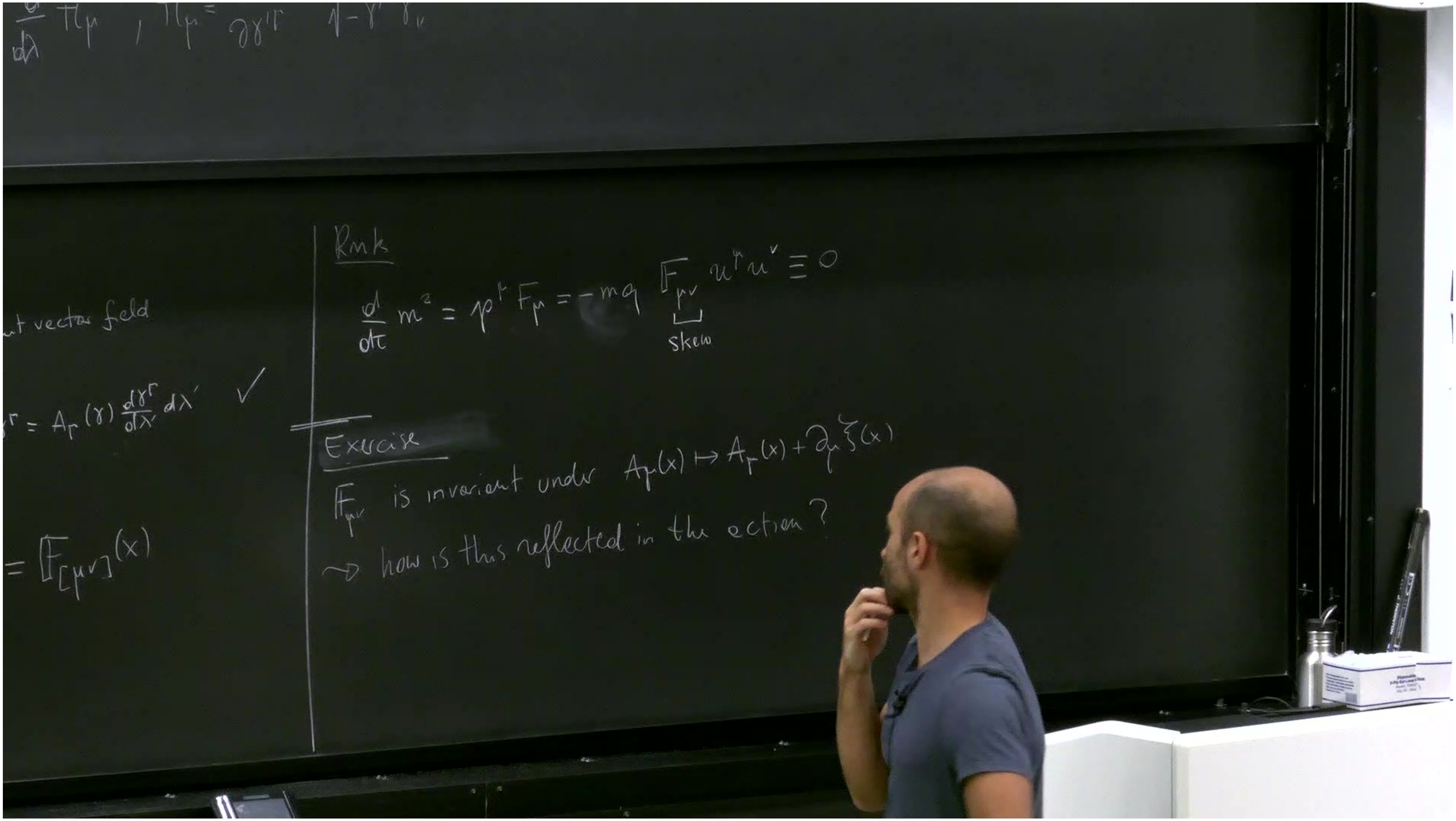
2) invariance under $\lambda \mapsto \lambda'$

$$A_\mu(\gamma)\frac{d\gamma^\mu}{d\lambda}d\lambda = A_\mu(\gamma)\,d\gamma^\mu = A_\mu(\gamma)\frac{d\gamma^\mu}{d\lambda'}d\lambda' \quad \checkmark$$

Exercise: Euler-Lagrange eqs, check!

$$\frac{d}{d\tau}p_\mu = -q\,\underbrace{F_{\mu\nu}(\gamma)\,u^\nu}_{\text{Lorentz force}}, \quad F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{[\mu\nu]}(x)$$

$(p_\mu \neq \pi_\mu)$


t vector field
Rmk
$\frac{d}{d\tau} m^2 = p^\mu F_\mu = -mq \, F_{\mu\nu} u^\mu u^\nu \equiv 0$
skew
Exercise
$F_{\mu\nu}$ is invariant under $A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \xi(x)$
→ how is this reflected in the action?
$= F_{[\mu\nu]}(x)$