Title: Beyond Horndeski theories

Speakers: David Langlois

Collection/Series: 50 Years of Horndeski Gravity: Exploring Modified Gravity

Subject: Cosmology, Strong Gravity, Mathematical physics

Date: July 17, 2024 - 9:45 AM

URL: https://pirsa.org/24070041

Abstract:

This talk will introduce scalar-tensor theories of gravity that contain a single scalar degree of freedom in addition to the usual tensor modes. These theories constitute the very broad family of Degenerate Higher-Order Scalar-Tensor (DHOST) theories, which include and extend Horndeski theories. Cosmological aspects of these theories will then be discussed. Finally, I will also present some results concerning black hole perturbations in the context of these models of modified gravity.

Beyond Horndeski theories

David Langlois
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Introduction

- **Horndeski theories**: most general family of scalar-tensor theories with a **single scalar** degree of freedom and at most **second-order** equations of motion.

- **Going beyond Horndeski**: most general family of scalar-tensor theories with a **single scalar** degree of freedom.

  Higher-order equations of motion are possible if the **Lagrangian is degenerate**.

- In this talk:
  1. DHOST (Degenerate Higher-Order Scalar-Tensor) theories
  2. Cosmological aspects
  3. Black hole perturbations

Degenerate theories

- In general, higher-order theories with a Lagrangian of the form $L(\ddot{q}, \dot{q}, q)$ contain an **extra dof.**

- **Degenerate** theories: EOM higher order, **no extra dof**

$$L = \frac{1}{2} a\, \ddot{\phi}^2 + b\, \ddot{\phi}\, \dot{q} + \frac{1}{2} c\, \dot{q}^2 + \frac{1}{2} \dot{\phi}^2 - V(\phi, q)$$

$$\begin{cases} ac - b^2 \neq 0 & : \textbf{3 dof} \\ ac - b^2 = 0 & : \textbf{2 dof} \end{cases}$$

$$\Longrightarrow \quad L_{\rm deg} = \frac{1}{2} c \left(\dot{q} + \frac{b}{c} \ddot{\phi} \right)^2 + \frac{1}{2} \dot{\phi}^2 - V(\phi, q) \qquad [x \equiv q + \frac{b}{c} \dot{\phi} \quad \text{and} \quad \phi]$$

- **DHOST** (Degenerate Higher-Order Scalar-Tensor)

$$\phi(t) \longrightarrow \phi(x^\mu)$$
$$q(t) \longrightarrow g_{\mu\nu}$$

DL & Noui '15

DHOST theories

DL & Noui '15
Ben Achour, Crisostomi, Koyama, DL, Noui & Tasinato '16

- Action

$$S[g,\phi] = \int d^4x \sqrt{-g} \left[F_{(2)}(X,\phi)\, {}^{(4)}R + P(X,\phi) + Q(X,\phi)\Box\phi + \sum_{I=1}^{5} A_I(X,\phi) L_I^{(2)} + F_{(3)}(X,\phi)\, G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X,\phi) L_I^{(3)} \right]$$

$$X \equiv \nabla_\mu\phi\nabla^\mu\phi$$
$$\phi_\mu \equiv \nabla_\mu\phi$$
$$\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$$

with 5 **quadratic** elementary Lagrangians:

$$L_1^{(2)} = \phi_{\mu\nu}\,\phi^{\mu\nu}\,, \quad L_2^{(2)} = (\Box\phi)^2\,, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu$$
$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2$$

and 10 **cubic** ones:

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad \ldots\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3$$

DHOST theories

- Action [DL & Noui '15 ; Ben Achour, Crisostomi, Koyama, DL, Noui & Tasinato '16]

$$S[g,\phi] = \int d^4x \sqrt{-g} \left[F_{(2)}(X,\phi)\, {}^{(4)}R + P(X,\phi) + Q(X,\phi)\Box\phi + \sum_{I=1}^{5} A_I(X,\phi) L_I^{(2)} + F_{(3)}(X,\phi)\, G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X,\phi) L_I^{(3)} \right]$$

- **Degeneracy conditions** (3 for quadratic DHOST)

  ➡ **Full classification**

- Particular subsets:
  - **Horndeski:**

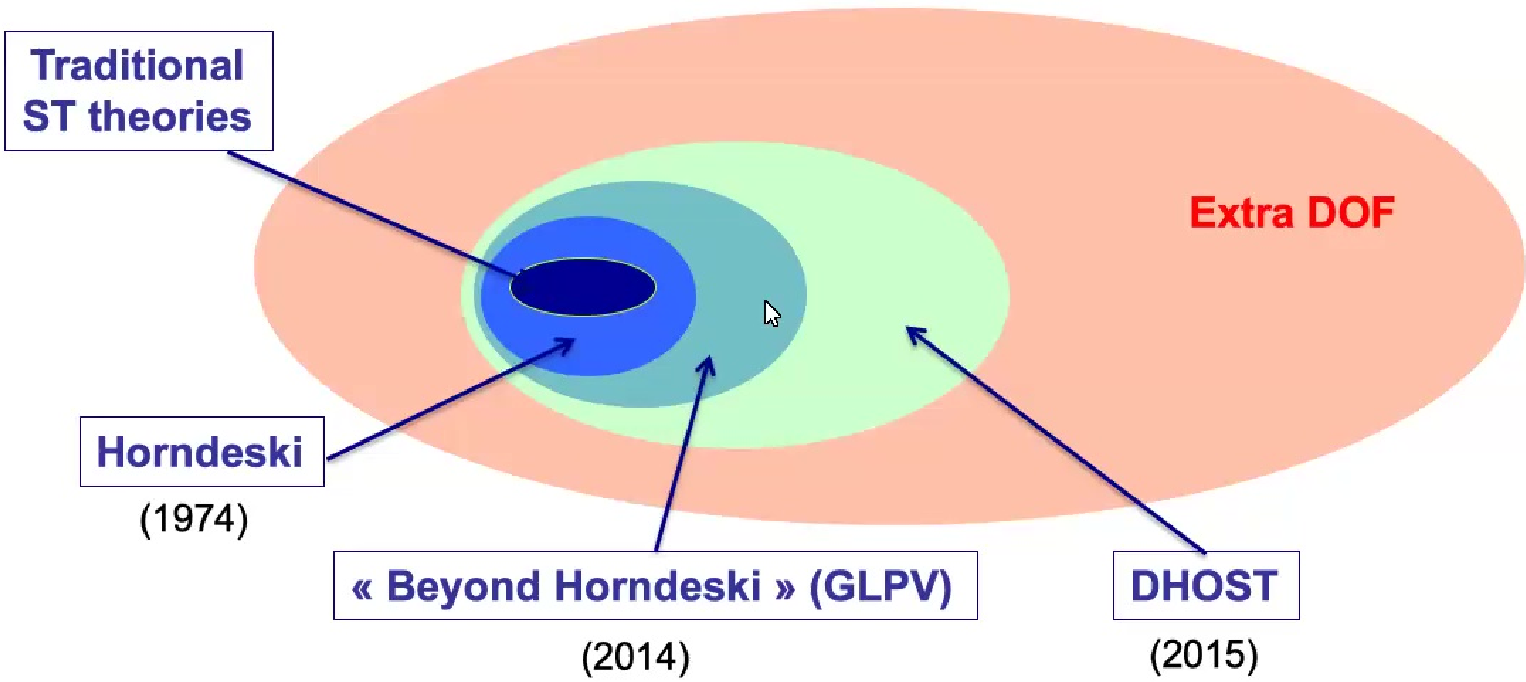
$$F_{(2)} = G_4\,, \quad A_1 = -A_2 = 2G_{4,X}\,, \quad A_3 = A_4 = A_5 = 0$$

$$F_{(3)} = G_5\,, \quad 3B_1 = -B_2 = \frac{3}{2}B_3 = G_{5,X}\,, \quad B_I = 0\ (I = 4, \dots, 10)$$

  - « Beyond Horndeski » (or GLPV) Gleyzes, DL, Piazza & Vernizzi '14

Scalar-tensor lanscape

- Traditional theories: $\mathcal{L}(\nabla_\lambda\phi, \phi)$
- Generalized theories: $\mathcal{L}(\nabla_\mu\nabla_\nu\phi,\ \nabla_\lambda\phi,\ \phi)$
  DHOST: most general family with a **single scalar DOF**



Disformal transformations

[Zumalacarregui & Garcia-Bellido '13; Ben Achour, DL & Noui '15]

- Transformation

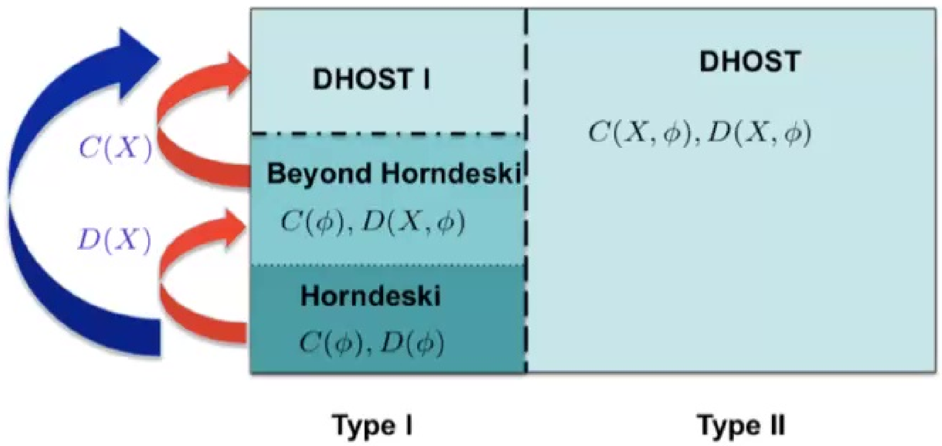
$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(X,\phi)\, g_{\mu\nu} + D(X,\phi)\, \partial_\mu\phi\, \partial_\nu\phi$$

- From $\tilde{S}\,[\phi, \tilde{g}_{\mu\nu}]$, one gets the new action

[Bekenstein '92]

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}\,[\phi, \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\, \phi_\mu \phi_\nu]$$

- DHOST families are **closed** under these transformations



- When **standard fields** are (minimally) included, two disformally related theories are **physically inequivalent** !

$$S[g_{\mu\nu}, \phi] + S_m[\Psi_m, g_{\mu\nu}] \quad \neq \quad \tilde{S}[\tilde{g}_{\mu\nu}, \phi] + S_m[\Psi_m, \tilde{g}_{\mu\nu}]$$

Dark energy & GW170817

- Constraint on the **speed of GW**: $|c_{\rm gw} - c|/c \lesssim 10^{-15}$

$c_{\rm gw} = c$ implies $A_1 = A_2 = 0$ (and no cubic term)

- To avoid **GW decay** into dark energy

Creminelli, Lewandowski, Tambalo & Vernizzi '18

$A_3 = 0$

$$L^{\rm DHOST}_{c_g=1,\text{no decay}} = F(X,\phi)\,R + P(X,\phi) + Q(X,\phi)\,\Box\phi + 6\frac{F_X^2}{F}\phi^\mu\phi_{\mu\nu}\phi^{\nu\sigma}\phi_\sigma$$

- However, constraints from GW170817 valid for $f \sim 1 - 100\,\text{Hz}$

Not directly relevant for cosmological scales

[Moreover, at the limit of the regime of validity De Rham & Melville '18]

Homogeneous cosmology of DHOST

Crisostomi, Koyama, DL, Noui & Steer '18; Boumaza, DL, Noui '20

- Quadratic DHOST theories:

$$L = F(X,\phi)\, R + P(X,\phi) + Q(X,\phi)\,\Box\phi + \sum_{I=1}^{5} A_I(X,\phi)\, L_I^{(2)}$$

- **Background:** $ds^2 = -N^2(t)dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j\,, \qquad \phi = \phi(t)$

  - Homogeneous action

$$\bar{S} = \int dt\, N a^3 \left\{ -6(F - XA_1)\left[\frac{\dot a}{Na} - \mathcal{V}\frac{\dot\phi}{N^2}\frac{d}{dt}\left(\frac{\dot\phi}{N}\right)\right]^2 - 3(Q+2F_\phi)\frac{\dot a\,\dot\phi}{N^2 a} - Q\frac{1}{N}\frac{d}{dt}\left(\frac{\dot\phi}{N}\right) + P \right\},$$

with $\mathcal{V} \equiv \dfrac{4F_X + XA_3 - 2A_1}{4(F - XA_1)} \qquad X = -\dfrac{\dot\phi^2}{N^2}$

  - Trick: auxiliary scale factor such that

$$\frac{\dot b}{b} = \frac{\dot a}{a} - \mathcal{V}\frac{\dot\phi}{N}\frac{d}{dt}\left(\frac{\dot\phi}{N}\right) + \dots \qquad a = e^{\lambda(X,\phi)}\, b\,, \quad \lambda_X = -\frac{1}{2}\mathcal{V}$$

Illustrative example

- Simple toy model

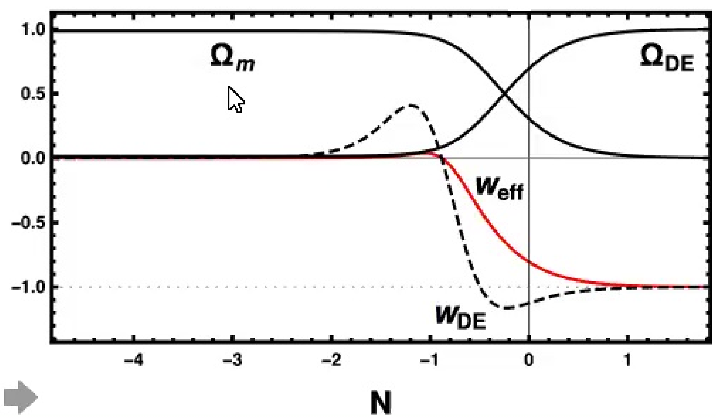
Boumaza, DL, Noui '20

$$F = \frac{1}{2}, \quad P = \alpha X, \quad Q = 0, \quad A_1 = -A_2 = -\beta X, \quad \lambda = \frac{1}{2} s\beta X^2$$

Choice of λ determines A_3 . Degeneracy imposes A_4 and A_5 .

- From matter era to dark energy era

$$3H^2 = \rho_m + \rho_{\rm DE} \qquad 2\dot{H} + 3H^2 = P_m + P_{\rm DE}$$



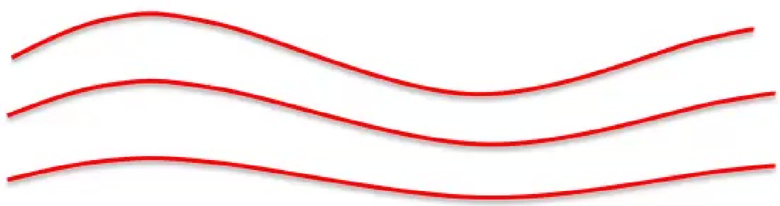
$$w_{\rm DE} \equiv \frac{P_{\rm DE}}{\rho_{\rm DE}}$$

$$w_{\rm eff} \equiv \frac{P_m + P_{\rm DE}}{\rho_m + \rho_{\rm DE}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}$$

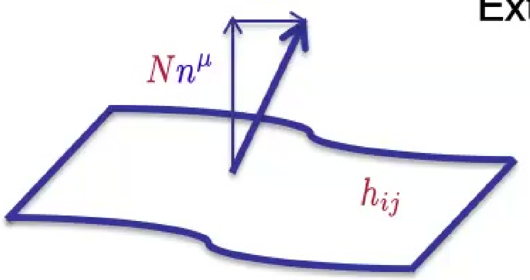
Effective description of Dark Energy

[Reviews: Gleyzes, DL & Vernizzi '14; DL '18]

- The scalar field defines a **preferred slicing**



- **3+1 decomposition** based on this uniform field slicing



Extrinsic curvature: $K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right)$

Intrinsic curvature: R_{ij}

$$X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = -\frac{\dot{\phi}^2(t)}{N^2}$$

Effective description of Dark Energy

- **Action of the form:** $S_g = \int d^4x \, N\sqrt{h}\, L(N, K_{ij}, R_{ij}; t)$

- **Homogeneous equations**
  - FLRW metric: $ds^2 = -\bar{N}^2(t)\, dt^2 + a^2(t)\, \delta_{ij} dx^i dx^j$
  - Lagrangian $$\bar{L}(a, \dot{a}, \bar{N}) \equiv L\left[K^i_j = \frac{\dot{a}}{\bar{N}a}\,\delta^i_j, R^i_j = 0, N = \bar{N}(t)\right]$$

- **Perturbations:** $\delta N \equiv N - \bar{N}\,,\ \delta K^i_j \equiv K^i_j - H\delta^i_j\,,\ \delta R^i_j \equiv R^j_i$

$$L(q_A) = \bar{L} + \frac{\partial L}{\partial q_A}\delta q^A + \frac{1}{2}\frac{\partial^2 L}{\partial q_A \partial q_B}\delta q_A \delta q_B + \dots$$

The quadratic action describes the dynamics of linear perturbations.

DHOST cosmological perturbations

- **Quadratic** action

$$S_{\rm quad} = \int d^3x\, dt\, a^3\, \frac{M^2}{2}\Big\{\delta K_{ij}\delta K^{ij} - \delta K^2 + (1+\alpha_{\rm T})\left(R\frac{\delta\sqrt{h}}{a^3} + \delta_2 R\right)$$

$$+H^2\alpha_K\delta N^2 + 4H\alpha_B\delta K\delta N + (1+\alpha_H)R\delta N + 4\beta_1\delta K\delta\dot{N} + \beta_2\delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i\delta N)^2\Big\}$$

	α_K	α_B	α_M	α_T	α_H	β_1
K-essence	✓					
Kinetic braiding, DGP	✓	✓				
Brans-Dicke, f(R)	✓	✓	✓			
Horndeski	✓	✓	✓	✓		
Beyond Horndeski	✓	✓	✓	✓	✓	
DHOST	✓	✓	✓	✓	✓	✓

$$\alpha_{\rm M} = \frac{1}{H}\frac{d}{dt}\ln M^2\,, \quad \beta_2 = -6\beta_1^2\,, \quad \beta_3 = -2\beta_1\left[2(1+\alpha_{\rm H}) + \beta_1(1+\alpha_{\rm T})\right]$$

Physical degrees of freedom

- **Scalar (** $\delta N\,, \quad N_i \equiv \partial_i \psi\,, \quad h_{ij} = a^2(t) e^{2\zeta} \delta_{ij}$ **)**

$$S_{\rm quad}[\zeta] \;=\; \int d^3x\, dt\, a^3\, \frac{M^2}{2} \left[\mathcal{A}_\zeta\, \dot{\zeta}^2 - \mathcal{B}_\zeta\, \frac{(\partial_i \zeta)^2}{a^2} \right]$$

with
$$\mathcal{A}_\zeta = \frac{1}{(1+\alpha_B - \dot{\beta}_1/H)^2} \left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} \left(a^3 H\, M^2 \alpha_B \beta_1 \right) \right]$$

$$\mathcal{B}_\zeta = -2(1+\alpha_T) + \frac{2}{aM^2} \frac{d}{dt} \left[\frac{aM^2 \left(1 + \alpha_H + \beta_1 (1+\alpha_T)\right)}{H(1+\alpha_B) - \dot{\beta}_1} \right]$$

- **Tensor:** $S_\gamma^{(2)} = \frac{1}{2} \int dt\, d^3x\, a^3 \left[\frac{M^2}{4} \dot{\gamma}_{ij}^2 - \frac{M^2}{4} (1+\alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right]$

- **Stability** (neither ghost nor gradient instability)

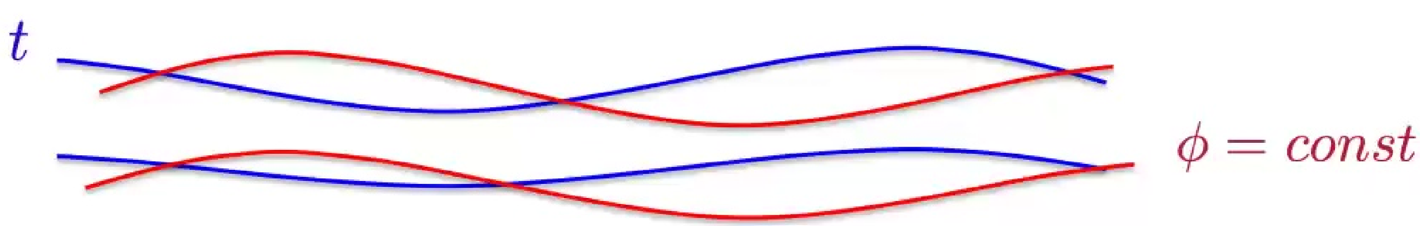
$$M^2 > 0\,, \quad \mathcal{A}_\zeta > 0\,, \quad \mathcal{B}_\zeta > 0\,, \quad c_T^2 \equiv (1+\alpha_T) > 0$$

Cosmological observations

- Use a traditional gauge, e.g. Newtonian gauge

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)\,(1-2\Psi)\,\delta_{ij}\,dx^i dx^j$$

- Description in an arbitrary slicing ?



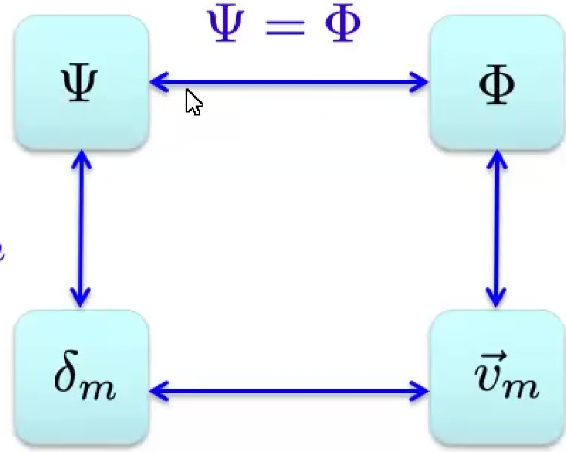
- Coordinate change $t \to t + \pi(t, \vec{x})$

- Perturbations: $\Phi\,,\ \Psi\,,\ \pi\,, \delta_m\,,\ \vec{v}_m$

Cosmological perturbations

- **Standard linear equations** (in GR)

$$\Psi = \Phi$$



$$\nabla^2 \Psi = 4\pi G a^2 \rho_m \delta_m$$

(relativistic Poisson)

$$\dot{\vec{v}}_m + H\,\vec{v}_m = -\vec{\nabla}\Phi$$

(Euler)

$$\dot{\delta}_m + \vec{\nabla} \cdot \vec{v}_m = 0$$

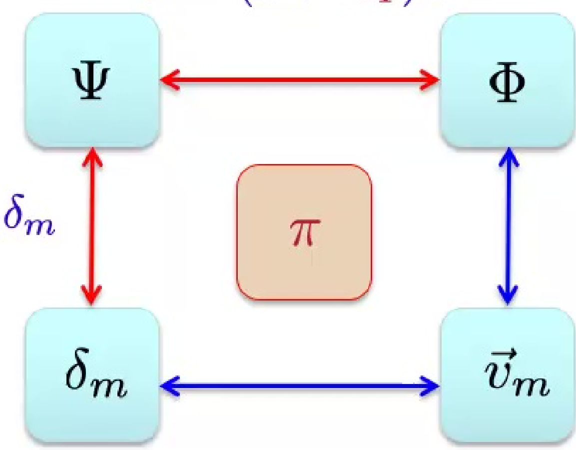
(continuity)

Cosmological perturbations

- **Modified equations (linear level)**

$$\Psi = (1 + \Upsilon_\Phi)\Phi$$

$$\nabla^2\Psi = 4\pi G(1 + \Upsilon_G)a^2\rho_m\delta_m$$



Quasi-static approximation
(valid on scales $kc_s \gg aH$)

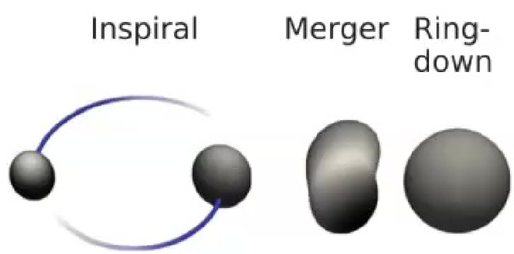
$$\Upsilon_G = \Upsilon_G(\alpha_i)\,, \quad \Upsilon_\Phi = \Upsilon_\Phi(\alpha_i)$$

which can be confronted to observations (galaxy clustering, weak lensing…)

Black hole perturbations

Black hole perturbations

- **GW astronomy** provides a new window to **test GR,** in particular in the strong field regime.



- **Ringdown phase** of a BH merger is interesting for models of modified gravity, as it can be described by the formalism of BH perturbations.

- **Deviations** in the context of **DHOST** theories ?

BH background solution

- Static spherically symmetric BH with a **nontrivial** scalar field

  – Metric: $ds^2 = -\mathcal{A}(r)dt^2 + \frac{dr^2}{\mathcal{B}(r)} + \mathcal{C}(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)$

  – Scalar field: $\phi(t,r) = q\,t + \psi(r)$ [Babichev & Charmousis '13]

  [$q \neq 0$ possible in shift-symmetric theories]

- **Examples:**

  – **« stealth » Schwarzschild:** $\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r}$

  – **« BCL »** [Babichev, Charmousis & Lehébel '17] $\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r} - \xi\frac{\mu^2}{2r^2}$

  – **« 4d Gauss-Bonnet »** [Glavan & Lin '19, Lu & Pang '20] $\mathcal{A} = \mathcal{B} = 1 - \frac{2\mu/r}{1 + \sqrt{1 + 4\alpha\mu/r^3}}$

  – **Several other solutions: cf Christos' talk**

Black hole perturbations

- In the frequency domain: $f(t,r) = f(r)\, e^{-i\omega t}$

- **Axial** (or odd) modes: $h_0(r), h_1(r)$ [Regge-Wheeler gauge]

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta} h_0^{\ell m} \partial_\varphi & -\sin\theta\, h_0^{\ell m} \partial_\theta \\ 0 & 0 & \frac{1}{\sin\theta} h_1^{\ell m} \partial_\varphi & -\sin\theta\, h_1^{\ell m} \partial_\theta \\ \mathrm{sym} & \mathrm{sym} & 0 & 0 \\ \mathrm{sym} & \mathrm{sym} & 0 & 0 \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

- **Polar** (or even) modes: H_0, H_1, H_2, K (and $\delta\phi$)

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} A(r) H_0^{\ell m}(r) & H_1^{\ell m}(r) & 0 & 0 \\ H_1^{\ell m}(r) & A^{-1}(r) H_2^{\ell m}(r) & 0 & 0 \\ 0 & 0 & K^{\ell m}(r)\, r^2 & 0 \\ 0 & 0 & 0 & K^{\ell m}(r)\, r^2 \sin^2\theta \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

Axial modes in GR

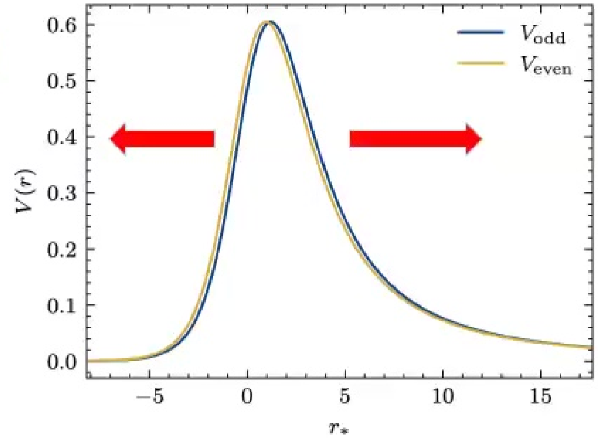
- The linearised metric eqs yield only 2 independent eqs

$$\frac{dY}{dr} = M(r)\,Y(r)\,, \quad Y = \begin{pmatrix} h_0 \\ h_1/\omega \end{pmatrix}$$

or, in a **Schroedinger** form, [Regge & Wheeler '57]

$$\frac{d^2\hat{Y}}{dr_*^2} + \left(\omega^2 - V(r)\right)\hat{Y} = 0$$

[r_* tortoise coordinate]



- **Asymptotically** ($r_* \to -\infty, +\infty$)

$$e^{-i\omega t}\,\hat{Y}(r) \approx a_+\,e^{-i\omega(t-r_*)} + a_-\,e^{-i\omega(t+r_*)}$$

outgoing ingoing

- **Quasi-normal modes**: $a_+^{\rm hor} = 0 \quad \text{and} \quad a_-^{\infty} = 0$

Axial modes in DHOST

- The equations have a similar structure: [DL, Noui & Roussille '21,'22]

$$\frac{dY}{dr} = MY\,, \quad M \equiv \begin{pmatrix} 2/r + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/r^2 \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix}$$

$$\lambda \equiv \frac{\ell(\ell+1)}{2} - 1$$

where Ψ, Φ, Γ and Δ depend on the Lagrangian's functions and on the background.

$$\mathcal{F} = \boxed{\mathcal{A}F_2} - (q^2 + \mathcal{A}X)A_1 - \frac{1}{2}\mathcal{A}\mathcal{B}\psi' X' F_{3X} - \frac{1}{2}\mathcal{B}\psi'(\mathcal{A}X)' B_2 - \frac{\mathcal{A}}{2\mathcal{B}}(\mathcal{B}\psi')^3 X' B_6\,,$$

$$\mathcal{F}\Psi = q\left[\psi' A_1 + \frac{1}{2}\left(\mathcal{B}\psi'^2\right)' F_{3X} + \frac{1}{2}\frac{(\mathcal{A}X)'}{\mathcal{A}}B_2 + \frac{1}{4}\left(\mathcal{B}^2\psi'^4\right)' B_6\right]\,,$$

$$\frac{\mathcal{F}}{\Phi} = \boxed{F_2} - XA_1 - \frac{1}{2}\mathcal{B}\psi' X' F_{3X} - \frac{1}{2}\mathcal{B}\psi'\frac{(\mathcal{C}X)'}{\mathcal{C}}B_2 - \frac{1}{2}\mathcal{B}\psi' X X' B_6\,,$$

$$\Gamma = \Psi^2 + \frac{1}{2\mathcal{A}\mathcal{B}\mathcal{F}}\left(2q^2 A_1 + \boxed{2\mathcal{A}F_2} + \mathcal{A}\mathcal{B}\psi' X' F_{3X} + q^2\frac{(\mathcal{A}X)'}{\mathcal{A}\psi'}B_2 + q^2\mathcal{B}\psi' X' B_6\right)\,,$$

$$\Delta = \boxed{-\frac{\mathcal{F}'}{\mathcal{F}} - \frac{\mathcal{B}'}{2\mathcal{B}} + \frac{\mathcal{A}'}{2\mathcal{A}}}$$

in GR

Axial modes in DHOST

- The equations have a similar structure: [DL, Noui & Roussille '21,'22]

$$\frac{dY}{dr} = MY\,, \quad M \equiv \begin{pmatrix} 2/r + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/r^2 \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix}$$

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where Ψ, Φ, Γ and Δ depend on the Lagrangian's functions and on the background.

- **Correspondence**

DHOST axial modes in $g_{\mu\nu}$	⟷	**GR** axial modes in $\tilde{g}_{\mu\nu}$

with the **effective metric**

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = |\mathcal{F}|\sqrt{\frac{\Gamma\mathcal{B}}{\mathcal{A}}}\left(-\Phi(dt - \Psi dr)^2 + \Gamma\Phi\, dr^2 + \mathcal{C}\, d\Omega^2\right)$$

Polar modes

- The linearised metric equations yield
  - 2 independent equations in GR (1 dof)
  - **4 independent equations** in DHOST theories (2 dof)

- In **GR**: 2-dimensional system $Y' = M\,Y$, which can be written in a Schroedinger form. [Zerilli '70]

- In **DHOST**, the system $Y' = M\,Y$ is now 4-dimensional, with

$$Y = {}^T(K\ \delta\phi\ H_1\ H_0)$$

- It is convenient to do an **asymptotic analysis** of the first-order system.

Asymptotics of a diffential system

DL, Noui & Roussille '21

- Instead of a Schroedinger-like approach, one can use directly the initial first-order equations of motion and their asymptotic limit:

$$\frac{dY}{dz} = M(z)\,Y\,, \quad M(z) = M_r z^r + M_{r-1} z^{r-1} + \dots \quad (z \to \infty)$$

- The generic solution is of the form

$$Y(z) = e^{\boldsymbol{\Upsilon}(z)}\, z^{\boldsymbol{\Delta}}\, \mathbf{F}(z)\, Y_0, \qquad (z \to \infty)$$

- There exists a well-defined algorithm to determine the diagonal matrices $\boldsymbol{\Upsilon}(z)$ and $\boldsymbol{\Delta}$. [Balser '99]

Idea: diagonalise, order by order, the matrix *M*, with $Y(z) = P(z)\,\tilde{Y}(z)$

$$\frac{d\tilde{Y}}{dz} = \tilde{M}(z)\,\tilde{Y}, \qquad \tilde{M}(z) \equiv P^{-1} M P - P^{-1}\frac{dP}{dz}$$

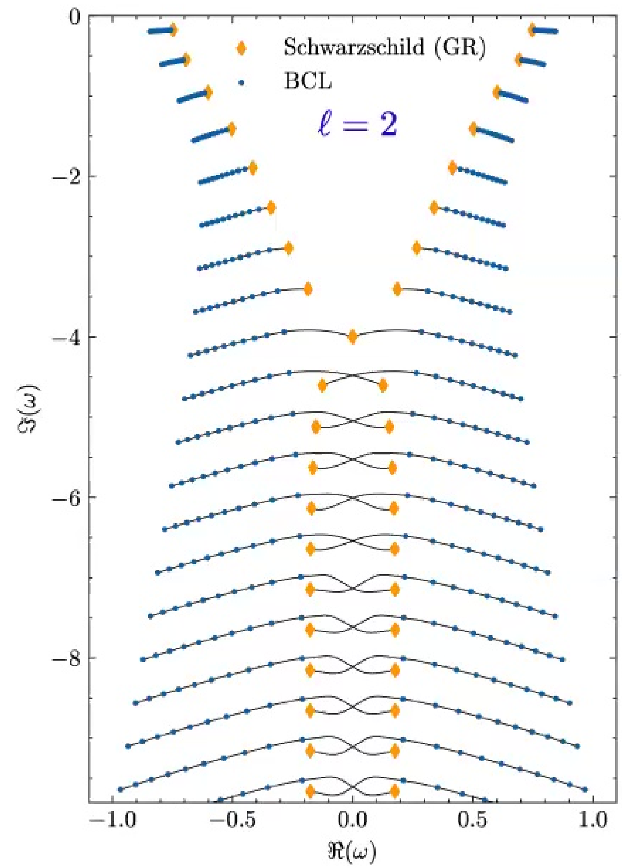
Numerical axial modes for BCL

Roussille, DL & Noui '23

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$A(r) = \left(1 - \frac{r_+}{r}\right)\left(1 + \frac{r_-}{r}\right)$$

- Axial QNM modes for $r_+ = 1$
- The parameter r_- varies between 0 (Schwarzschild) and 0.5.
- Migration of the modes in the complex plane



Conclusion

- **DHOST theories:** most general framework for scalar-tensor theories propagating a single scalar dof.

- In cosmology, DHOST theories provide a **very rich phenomenology**, which can **parametrise deviations** from General Relativity and be tested in future cosmological observations.

- **Deviations from GR** can also be **explored in** compact objects, in particular **black holes**. **QNMs modes** are a natural target to **detect or constrain** deviations from GR in the strong gravity regime.

Physical degrees of freedom

- **Scalar (** $\delta N\,,\quad N_i \equiv \partial_i \psi\,,\quad h_{ij} = a^2(t) e^{2\zeta} \delta_{ij}$ **)**

$$S_{\rm quad}[\zeta] \;=\; \int d^3x\,dt\,a^3\,\frac{M^2}{2}\left[\mathcal{A}_\zeta\,\dot\zeta^2 - \mathcal{B}_\zeta\,\frac{(\partial_i\zeta)^2}{a^2}\right]$$

with $\mathcal{A}_\zeta = \frac{1}{(1+\alpha_B-\dot\beta_1/H)^2}\left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3H^2M^2}\frac{d}{dt}\left(a^3 H\, M^2 \alpha_B \beta_1\right)\right]$

$$\mathcal{B}_\zeta = -2(1+\alpha_T) + \frac{2}{aM^2}\frac{d}{dt}\left[\frac{aM^2\,(1+\alpha_H+\beta_1(1+\alpha_T))}{H(1+\alpha_B)-\dot\beta_1}\right]$$

- **Tensor:** $S_\gamma^{(2)} = \frac{1}{2}\int dt\,d^3x\,a^3\left[\frac{M^2}{4}\dot\gamma_{ij}^2 - \frac{M^2}{4}(1+\alpha_T)\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]$

- **Stability** (neither ghost nor gradient instability)

$$M^2 > 0\,,\quad \mathcal{A}_\zeta > 0\,,\quad \mathcal{B}_\zeta > 0\,,\quad c_T^2 \equiv (1+\alpha_T) > 0$$

Illustrative example

- Simple toy model

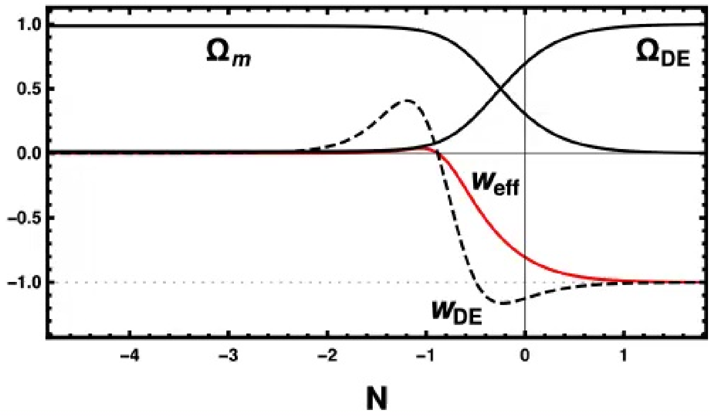
Boumaza, DL, Noui '20

$$F = \frac{1}{2}, \quad P = \alpha X, \quad Q = 0, \quad A_1 = -A_2 = -\beta X, \quad \lambda = \frac{1}{2} s \beta X^2$$

Choice of λ determines A_3 . Degeneracy imposes A_4 and A_5 .

- From matter era to dark energy era

$$3H^2 = \rho_m + \rho_{\rm DE} \qquad 2\dot{H} + 3H^2 = P_m + P_{\rm DE}$$



$$w_{\rm DE} \equiv \frac{P_{\rm DE}}{\rho_{\rm DE}}$$

$$w_{\rm eff} \equiv \frac{P_m + P_{\rm DE}}{\rho_m + \rho_{\rm DE}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}$$

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- From matter era to dark energy era

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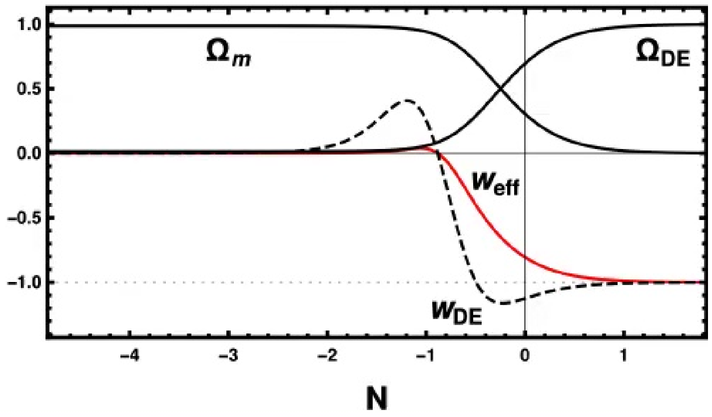

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