Title: Lecture - Classical Physics, PHYS 776

Speakers: Aldo Riello

Collection/Series: Classical Physics (Core), PHYS 776, September 3 - October 4, 2024

Date: September 04, 2024 - 10:45 AM

URL: https://pirsa.org/24090001

ACTION PRINCIPLE

Configuration space Q, described by n (generalized) coordinates $(q^1, \ldots, q^n) \in Q$

Changes of configuration in time: (generalized) velocities $(v^1, \ldots, v^n)$

Rmk Velocities are always unbounded "arrows over Q" (VECTOR SPACE)

The set of pairs

$$(q^i, v^i) \in TQ$$

"tangent space of config. sp. Q"

Ex • particle in $\mathbb{R}^3$: $Q = \mathbb{R}^3$

$$TQ = \mathbb{R}^3 \times \mathbb{R}^3$$

• pendulum : $Q = S^1$ (cpt)

$$TQ = S^1 \times \mathbb{R} \ni (\theta, \omega)$$

$\times \mathbb{R}^3$

(θ, ω)

History (trajectory) of the system

$$\gamma \in \Gamma_Q = C^\infty(\underset{\substack{\uparrow \\ \text{time}}}{\mathbb{R}}, Q)$$

$$\gamma : \mathbb{R} \longrightarrow Q$$
$$t \longmapsto q^i = \gamma^i(t)$$

configuration at time t

$$\mathbb{R} \longrightarrow TQ$$
$$t \longmapsto (q^i, v^i) = \Big(\gamma^i(t), \underbrace{\frac{d}{dt}\gamma^i(t)}_{\dot\gamma^i(t)}\Big)$$

- Lagrangian function

$$L : \underset{\substack{\uparrow \\ \text{time}}}{\mathbb{R}} \times TQ \longrightarrow \mathbb{R}$$

$$(t, q^i, v^i) \mapsto L(q^i, v^i, t)$$

- Action functional

$$S : \mathbb{R} \times \mathbb{R} \times \Gamma_Q \longrightarrow \mathbb{R}$$

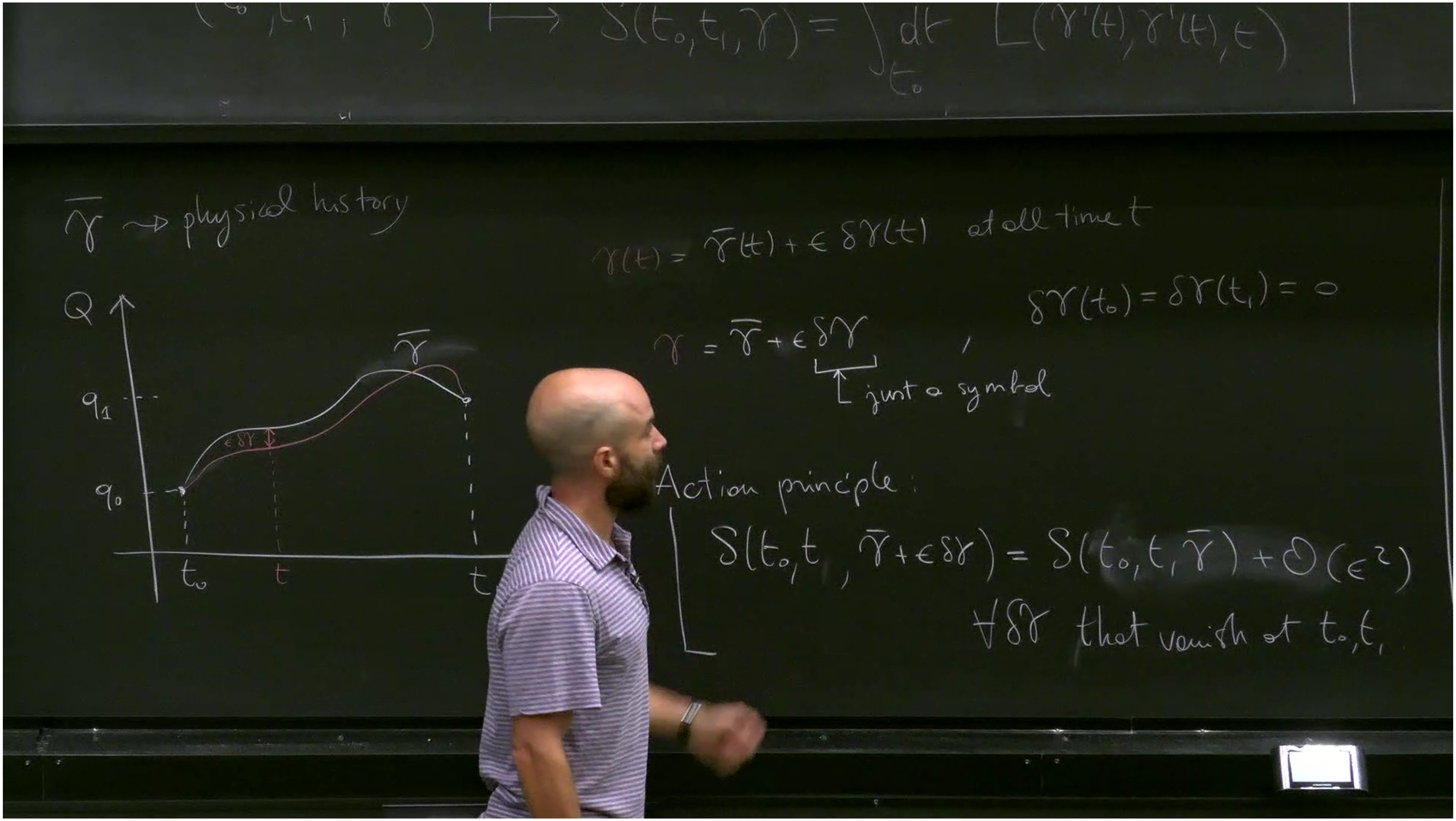
$$(t_0, t_1, \gamma) \longmapsto S(t_0, t_1, \gamma) = \int_{t_0}^{t_1} dt \; L(\gamma^i(t), \dot{\gamma}^i(t), t)$$

Rmk (notation) For brevity

$$S[q(t)] = \int L(q, \dot{q}, t)$$

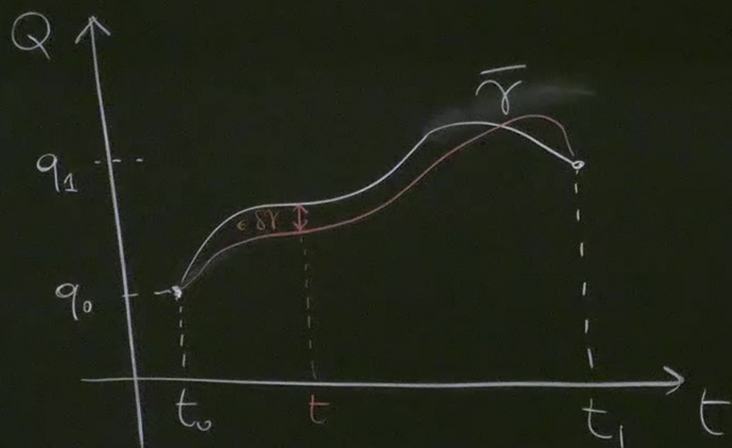
Action principle (Hamilton)

The dynamics of (any ?) physical system is fully characterized by a Lagrangian function. Its physical histories are extrema of associated action functional at fixed boundary conditions (in time).


$(t_0, t_1, \gamma) \longmapsto S(t_0, t_1, \gamma) = \int_{t_0} dt \; L(\gamma'(t), \gamma'(t), t)$
$\bar{\gamma} \rightsquigarrow$ physical history
$\gamma(t) = \bar{\gamma}(t) + \epsilon\, \delta\gamma(t)$ at all time t
$\delta\gamma(t_0) = \delta\gamma(t_1) = 0$
$\gamma = \bar{\gamma} + \underbrace{\epsilon\, \delta\gamma}_{\text{just a symbol}}$
Q
q_1
q_0
$\bar{\gamma}$
$\epsilon\delta\gamma$
t_0
t
t_1
Action principle:
$S(t_0, t_1, \bar{\gamma} + \epsilon\delta\gamma) = S(t_0, t_1, \bar{\gamma}) + \mathcal{O}(\epsilon^2)$
$\forall \delta\gamma$ that vanish at t_0, t_1

$S(t_0, t_1, \gamma) = \int_{t_0} dt \; L(\gamma(t), \dot\gamma(t), t)$

$\bar\gamma \rightsquigarrow$ physical history



$\gamma(t) = \bar\gamma(t) + \epsilon\,\delta\gamma(t)$ at all time t

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Action principle:

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$\forall\,\delta\gamma$ that vanish at t_0, t_1

$) = 0$

$+ \mathcal{O}(\epsilon^2)$

at t_0, t_1

Formally.

$$\left.\frac{\delta S}{\delta \gamma}\right|_{\bar{\gamma}} = 0$$

$$\left(\frac{d}{dt}\delta\gamma\right)(t)$$

$$\equiv$$

$$L(\gamma(t), \dot{\gamma}(t), t) = L(\bar{\gamma}(t) + \epsilon\,\delta\gamma(t), \dot{\bar{\gamma}}(t) + \epsilon \boxed{\delta\dot{\gamma}(t)}, t)$$

$$= L(\bar{\gamma}(t), \dot{\bar{\gamma}}(t), t)$$

$$+ \epsilon\left(\sum_i \left.\frac{\partial L}{\partial q^i}\right|_{(\bar{\gamma}(t), \dot{\bar{\gamma}}(t), t)} \delta\gamma^i(t) + \left.\frac{\partial L}{\partial v^i}\right|_{(\bar{\gamma}(t), \dot{\bar{\gamma}}(t), t)} \delta\dot{\gamma}^i(t)\right)$$

$$+ \mathcal{O}(\epsilon^2)$$

In short-hand notation

$$L(\bar q+\epsilon\delta q,\ \dot{\bar q}+\epsilon\delta\dot q,\ t) = L(\bar q,\dot{\bar q},t)+\epsilon\,\delta L(\bar q,\dot{\bar q},t)+O(\epsilon^2)$$

$$\delta L = \sum_i \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial v^i}\underbrace{\delta\dot q^i}_{\frac{d}{dt}\delta q^i}$$

Integr. by parts

$$\delta S[\bar q(t)] = \int_{t_0}^{t_1} dt\ \delta L$$

$$= \int_{t_0}^{t_1}\sum_i\left(\frac{\partial L}{\partial q^i}-\frac{d}{dt}\frac{\partial L}{\partial v^i}\right)\delta q^i + \left[\sum_i \frac{\partial L}{\partial v^i}\delta q^i\right]_{t_0}^{t_1}$$

Boundary cond

Vanishes

$\delta q(t_0)=\delta q(t_1)=0$

Hamilton's principle $\iff$ Euler-Lagrange eqs

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i}\bigg|_{(\gamma,\dot{\gamma},t)} = 0$$

Exercise $Q=\mathbb{R}^3$, $(Q=\mathbb{R}^{3N})$

Newton's equations in a conservative potential $V(\vec{x})$

$$m\vec{a} = \vec{F}(\vec{x}) = -\vec{\nabla}V(\vec{x})$$

check are the EL eom $L = \frac{1}{2}m|\vec{v}|^2 - V(\vec{x})$.

...ary cond

...anishes $\delta q(t_0) = \delta q(t_1) = 0$

$\int_{t_0}^{t_1}$

$= 0$

Hamilton's principle $\iff$ Euler-Lagrange eqs

$$\frac{\delta S}{\delta \gamma}\Big|_{\bar{\gamma}} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\Big|_{(\bar{\gamma}, \dot{\bar{\gamma}}, t)} = 0$$

Exercise $Q = \mathbb{R}^3, (Q = \mathbb{R}^{3N})$

Newton's equations in a conservative potential $V(\vec{x})$

$$m\vec{a} = \vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x})$$

check are the EL eom $L = \frac{1}{2} m |\dot{\vec{r}}|^2 - V(\vec{x})$.

Rmk

Boundary conditions vs Initial value problem.

the EL eqs are
2nd order ODE
they have unique sols
in terms initial pos. & vel.

$$\langle q_1, t_1 | e^{-iH\Delta t} | q_0, t_0 \rangle$$

if all we care about L is the E.L. eqs that we get from it, then we can redifine L as follows

"$L \rightsquigarrow \tilde{L} = L + \frac{d}{dt} \ell$", $\ell = \ell(q,t)$

more precisely:

$$\tilde{L}(q,v,t) = L(q,v,t) + \frac{\partial \ell}{\partial t}(q,t) + \sum_i v^i \frac{\partial \ell}{\partial q^i}(q,t)$$

$$\tilde{L}(\gamma(t),\dot{\gamma}(t),t) = L(\gamma,\dot{\gamma},t) + \frac{d}{dt}\ell(\gamma(t),t)$$

$\tilde{L}(\gamma(t), \dot{\gamma}(t), t) = L($

Pf 1 Brute force:

$$\frac{\partial \tilde{L}}{\partial v^i} = \frac{\partial L}{\partial v^i} + \frac{\partial \ell}{\partial q^i}$$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^i}\Big|_\gamma = \frac{d}{dt}\frac{\partial L}{\partial v^i}\Big|_\gamma + \frac{\partial}{\partial t}\frac{\partial \ell}{\partial q^i}\Big|_\gamma + \sum_k \dot{\gamma}^k \frac{\partial^2 \ell}{\partial q^k \partial q^i}\Big|_\gamma$$

$$\frac{\partial \tilde{L}}{\partial q^i}\Big|_\gamma = \frac{\partial L}{\partial q^i}\Big|_\gamma + \frac{\partial^2 \ell}{\partial q^i \partial t}\Big|_\gamma + \sum_k \underbrace{v^k}_{\dot{\gamma}^k} \frac{\partial^2 \ell}{\partial q^i \partial q^k}\Big|_\gamma$$

$$(t_0, t_1, \gamma) \longmapsto S(t_0, t_1, \gamma) = \int_{t_0} dt \; L(\gamma^i(t), \dot\gamma^i(t), t)$$

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$S(t_0, t_1, \gamma) = \int_{t_0} dt \; L(\gamma(t), \gamma'(t), t)$

Cyclical coordinates

Def q^1 is cyclical if $\frac{\partial L}{\partial q^1} = 0$

Prop. if q^1 is cyclical, then

$$P_1 \equiv P(q, v, t) := \frac{\partial L}{\partial v^1}(q, v, t)$$

is conserved along a physical history.

Pf. E.L. eqs: $0 = \frac{d}{dt}\frac{\partial L}{\partial v^1} - \frac{\partial L}{\partial q^1}\Big|_{\overline{\gamma}} \underset{\text{cycl hyp}}{=} \frac{d}{dt}\frac{\partial L}{\partial v^1} \equiv \frac{d}{dt} P_1$ ☒

$$L = \frac{1}{2} m |\vec{v}|^2 - V(\vec{x})$$

Suppose $\frac{\partial V}{\partial x^1} = 0 \Rightarrow$

$$\underset{\displaystyle \frac{d}{dt} m v^1}{\underset{\shortparallel}{m a^1}} = F^1 = -\frac{\partial V}{\partial x^1} = 0$$

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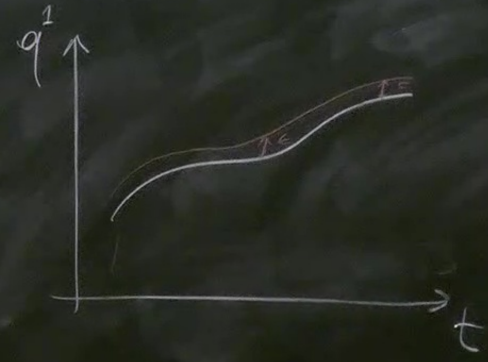

$L =$

Suppose