Title: Gauss-Bonnet Gravity in 4D and the connection to Horndeski's Theory

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Collection/Series: 50 Years of Horndeski Gravity: Exploring Modified Gravity

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Abstract:

In this talk I will review the topic of 4D Einstein-Gauss-Bonnet gravity, which has been the subject of considerable interest over the past years. I will discuss the mathematical complexities involved in implementing this idea, and review recent attempts at constructing well-defined, self-consistent theories that enact it, and their relation to Horndeski gravity. I then move on to consider the interesting phenomenology that results from these theories.

Gauss-Bonnet Gravity in 4D and the connection to Horndeski's Theory

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50 Years of Horndeski Gravity: Exploring Modified Gravity
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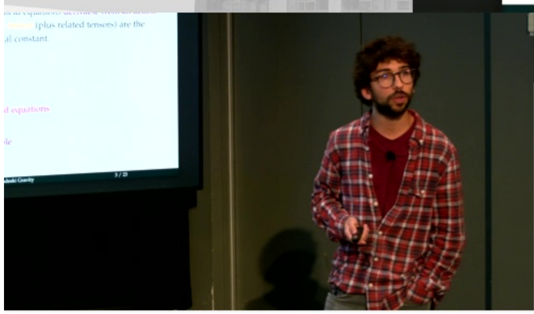
Lovelock's Theorem

The only second-order, local gravitational field equations derivable from an action containing solely the 4-dimensional metric tensor (plus related tensors) are the Einstein field equations with a cosmological constant.

Our options for modifying gravity:

1. New field content
2. Higher dimensions
3. Derivatives of higher order in the field equations
4. Non-locality
5. Non-derivable from an action principle



The Gauss-Bonnet Term

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

- Natural generalization of Einstein's gravity in higher-dimensions (EGB gravity)
- Part of Horndeski's gravity ($F(\phi)\mathcal{G}$)
- Motivated by quantum gravity (string theory low-energy EFTs, trace anomaly...)

The Gauss-Bonnet Term in 4D

Gauss-Bonnet action in D-dimensions

$$S_{\mathcal{G}} = \alpha \int d^D x \sqrt{-g}\,\mathcal{G}$$

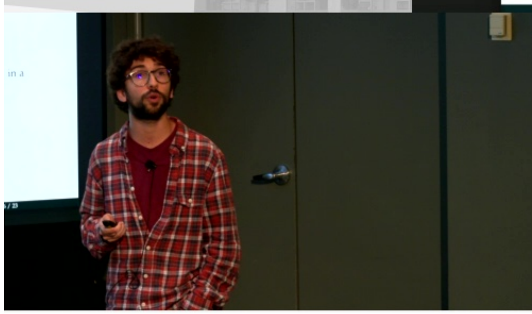
In $D = 4$ we have that

$$\frac{\delta}{\delta g^{\mu\nu}} S_{\mathcal{G}} \equiv H_{\mu\nu}$$

is identically vanishing (Euler characteristic $\chi \propto S_{\mathcal{G}} \to H_{\mu\nu} = 0$). Manifest in the trace:

$$g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} S_{\mathcal{G}} = g^{\mu\nu} H_{\mu\nu} = \alpha \frac{(D-4)}{2} \mathcal{G}$$

What is the *most natural* way of including the Gauss-Bonnet term in a four-dimensional theory?



4D Einstein-Gauss-Bonnet Gravity

Dimensional regularization procedure introduced by Glavan & Lin in **PRL 124 (2020) 8, 081301**

PHYSICAL REVIEW LETTERS **124,** 081301 (2020)

Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime

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In this Letter we present a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses Lovelock's theorem. The theory we present is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to nontrivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. We report several appealing new predictions of this theory, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions, and others.

4D Einstein-Gauss-Bonnet Gravity

1. Start with the Einstein-Gauss-Bonnet action in D dimensions and introduce a divergent factor for the Gauss-Bonnet sector

$$S = \lim_{D \to 4} \left[\frac{1}{16\pi} \int d^D x \sqrt{-g} \left(R + \frac{\alpha}{D-4} \mathcal{G} \right) + S_M \right]$$

4D Einstein Gauss-Bonnet Gravity

3. The trace equation, for instance, takes the finite form

$$\lim_{D\to 4}\left[g^{\mu\nu}\left(G_{\mu\nu}+\frac{\alpha}{D-4}H_{\mu\nu}\right)=8\pi T\right]\Leftrightarrow$$

$$\lim_{D\to 4}\left[\frac{(D-2)}{2}R+\frac{\alpha}{(D-4)}\frac{(D-4)}{2}\mathcal{G}=-8\pi T\right]\Leftrightarrow$$

$$R+\frac{\alpha}{2}\mathcal{G}=-8\pi T,$$

4D Einstein Gauss-Bonnet Gravity

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$$R+\frac{\alpha}{2}\mathcal{G}=-8\pi T,$$

4. The equations of motion for certain line elements, such as static spherically symmetric or FLRW do not diverge in the 4D limit. Solutions similar to the higher-dimensional EGB ones.

4D Einstein Gauss-Bonnet Gravity – Black holes and Friedmann equations

Gauss-Bonnet corrected black holes:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$
$$f(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8M\alpha}{r^3}}\right)$$

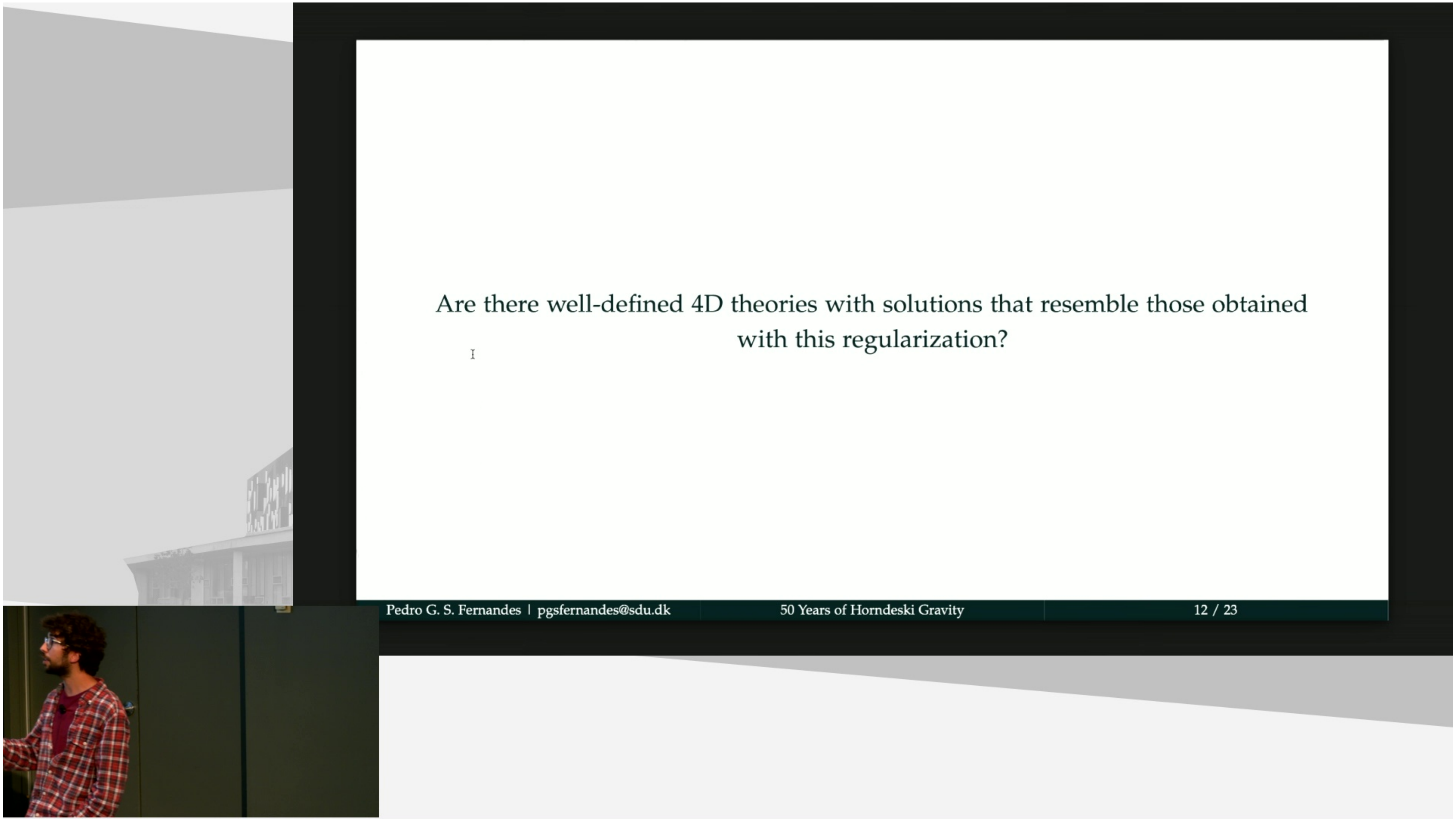
Gauss-Bonnet corrected Friedmann equations:

$$H^2 + \alpha H^4 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$
$$\left(1 + 2\alpha H^2\right)\dot{H} = -4\pi G\left(\rho + p\right).$$

4D Einstein Gauss-Bonnet Gravity – Shortcomings

- The process relies on the specification of the geometry of the extra dimensional space before taking the 4D limit. There are countless ways to do this.
- In spacetimes that lack explicit symmetries, the equations of motion (other than the trace equation) are not well-defined in the 4D limit

$$\lim_{D\to 4} \frac{H_{\mu\nu}}{D-4} = \text{finite term} + \lim_{D\to 4} \frac{1}{D-4}\left(C_{\mu\alpha\beta\rho}C_{\nu}{}^{\alpha\beta\rho} - \frac{1}{4}g_{\mu\nu}C^2\right)$$


Are there well-defined 4D theories with solutions that resemble those obtained with this regularization?
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Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

- Problematic terms depend on the Weyl tensor
- The Weyl tensor is conformally invariant, i.e., ${C^{\mu}}_{\alpha\beta\rho} = {\tilde{C}^{\mu}}_{\alpha\beta\rho}$ given two conformally related metrics $g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu}$
- We will then remove the divergences of the theory by adding to our action the counterterm

$$S = \lim_{D\to 4}\int d^D x\sqrt{-g}\left(R + \frac{\alpha}{D-4}\mathcal{G}\right) - \lim_{D\to 4}\int d^D x \frac{\alpha}{D-4}\sqrt{-\tilde{g}}\tilde{\mathcal{G}}$$

Dimensional Regularization of the Gauss-Bonnet term (2004.08362)

The limit is well-defined and results in

$$S = \int d^4x \sqrt{-g} \left[R - \alpha \left(\phi \mathcal{G} - 4G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 4\Box\phi (\nabla\phi)^2 - 2(\nabla\phi)^4 \right) \right]$$

The theory belongs to the shift-symmetric ($\phi \to \phi + c$) Horndeski class of theories

$$G_2 = 8\alpha X^2, \qquad G_3 = 8\alpha X, \qquad G_4 = 1 + 4\alpha X, \qquad G_5 = 4\alpha \log X$$

Remarkably, a linear combination of the trace and scalar field equations leads to

$$R + \frac{\alpha}{2}\mathcal{G} = -8\pi T$$

In this well-defined regularized theory we can recover the same static black holes and Friedmann equations as in the original regularization procedure!

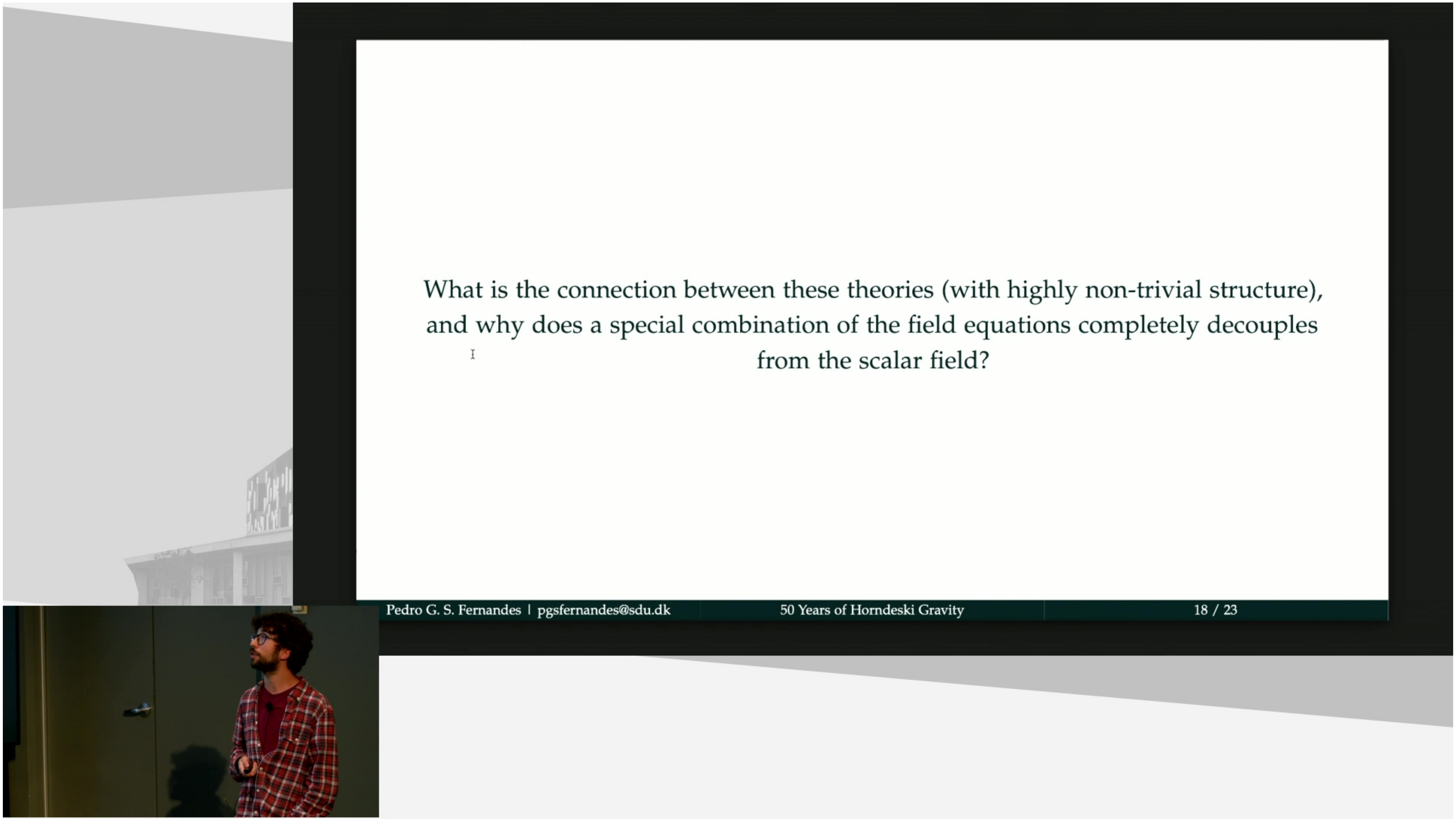
Dimensional Regularization of the Gauss-Bonnet term

Regularized Kaluza-Klein reduction [Lu and Pang, 2003.11552]

$$S = \int d^4x\sqrt{-g}\Big[R - \alpha\Big(\phi\mathcal{G} - 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4 + 2\lambda e^{2\phi}\left[R + 6(\nabla\phi)^2 + 3\lambda e^{2\phi}\right]\Big)\Big]$$

The same combination of the trace and scalar field equations leads to the purely geometrical equation. Solutions also present.

See also [Aoki, Gorji, Mukohyama, 2005.03859, 2020] for diffeomorphism breaking regularization


What is the connection between these theories (with highly non-trivial structure), and why does a special combination of the field equations completely decouples from the scalar field?
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Conformally coupled scalar field (2105.04687)

PHYSICAL REVIEW D **103,** 104065 (2021)

Gravity with a generalized conformal scalar field: Theory and solutions

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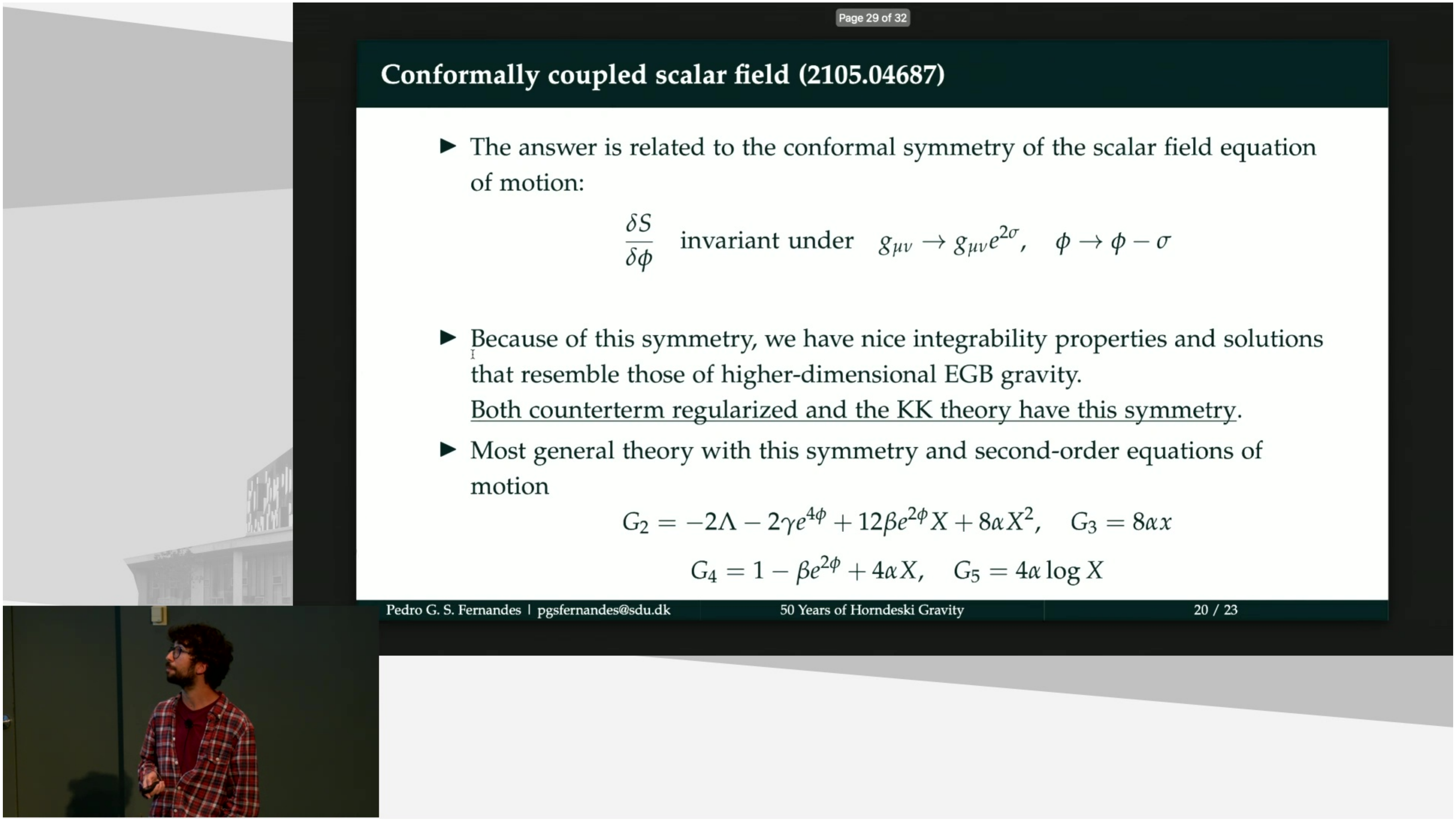
We naturally extend general relativity with a conformally coupled scalar field by only requiring conformal invariance of the scalar field equation of motion and not of the action. The classically extended theory incorporates a scalar-Gauss-Bonnet sector and has second-order equations of motion, belonging to the Horndeski class. Remarkably, the theory features a purely geometrical field equation that allows for closed-form black hole solutions and cosmologies to be easily found. These solutions permit investigations of in-vogue scalar-Gauss-Bonnet corrections to the gravitational action without the need of resorting to approximations or numerical methods.

DOI: 10.1103/PhysRevD.103.104065

Conformally coupled scalar field (2105.04687)

- The answer is related to the conformal symmetry of the scalar field equation of motion:

$$\frac{\delta S}{\delta \phi} \text{ invariant under } \quad g_{\mu\nu} \to g_{\mu\nu} e^{2\sigma}, \quad \phi \to \phi - \sigma$$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Conformally coupled scalar field (2105.04687)
The answer is related to the conformal symmetry of the scalar field equation of motion:
$\frac{\delta S}{\delta \phi}$ invariant under $g_{\mu\nu} \to g_{\mu\nu} e^{2\sigma}, \quad \phi \to \phi - \sigma$
Because of this symmetry, we have nice integrability properties and solutions that resemble those of higher-dimensional EGB gravity.
Both counterterm regularized and the KK theory have this symmetry.
Most general theory with this symmetry and second-order equations of motion
$G_2 = -2\Lambda - 2\gamma e^{4\phi} + 12\beta e^{2\phi} X + 8\alpha X^2, \quad G_3 = 8\alpha x$
$G_4 = 1 - \beta e^{2\phi} + 4\alpha X, \quad G_5 = 4\alpha \log X$
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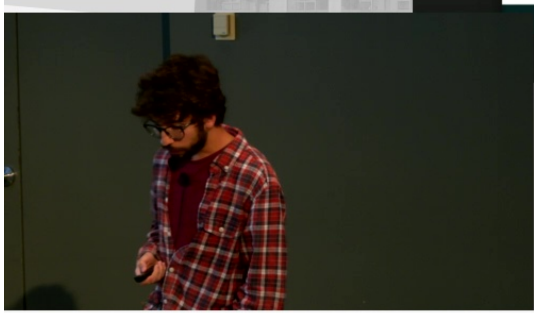
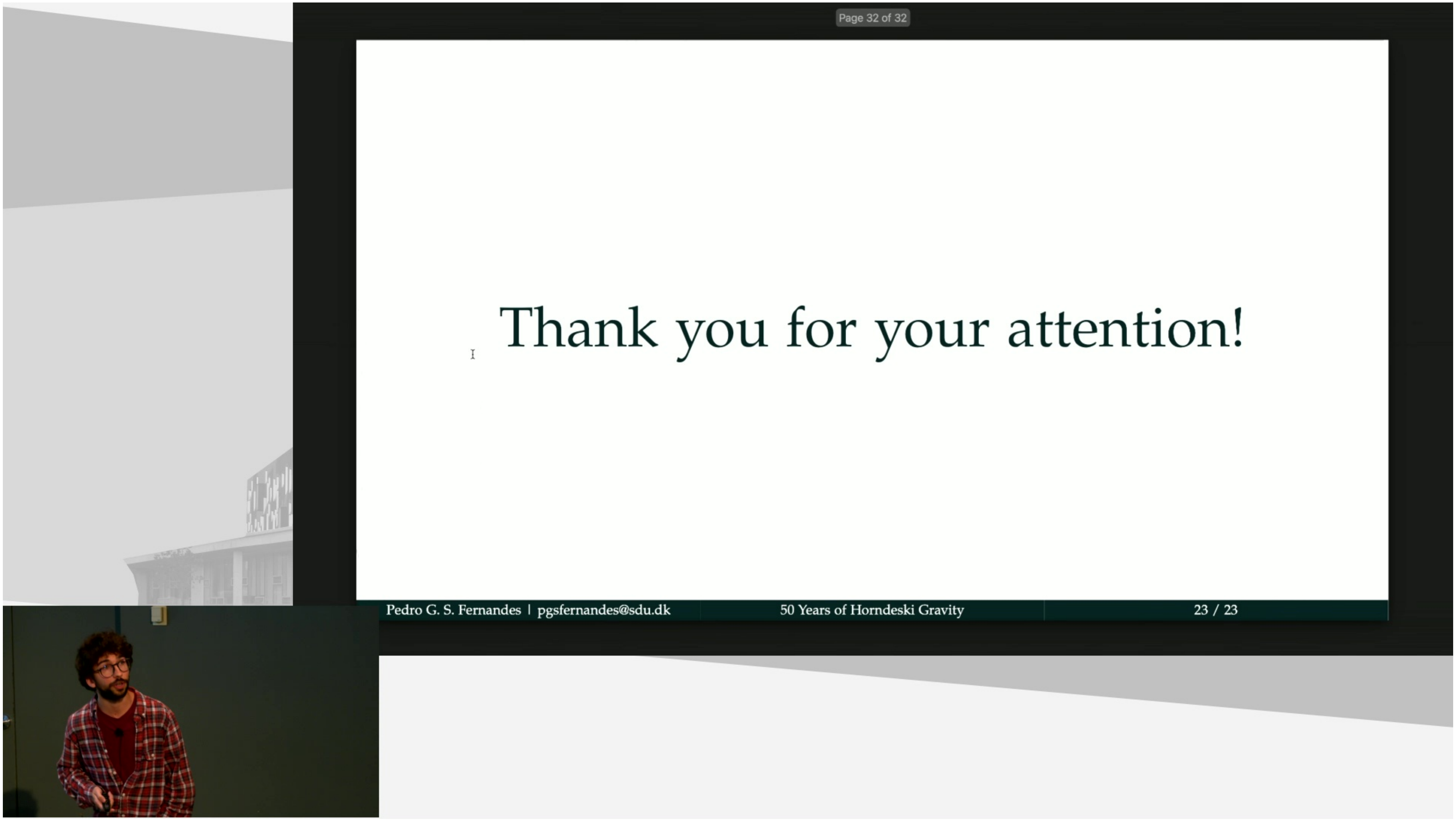
Message to take home

What 4D theory is most similar to higher-dimensional Einstein-Gauss-Bonnet gravity?

The 4D theory most similar to higher-dimensional Einstein-Gauss-Bonnet gravity belongs to the Horndeski class with (generalized) conformal symmetry



If you're interested in learning more please check the review "The 4D Einstein–Gauss–Bonnet theory of gravity: a review", Fernandes et. al, arXiv:2202.13908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Thank you for your attention!
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