Title: How we rediscovered Horndeski gravity

Speakers: Cédric Deffayet

Collection/Series: 50 Years of Horndeski Gravity: Exploring Modified Gravity

Subject: Cosmology, Strong Gravity, Mathematical physics

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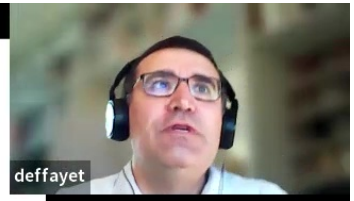
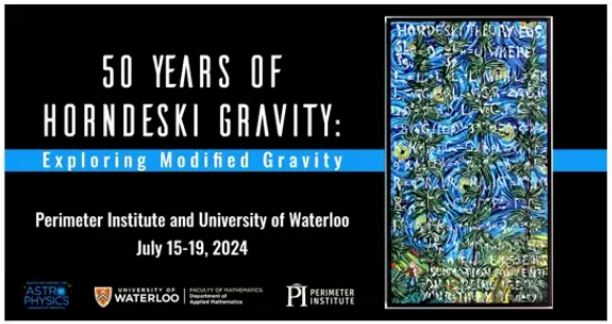
Abstract:

I will review the steps that led to the rediscovery and the wide use of Horndeski theories as well as stress some left over open questions and ideas.

How we rediscovered Horndeski gravity

Some history and (open) questions





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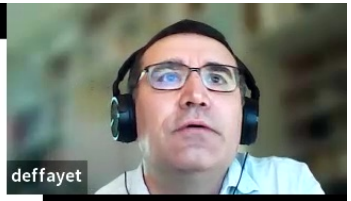


How we rediscovered Horndeski gravity

Some history and (open) questions

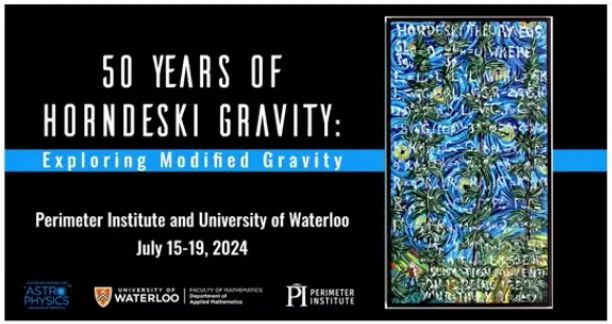
I. The starting point: the DGP model

II. Galileons and covariantized Galileons

III. From k-essence to generalized Galileons

VI. Relation to graph theories

V. P-forms



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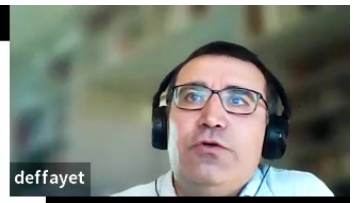
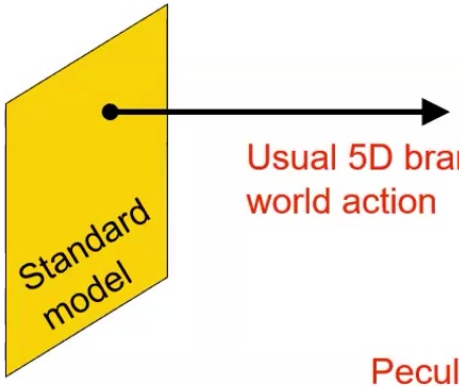
I. The starting point: the DGP model

Soon the 25th anniversary!



Dvali, Gabadadze, Porrati, 2000



Usual 5D brane world action

$$S = M_{(5)}^3 \int d^5x\sqrt{g}\left(\tilde{R} + \cdots\right) + \int_{\text{brane}} d^4x\sqrt{g}\mathcal{L}_{\text{matter}}$$

Peculiar to DGP model

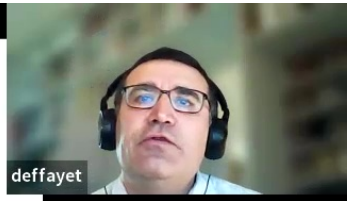
$$\underbrace{+M_P^2 \int_{\text{brane}} d^4x\sqrt{g}\left(R + \cdots\right)}_{\text{Brane localized kinetic term for the graviton}}$$

induced by quantum corrections
early name: « brane induced gravity »

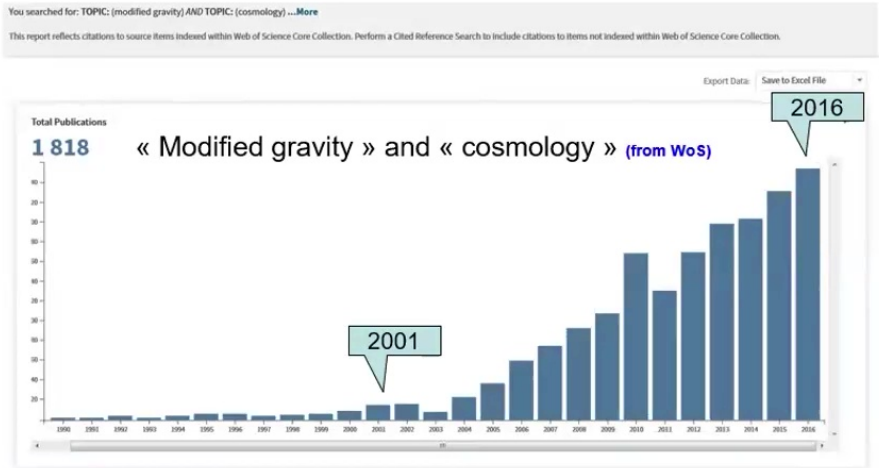
A special hierarchy between $M_{(5)}$ and M_P is required to make the model phenomenologically interesting

Leads to the e.o.m.

$$G_{AB}^{(5)} = \delta\left(\text{brane}\right)\frac{1}{M_{(5)}^3}\left[\, G_{\mu\nu} - \frac{1}{M_P^2}T_{\mu\nu}\,\right]$$

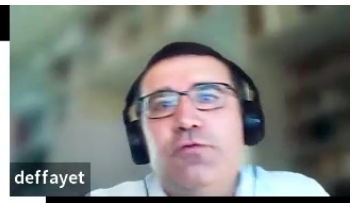


A new way to modify gravity at large distance, with a new type of phenomenology … The **first framework where cosmic acceleration was proposed (and showed) to be (possibly) linked to a large distance modification of gravity** (CD 2001)



Rebirth of massive gravity, Vainshtein mechanism …

Lead to many subsequent developments including, Galileons, the rediscovery and wide use of Horndeski gravity, generalized Vainshtein screening and k-mouflage, dRGT theories ….



Homogeneous cosmology of DGP model

One obtains the following modified Friedmann equations (CD 2001)

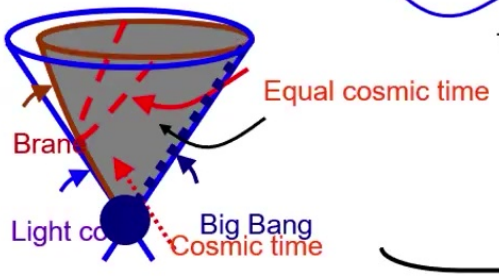
Two branches of solutions

Energy density of brane localized matter

$$\sqrt{H^2 + \frac{k}{a^2}} = \frac{\epsilon}{2r_c} + \sqrt{\frac{\rho_{(M)}}{3M_P^2} + \frac{1}{4r_c^2}}$$

$$\text{with } \epsilon = \pm 1$$

$$\dot{\rho} = -3H\left(P + \rho\right)$$

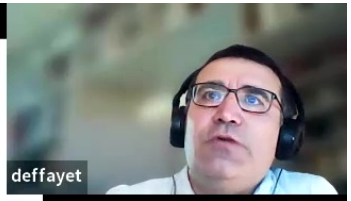


- Analogous to standard (4D) Friedmann equations

$$H^2 + \frac{k}{a^2} = \frac{\rho_{(M)}}{3M_P^2}$$

In the early Universe (small Hubble radii $H^{-1} \ll r_c$)

- Deviations at late time (self-acceleration)



In the DGP model :

- Newtonian potential on the brane behaves as

$$V(r) \propto \frac{1}{r}$$ ⟵ 4D behavior at small distances

$$V(r) \propto \frac{1}{r^2}$$ ⟵ 5D behavior at large distances

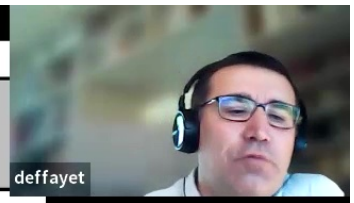
- The crossover distance between the two regimes is given by

$$r_c = \frac{M_P^2}{2M_{(5)}^3}$$ ⇨ This enabled to get a "4D looking" theory of gravity out of one which is not, without having to assume a compact (Kaluza-Klein 1920-1930) or "curved" (Randall-Sundrum 1999) bulk space-time.

- But the tensorial structure of the graviton propagator is that of a **massive graviton** (gravity is mediated by a continuum of massive modes)

Leads to the « van Dam-Veltman-Zakharov (vDVZ) discontinuity » (a feature of linear massive gravity)

The vDVZ discontinuity was discovered in the Fierz-Pauli theory (linear theory of a massive graviton on flat space-time)



In the generic case (with 5 DOF) the scalar polarization of the Fierz-Pauli graviton is an obstacle to real world applications.

Wrong light bending

Stays true sending the mass of the graviton smoothly to zero (van Dam, Veltman, Zakharov, Iwasaki 1970)

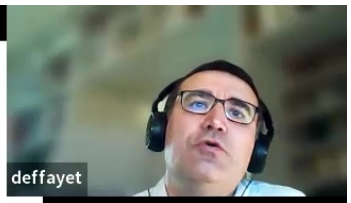
A way out relying on non linearities was suggested by Arkady Vainshtein in 1972 (« the Vainshtein mechanism »)

Vainshtein's suggestion was criticized by Boulware and Deser (BD) in 1972

- BD pointed out that one needed to find an everywhere existing non singular solution of in amassive gravity theory featuring the behaviour conjectured by Vainshtein
- It was only much later that such solutions were found (CD, Dvali, Gabadadze, Vainshtein 2002, Babichev, CD, Ziour, 2009-2010)

In the same paper BD also pointed out another generic (they thought) pathology of non linear massive gravity: the « BD ghost », a 6th DOF propagating at non linear level with negative energies

- This was overcome in 2010-2011 by de Rham, Gababadze, Tolley (« dRGT theory »)



Van Dam-Veltman-Zakharov discontinuity in the DGP model…

Exact cosmological solutions provided an explicit example of interpolation between theories with different tensor structure for the graviton propagator.

CD,Gabadadze, Dvali, Vainshtein (2002)

$$H^2 = \frac{\rho}{3M_P^2} \xleftrightarrow[\text{small } r_c]{\text{large } r_c} H^2 = \frac{\rho^2}{36M_{(5)}^6}$$

Solution of 4D GR with cosmic fluid

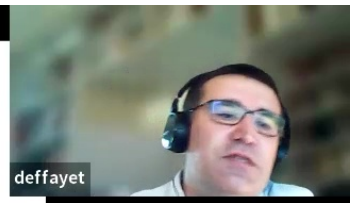
$$S_0 = M_P^2 \int d^4x\sqrt{-g}R$$

Solution of 5D GR with a brane source

$$S_5 = M_{(5)}^3 \int d^5x\sqrt{-g}R$$

Came as the first support ever of a « Vainshtein mechanism (at work in DGP…)

(NB: soon after Damour, Kogan and Papazoglou (2003) concluded it does not work (in some version of non linear massive gravity))



The non linearities associated to the possible Vainshtein curing of the vDVZ discontinuity were shown to exist in the DGP model in CD,Gabadadze, Dvali, Vainshtein (2002)

These non linearities and some crucial phenomenological properties of the DGP model were captured taking a « decoupling limit » in
Luty, Porrati, Rattazzi (2003)
(with an almost simultaneous paper by Rubakov)

This decoupling limit marks the first step of the rebirth of Horndeski gravity

DGP decoupling limit

Luty, Porrati, Rattazzi, 2003

A good description of many DGP key properties is given by the action

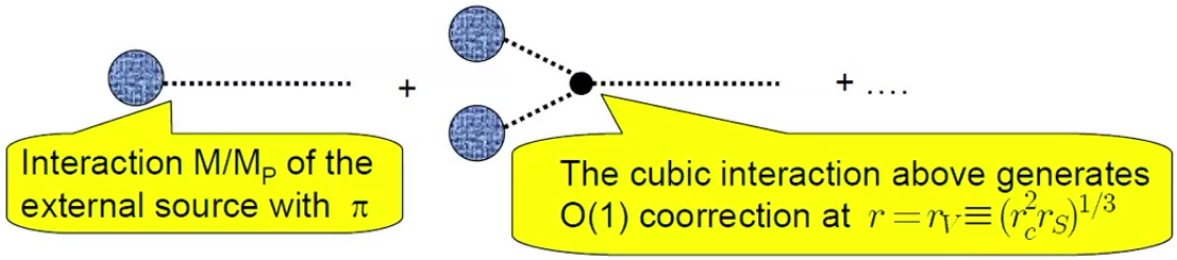
$$3\pi\Box\pi - \frac{1}{\Lambda^3}(\partial_\mu\pi\partial^\mu\pi)\Box\pi - \frac{2}{M_P}\pi T$$

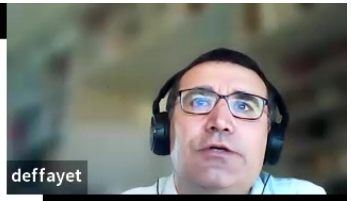
Scalar sector of the model

Energy scale $\Lambda = \left(r_c^2/M_P\right)^{1/3}$

Yielding the e.o.m. $\Box\pi - \frac{1}{3\Lambda^3}\left[\left(\Box\pi\right)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right] = \frac{T}{3M_P}$

- Leads in vacuum to two branches of solutions, $\pi \sim 0$ and $\Box\pi \sim \Lambda^3$ representing the two branches of solutions of the original model…
- Around massive body, the cubic self interaction of π becomes of the same order as the quadratic term at the « Vainshtein radius »





The action $$3\pi\Box\pi - \frac{1}{\Lambda^3}(\partial_\mu\pi\partial^\mu\pi)\Box\pi - \frac{2}{M_P}\pi T$$

Is obtained taking the « decoupling limit »

Luty, Porrati, Rattazzi;
Nicolis, Rattazzi

$$\begin{cases} M_P \to \infty \\ M_{(5)} \to \infty \\ \Lambda \ \text{fixed} \\ T/M_P \ \text{fixed} \end{cases}$$

It can be obtained from the (5D) « Hamiltonian » constraint

$$R = K^2 - K_{\mu\nu}^2$$

Where one substitutes the Israel junction condition

$$K = \frac{1}{6M_{(5)}^3}\left(T + M_P^2 R\right)$$

To obtain $$\frac{3}{r_c}K - K^2 + K_{\mu\nu}^2 = \frac{T}{M_P^2}$$

A last substitution $$K_{\mu\nu} = \frac{r_c}{M_P}\partial_\mu\partial_\nu\pi$$

Yields the e.o.m. for π deduced from above action :

$$\Box\pi - \frac{1}{3\Lambda^3}\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right] = \frac{T}{3M_P}$$

Expanding around the vacuum solutions of

$$\Box\pi - \frac{1}{3\Lambda^3}\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right] = \frac{T}{3M_P}$$

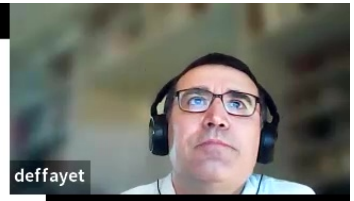
$\pi \sim 0$ Positive energy fluctuations

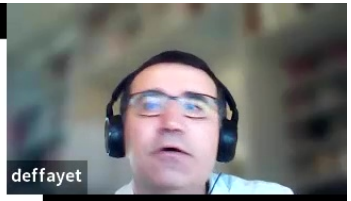
$\Box\pi \sim \Lambda^3$ Negative energy fluctuations

i.e. widely discussed ghost of DGP self accelerating branch

Luty, Porrati, Rattazzi, 2003; Koyama, 2005; Gorbunov, Koyama, Sibiryakov, 2006; Charmousis, Gregory, Kaloper, Padilla, 2006. CD, Gabadadze, Iglesias, 2006…

Note however that the background solution is at the scale Λ, believed (most of the time) to be the UV cutoff (Luty, Porrati, Rattazzi; Dvali; CD) . One should keep this in mind (often forgotten when addressing the DGP ghost and stability)!





The action $$3\pi\Box\pi - \frac{1}{\Lambda^3}(\partial_\mu\pi\partial^\mu\pi)\Box\pi - \frac{2}{M_P}\pi T$$

Yields second order EOM

The first « **Galileon** » (belonging to the Horndeski theories)

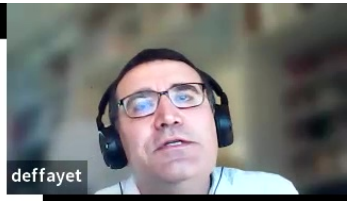
In contrast, generic massive gravities have « decoupling limits » yielding actions such as (Arkani-Hamed, Georgi, Schwartz, 2003) …

$$\frac{1}{2}(\nabla\tilde{\phi})^2 - \frac{1}{M_P}\tilde{\phi}T + \frac{1}{\Lambda^5}\left\{(\nabla^2\tilde{\phi})^3 + \quad \dots \quad\right\}$$

… yielding higher order equations of motion (and different strong coupling scale $\Lambda = (m^4 M_P)^{1/5}$)

These contrasting properties can be related to the absence/presence of the BD ghost (CD, Rombouts, 2005)

This last remark played an instrumental role in the construction of the BG ghost free dRGT massive gravity (de Rahm, Gabadadze; de Rham, Gababadze, Tolley 2010, 2011)



II. Galileons

II. 1 Flat space-time Galileon in 4 D

- Galileon were originally (Nicolis, Rattazzi, Trincherini 2009) defined in flat space-time as the most general scalar theory which has **(strictly) second order fields equations**

- In 4D, there is only 4 non trivial such theories

(with $\pi_\mu = \partial_\mu \pi \quad \pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$)

$$\mathcal{L}_{(2,0)} = \pi_\mu \pi^\mu$$

$$\mathcal{L}_{(3,0)} = \pi^\mu \pi_\mu \Box \pi$$

$$\begin{aligned}\mathcal{L}_{(4,0)} = {} & (\Box\pi)^2 (\pi_\mu \pi^\mu) - 2 (\Box\pi) (\pi_\mu \pi^{\mu\nu} \pi_\nu) \\ & - (\pi_{\mu\nu} \pi^{\mu\nu}) (\pi_\rho \pi^\rho) + 2 (\pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^\rho)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{(5,0)} = {} & (\Box\pi)^3 (\pi_\mu \pi^\mu) - 3 (\Box\pi)^2 (\pi_\mu \pi^{\mu\nu} \pi_\nu) - 3 (\Box\pi) (\pi_{\mu\nu} \pi^{\mu\nu}) (\pi_\rho \pi^\rho) \\ & +6 (\Box\pi) (\pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^\rho) + 2 \left(\pi_\mu{}^\nu \pi_\nu{}^\rho \pi_\rho{}^\mu\right) (\pi_\lambda \pi^\lambda) \\ & +3 (\pi_{\mu\nu} \pi^{\mu\nu}) \left(\pi_\rho \pi^{\rho\lambda} \pi_\lambda\right) - 6 \left(\pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^{\rho\lambda} \pi_\lambda\right)\end{aligned}$$

Simple rewriting of those Lagrangians with epsilon tensors
(up to integrations by part):
(C.D., S.Deser, G.Esposito-Farese, 2009)

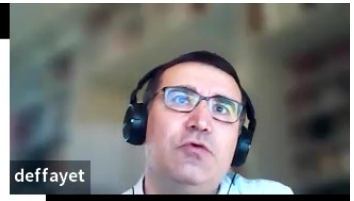
$$\mathcal{L}_{(2,0)} = \epsilon^{\mu_1\lambda_1\lambda_2\lambda_3}\epsilon^{\nu_1}{}_{\lambda_1\lambda_2\lambda_3}\pi_{\mu_1}\pi_{\nu_1}$$

$$\mathcal{L}_{(3,0)} = \epsilon^{\mu_1\mu_2\lambda_1\lambda_2}\epsilon^{\nu_1\nu_2}{}_{\lambda_1\lambda_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}$$

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

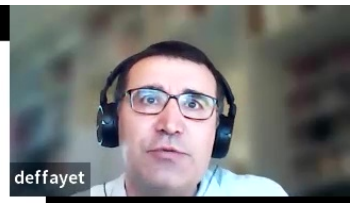
$$\mathcal{L}_{(5,0)} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\pi_{\mu_4\nu_4}$$

This leads to (exactly) second order field equations



Indeed, consider e.g.

$$\begin{aligned}\mathcal{L}_{(4,0)} &= \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} \\ &\propto (\Box\pi)^2(\pi_\mu\pi^\mu) - 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu) \\ &\quad - (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) + 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho)\end{aligned}$$



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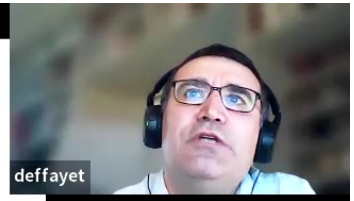
$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Varying this Lagrangian with respect to π yields (after integrating by part)

$$\delta\mathcal{L}_{(4,0)} \supset -\epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\partial_{\mu_1}\left\{\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right\}$$

Second order derivative



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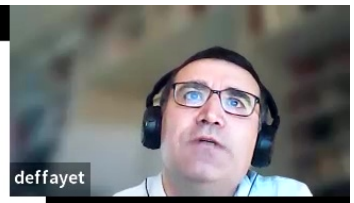
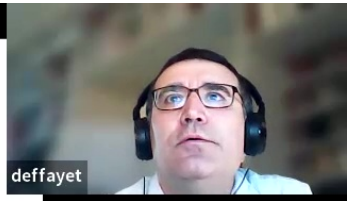
Third order derivative…

… killed by the contraction with epsilon tensor





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Similarly, one also have in the field equations

$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\partial_{\nu_2}\partial_{\mu_2}\left\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\right\}$$

Yields third and fourth
order derivative…
killed by the epsilon tensor

Indeed, consider e.g.

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Varying this Lagrangian with respect to π yields (after integrating by part)

$$\delta\mathcal{L}_{(4,0)} \supset -\epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\partial_{\mu_1}\left\{\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right\}$$

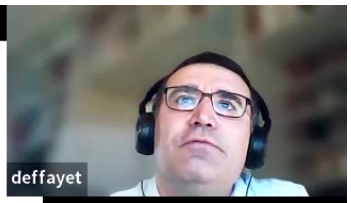
Similarly, one also have in the field equations

$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\partial_{\nu_2}\partial_{\mu_2}\left\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\right\}$$

Hence the field equations are proportional to

$$\mathcal{E}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Which does only contain second derivatives



The field equations

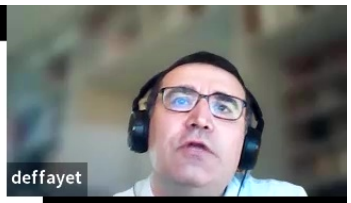
$$\mathcal{E}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} = 0$$

containing only second derivatives, have the « Galilean » symmetry

$$\pi \longrightarrow \pi + C + D_\mu x^\mu$$

They also can be written using 4 (and lower) dimensional determinants and are in fact **generalized Monge Ampere equation** of the form *det* (π_{ij}) = 0 (Monge Ampere equation have various interesting properties - Fairlie)

II. 2 Flat space-time Galileon in arbitrary Dimension



In D dimensions, D non trivial Galileons can be defined as

$$\mathcal{L}_{(n+1,0)} = \sum_{\sigma\in S_n} \epsilon(\sigma) g^{\mu_{\sigma(1)}\nu_1} g^{\mu_{\sigma(2)}\nu_2} \dots g^{\mu_{\sigma(n)}\nu_n} (\pi_{\nu_1}\pi_{\mu_1})(\pi_{\nu_2\mu_2}\pi_{\nu_3\mu_3}\dots\pi_{\nu_n\mu_n}).$$

Only the Lagrangians with $D \geq n$ are non vanishing.

Using the tensors

$$\mathcal{A}^{\mu_1\mu_2\dots\mu_{2n}}_{(2n)} \equiv \frac{1}{(D-n)!} \varepsilon^{\mu_1\mu_3\mu_5\dots\mu_{2n-1}\,\nu_1\nu_2\dots\nu_{D-n}} \varepsilon^{\mu_2\mu_4\mu_6\dots\mu_{2n}}{}_{\nu_1\nu_2\dots\nu_{D-n}}$$

$\mathcal{A}_{(2n)}$ is antisymmetric (separately) in odd and even indices

Or to alleviate notations

$$\mathcal{A}^{1234\dots}_{(2n)} = \frac{1}{(D-n)!} \varepsilon^{135\dots\,\nu_1\nu_2\dots\nu_{D-n}} \varepsilon^{246\dots}{}_{\nu_1\nu_2\dots\nu_{D-n}}$$

One has

$$\mathcal{L}_{(n+1,0)} = -\mathcal{A}_{(2n)}(\pi_1\pi_2)(\pi_{34}\pi_{56}\pi_{78}\dots\pi_{\mu_{2n-1}\mu_{2n}})$$

All free indices are contracted with those of $\mathcal{A}_{(2n)}$

Up to total derivatives, the following Lagrangians are equivalent (where N=n+2 is the number of field π)

$$\mathcal{L}_N^{\mathrm{Gal},1} = \left(\mathcal{A}_{(2n+2)}^{\mu_1\dots\mu_{n+1}\nu_1\dots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}}\right)\pi_{\mu_1\nu_1}\dots\pi_{\mu_n\nu_n}$$

$$\mathcal{L}_N^{\mathrm{Gal},2} = \left(\mathcal{A}_{(2n)}^{\mu_1\dots\mu_n\nu_1\dots\nu_n}\pi_{\mu_1}\pi_\lambda\pi^\lambda{}_{\nu_1}\right)\pi_{\mu_2\nu_2}\dots\pi_{\mu_n\nu_n}$$

$$\mathcal{L}_N^{\mathrm{Gal},3} = \left(\mathcal{A}_{(2n)}^{\mu_1\dots\mu_n\nu_1\dots\nu_n}\pi_\lambda\pi^\lambda\right)\pi_{\mu_1\nu_1}\dots\pi_{\mu_n\nu_n}$$

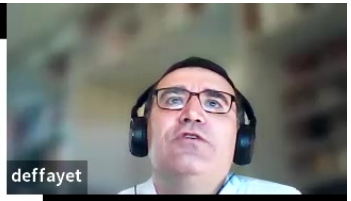
One has the exact relation

$$(N-2)\mathcal{L}_N^{\mathrm{Gal},2} = \mathcal{L}_N^{\mathrm{Gal},3} - \mathcal{L}_N^{\mathrm{Gal},1}$$

One special case: the cubic Galileon (DGP decoupling limit) :

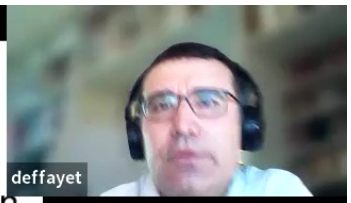
$$\mathcal{L}_{(3,0)} = \Box\pi\partial_\mu\pi\partial^\mu\pi$$

$$\Leftrightarrow \quad -2\ \partial_\mu\pi\partial^\mu\partial^\nu\pi\partial_\nu\pi$$



II. 3 Curved space-time Galileon

CD, Esposito-Farese, Vikman, 2009
CD, Deser, Esposito-Farese, 2009



A naive covariantization leads to the loss of the distinctive properties of the Galileon

Indeed, consider now in curved space-time (with $\pi_\mu = \nabla_\mu \pi$ and $\pi_{\mu\,\nu} = \nabla_\mu \nabla_\nu \pi$)

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Variation yields in particular

$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\nabla_{\nu_2}\nabla_{\mu_2}\left\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\right\}$$

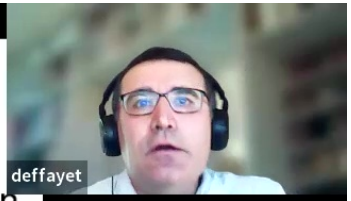
Third derivatives generate
now Riemann tensors …
and fourth derivatives,
derivatives of the Riemann

Indeed the (naively covariantized) $\mathcal{L}_{(4,0)}$ has the field equations

$$\begin{aligned}\mathcal{E}_{(4,0)} = {} & 2\left(\pi_\mu\pi^\mu\right)\left(\pi_\nu{}^\nu{}_\rho{}^\rho - \pi_{\nu\rho}{}^{\nu\rho}\right) + 2\,\pi^\mu\pi^\nu\left(2\,\pi_{\mu\rho\nu}{}^\rho - \pi_{\mu\nu\rho}{}^\rho - \pi_\rho{}^\rho{}_{\mu\nu}\right)\\ & +10\left(\Box\pi\right)\pi^\mu\left(\pi_{\mu\nu}{}^\nu - \pi_\nu{}^\nu{}_\mu\right) + 12\,\pi_\mu\,\pi^{\mu\nu}\left(\pi_\rho{}^\rho{}_\nu - \pi_{\nu\rho}{}^\rho\right)\\ & +8\,\pi^\mu\,\pi^{\nu\rho}\left(\pi_{\nu\rho\mu} - \pi_{\mu\nu\rho}\right) - 4\left(\Box\pi\right)^3 - 8\left(\pi_\mu{}^\nu\,\pi_\nu{}^\rho\,\pi_\rho{}^\mu\right) + 12\left(\Box\pi\right)\left(\pi_{\mu\nu}\,\pi^{\mu\nu}\right).\end{aligned}$$

II. 3 Curved space-time Galileon

CD, Esposito-Farese, Vikman, 2009
CD, Deser, Esposito-Farese, 2009



A naive covariantization leads to the loss of the distinctive properties of the Galileon

Indeed, consider now in curved space-time (with $\pi_\mu = \nabla_\mu \pi$ and $\pi_{\mu\,\nu} = \nabla_\mu \nabla_\nu \pi$)

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Variation yields in particular

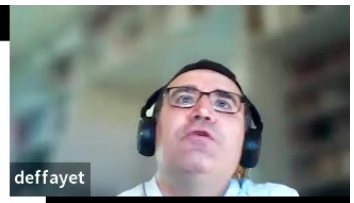
$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\nabla_{\nu_2}\nabla_{\mu_2}\left\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\right\}$$

Third derivatives generate now Riemann tensors …
and fourth derivatives, derivatives of the Riemann

Indeed the (naively covariantized) $\mathcal{L}_{(4,0)}$ has the field equations

$$\begin{aligned}\mathcal{E}_{(4,0)} = & -4\left(\Box\pi\right)^3 - 8\left(\pi_\mu{}^\nu\,\pi_\nu{}^\rho\,\pi_\rho{}^\mu\right) + 12\left(\Box\pi\right)\left(\pi_{\mu\nu}\pi^{\mu\nu}\right) - \left(\pi_\mu\,\pi^\mu\right)\left(\pi_\nu R^{;\nu}\right) \\ & +2\left(\pi_\mu\,\pi_\nu\,\pi_\rho R^{\mu\nu;\rho}\right) + 10\left(\Box\pi\right)\left(\pi_\mu R^{\mu\nu}\pi_\nu\right) - 8\left(\pi_\mu\,\pi^{\mu\nu} R_{\nu\rho}\pi^\rho\right) \\ & -2\left(\pi_\mu\,\pi^\mu\right)\left(\pi_\nu R^{\nu\rho}\right) - 8\left(\pi_\mu\,\pi_\nu\,\pi_{\rho\sigma} R^{\mu\rho\nu\sigma}\right).\end{aligned}$$

Kinetic mixing



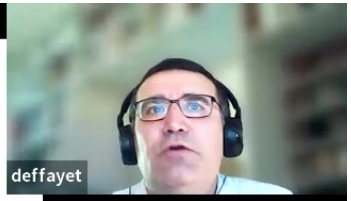
Similarly, varying w.r.t. the metric

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\,\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\,\pi_{\mu_1}\,\pi_{\nu_1}\,\pi_{\mu_2\nu_2}\,\pi_{\mu_3\nu_3}$$

Yields third order derivatives of the scalar π in the energy momentum tensor

Do not vanish in flat space-time !

$$\begin{aligned}
\mathcal{T}^{\mu\nu}_{(4,0)} = {} & (\pi^\mu\,\pi^\nu)\,\pi^\lambda\,\boxed{\left(2\,\pi_{\lambda\rho}{}^{\rho} - \pi^{\rho}{}_{\rho\lambda}\right)} \\
& + \left(\pi_\lambda\,\pi^\lambda\right)\pi^\mu\,\boxed{\left(\pi_\rho{}^{\rho\nu} - \pi^{\nu\rho}{}_{\rho}\right)} + \left(\pi_\lambda\,\pi^\lambda\right)\pi^\nu\,\boxed{\left(\pi_\rho{}^{\rho\mu} - \pi^{\mu\rho}{}_{\rho}\right)} \\
& - \pi^\lambda\,\pi^\rho\,\boxed{\left(\pi^\mu\,\pi_{\lambda\rho}{}^{\nu} + \pi^\nu\,\pi_{\lambda\rho}{}^{\mu}\right)} + \left(\pi_\lambda\,\pi^\lambda\right)\left(\pi_\rho\,\boxed{\pi^{\mu\nu\rho}}\right) \\
& + \left(\pi_\lambda\,\pi_\rho\,\pi_\sigma\,\boxed{\pi^{\lambda\rho\sigma}}\right)g^{\mu\nu} - \left(\pi_\lambda\,\pi^\lambda\right)\left(\pi_\rho\,\boxed{\pi_\sigma{}^{\sigma\rho}}\right)g^{\mu\nu} \\
& + (\pi^\mu\,\pi^\nu)\left[3\left(\pi_{\lambda\rho}\,\pi^{\lambda\rho}\right) - 2\,(\Box\pi)^2\right] + (\pi^{\mu\nu})\,\pi_\lambda\left(2\,\pi^{\lambda\rho}\,\pi_\rho + \pi^\lambda\,\Box\pi\right) \\
& + 3\,(\Box\pi)\,\pi_\lambda\left(\pi^{\lambda\mu}\,\pi^\nu + \pi^{\lambda\nu}\,\pi^\mu\right) - 4\,\pi_\lambda\,\pi^{\lambda\rho}\left(\pi_\rho{}^{\mu}\,\pi^\nu + \pi_\rho{}^{\nu}\,\pi^\mu\right) \\
& - 2\left(\pi_\lambda\,\pi^{\lambda\mu}\right)\left(\pi_\rho\,\pi^{\rho\nu}\right) - \frac{1}{2}\left(\pi_\lambda\,\pi^\lambda\right)\left[(\Box\pi)^2 + \left(\pi_{\rho\sigma}\,\pi^{\rho\sigma}\right)\right]g^{\mu\nu} \\
& + \pi_\lambda\,\pi_\rho\left[3\pi^{\lambda\sigma}\,\pi_\sigma{}^{\rho} - 2\,(\Box\pi)\,\pi^{\lambda\rho}\right]g^{\mu\nu}.
\end{aligned}$$



This can be cured by a non minimal coupling to the metric

Adding to

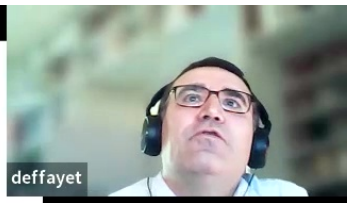
$$\mathcal{L}_{(4,0)} = (\Box\pi)^2 (\pi_\mu \pi^\mu) - 2 (\Box\pi) (\pi_\mu \pi^{\mu\nu} \pi_\nu) - (\pi_{\mu\nu} \pi^{\mu\nu}) (\pi_\rho \pi^\rho) + 2 (\pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^\rho)$$

The Lagrangian

$$\mathcal{L}_{(4,1)} = (\pi_\lambda \pi^\lambda) \pi_\mu \Big[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \Big] \pi_\nu$$

Yields second order field equations for the scalar and the metric
(but loss of the « Galilean » symmetry)
e.g. one has now the energy momentum tensor

$$\begin{aligned}
\mathcal{T}_4^{\mu\nu} = {} & 4 (\Box\pi) \pi_\rho \big[\pi^\mu \pi^{\rho\nu} + \pi^\nu \pi^{\rho\mu} \big] - 2 (\Box\pi)^2 (\pi^\mu \pi^\nu) + 2 (\Box\pi) \big(\pi_\lambda \pi^\lambda \big) (\pi^{\mu\nu}) \\
& + 4 \big(\pi_\lambda \pi^{\lambda\rho} \pi_\rho \big) (\pi^{\mu\nu}) - 4 \big(\pi_\lambda \pi^{\lambda\mu} \big) (\pi_\rho \pi^{\rho\nu}) + 2 \big(\pi_{\lambda\rho} \pi^{\lambda\rho} \big) (\pi^\mu \pi^\nu) \\
& - 2 \big(\pi_\lambda \pi^\lambda \big) \big(\pi^\mu{}_\rho \pi^{\rho\nu} \big) - 4 \pi^\lambda \pi_{\lambda\rho} \big[\pi^{\rho\mu} \pi^\nu + \pi^{\rho\nu} \pi^\mu \big] - (\Box\pi)^2 \big(\pi_\lambda \pi^\lambda \big) g^{\mu\nu} \\
& - 4 (\Box\pi) \big(\pi_\lambda \pi^{\lambda\rho} \pi_\rho \big) g^{\mu\nu} + 4 \big(\pi_\lambda \pi^{\lambda\rho} \pi_{\rho\sigma} \pi^\sigma \big) g^{\mu\nu} \\
& + \big(\pi_\lambda \pi^\lambda \big) (\pi_{\rho\sigma} \pi^{\rho\sigma}) g^{\mu\nu} + \big(\pi_\lambda \pi^\lambda \big) (\pi^\mu \pi^\nu) R - \frac{1}{4} \big(\pi_\lambda \pi^\lambda \big) (\pi_\rho \pi^\rho) g^{\mu\nu} R \\
& - 2 \big(\pi_\lambda \pi^\lambda \big) \pi_\rho \big[R^{\rho\mu} \pi^\nu + R^{\rho\nu} \pi^\mu \big] + \frac{1}{2} \big(\pi_\lambda \pi^\lambda \big) (\pi_\rho \pi^\rho) R^{\mu\nu} \\
& + 2 \big(\pi_\lambda \pi^\lambda \big) (\pi_\rho R^{\rho\sigma} \pi_\sigma) g^{\mu\nu} - 2 \big(\pi_\lambda \pi^\lambda \big) (\pi_\rho \pi_\sigma R^{\mu\rho\nu\sigma}) ,
\end{aligned}$$



This can be generalized to arbitrary Galileons
(arbitrary number of fields and dimensions)

Introducing $\mathcal{L}_{(n+1,p)} = -\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}$

With $$\mathcal{R}_{(p)} \equiv \left(\pi_\lambda\pi^\lambda\right)^p \prod_{i=1}^{i=p} R_{\mu_{4i-1}\,\mu_{4i+1}\,\mu_{4i}\,\mu_{4i+2}},$$

$$\mathcal{S}_{(q)} \equiv \prod_{i=0}^{i=q-1} \pi_{\mu_{2n-1-2i}\,\mu_{2n-2i}},$$

The action

$$I = \int d^D x\sqrt{-g}\sum_{p=0}^{p_{\max}} \mathcal{C}_{(n+1,p)}\mathcal{L}_{(n+1,p)}$$

with

$$\mathcal{C}_{(n+1,p)} = \left(-\frac{1}{8}\right)^p \frac{(n-1)!}{(n-1-2p)!\,(p!)^2} = \left(-\frac{1}{8}\right)^p \binom{n-1}{2p}\binom{2p}{p}$$

Yields second order field equations.

Heuristically, one needs to replace sucessively pairs of twice differentiated π by Riemanns

This can be understood as follows. Considers e.g.

$$\mathcal{L}_{(5,0)} = -\varepsilon^{\mu_1\mu_3\mu_5\mu_7}\,\varepsilon^{\mu_2\mu_4\mu_6\mu_8}\,\pi_{\mu_1}\pi_{\mu_2}\,\pi_{\mu_3\mu_4}\pi_{\mu_5\mu_6}\pi_{\mu_7\mu_8} = -\mathcal{A}_{(8)}\,\pi_1\pi_2\,\pi_{34}\pi_{56}\pi_{78}.$$

⇩ Vary w.r.t. π

$$\delta_\pi\mathcal{L}_{(5,0)} = -2\mathcal{A}_{(8)}\delta\pi_1\pi_2\pi_{34}\pi_{56}\pi_{78} - 3\mathcal{A}_{(8)}\pi_1\pi_2\delta\pi_{34}\pi_{56}\pi_{78}$$

⇩ Only « dangerous » term (i.e. term leading to higher derivatives)

$$\delta_\pi\mathcal{L}_{(5,0)} \sim -3\mathcal{A}_{(8)}\pi_1\pi_2\delta\pi_{34}\pi_{56}\pi_{78}.$$

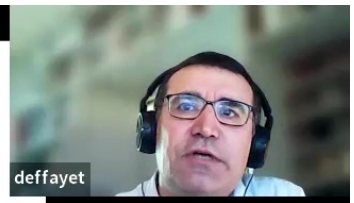
⇩ Integrating by part

$$\delta_\pi\mathcal{L}_{(5,0)} \sim -3\times 2\,\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi_{5643}\pi_{78}$$

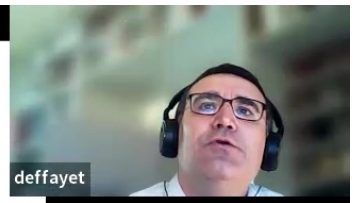
⇩ Using the antisymmetry of $\mathcal{A}_{(8)}$ and the Riemann Bianchi identity

$$\begin{aligned}\delta_\pi\mathcal{L}_{(5,0)} &\sim -3\,\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\left(\pi_{5643}-\pi_{5463}\right)\pi_{78}\\ &\sim -3\,\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda R_{465\lambda;3}\pi_{78}\\ &\sim -\frac{3}{2}\,\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda\left(R_{465\lambda;3}+R_{46\lambda3;5}\right)\pi_{78}\\ &\sim \frac{3}{2}\,\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda R_{3546;\lambda}\pi_{78},\end{aligned}$$

Can be cancelled by varying $\mathcal{L}_{(5,1)} = \frac{3}{4}\mathcal{A}_{(8)}\pi_1\pi_2\left(\pi_\lambda\pi^\lambda\right)R_{3546}\,\pi_{78}.$



Note that the extra term $\mathcal{L}_{(5,1)} = \frac{3}{4}\mathcal{A}_{(8)}\pi_1\pi_2\left(\pi_\lambda\pi^\lambda\right)R_{3546}\,\pi_{78}.$



Does not generate unwanted derivatives of the curvature thanks to Bianchi identity

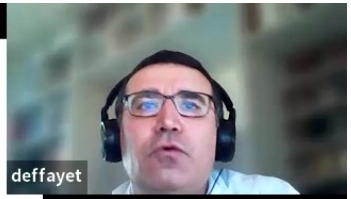
Indeed one has
$$\begin{cases} \mathcal{A}_{(8)}R_{3546;7} = 0 \\ \mathcal{A}_{(8)}R_{3546;8} = 0 \end{cases}$$

Similarly to
$$\begin{cases} \mathcal{A}_{(8)}\partial_7\pi_{35} = 0 \\ \mathcal{A}_{(8)}\partial_8\pi_{46} = 0 \end{cases}$$

Yields an easy generalization to (even-)p-forms

CD, Deser, Esposito-Farese, 2010

III. From k-essence to generalized Galileons



CD, X.Gao, D. Steer, G. Zahariade 2011

What is the most general scalar theory which has (not necessarily exactly) second order field equations in **flat space** ?

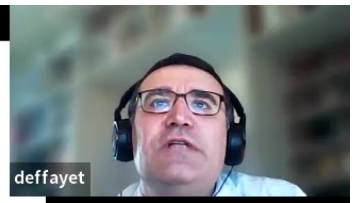
Specifically we looked for the most general scalar theory such that (in flat space-time)

i/ Its **Lagrangian contains derivatives of order 2** or less of the scalar field π

ii/ Its **Lagrangian is polynomial in second derivatives of** π (can be relaxed: Padilla, Sivanesan; Sivanesan)

iii/ The **field equations are of order 2 or lower** in derivatives

(NB: those hypothesis cover k-essence, simple Galileons ,…)

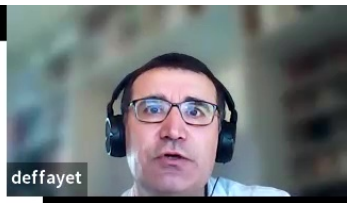


Answer: the most general such theory is given by a linear combination of the Lagrangians $\mathcal{L}_n\{f\}$

Free function of π and X

defined by
$$\begin{aligned}\mathcal{L}_n\{f\} &= f(\pi, X) \times \mathcal{L}^{\mathrm{Gal},3}_{N=n+2}, \\ &= f(\pi, X) \times \left(X \mathcal{A}^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n}_{(2n)} \pi_{\mu_1 \nu_1} \ldots \pi_{\mu_n \nu_n} \right)\end{aligned}$$

where $X \equiv \pi_\mu \pi^\mu$



Sketch of the proof

Consider « cycles » containing second derivatives

$$\begin{cases} [i] \equiv \pi^{\mu_1}{}_{\mu_2}\pi^{\mu_2}{}_{\mu_3}\pi^{\mu_3}{}_{\mu_4}\cdots\pi^{\mu_i}{}_{\mu_1} \\ \langle i\rangle \equiv \pi_{\mu_1}\pi^{\mu_1}{}_{\mu_2}\pi^{\mu_2}{}_{\mu_3}\pi^{\mu_3}{}_{\mu_4}\cdots\pi^{\mu_i}{}_{\mu_{i+1}}\pi^{\mu_{i+1}} \end{cases}$$

then

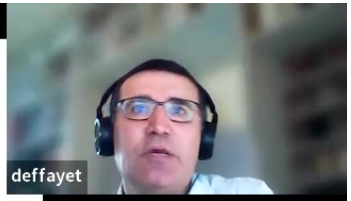
$$\begin{cases} \begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix} = [1]^{p_1}[2]^{p_2}\cdots[r]^{p_r} \\ \left\langle \begin{matrix} q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s \end{matrix} \right\rangle = \langle 1\rangle^{q_1}\langle 2\rangle^{q_2}\cdots\langle s\rangle^{q_s} \end{cases}$$

The most general theory we look for is made out of linear combinations of

$$\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s} = f(\pi, X) \times \begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix} \left\langle \begin{matrix} q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s \end{matrix} \right\rangle$$

Recursion relation leads to the previously decribed theory

Our most general flat space-time theories can easily be « covariantized » using the previously described technology



The covariantized theory is given by a linear combination of the Lagrangians

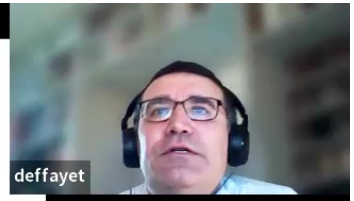
$$\mathcal{L}_{n,p}\{f\} = \mathcal{P}_{(p)}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\mathcal{R}_{(p)}\mathcal{S}_{(q\equiv n-2p)}$$

with

$$\begin{cases} \mathcal{S}_{(q\equiv n-2p)} \equiv \prod_{i=1}^{q-1} \pi_{\mu_{n-i}\nu_{n-i}} \\ \mathcal{R}_{(p)} \equiv \prod_{i=1}^{p} R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}} \\ \mathcal{P}_{(p)} \equiv \int_{X_0}^{X} dX_1 \int_{X_0}^{X_1} dX_2 \cdots \int_{X_0}^{X_{p-1}} dX_p\, \mathcal{T}_{(2n)}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(\pi, X_1) \\ \mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, X) \\ \qquad = f(\pi, X) \times X\mathcal{A}_{(2n)} \end{cases}$$

Specifically the covariantized theory is given by

$$\mathcal{L}_n^{\mathrm{cov}}\{f\} = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{C}_{n,p}\mathcal{L}_{n,p}\{f\} \quad \text{with} \quad \mathcal{C}_{n,p} = \left(-\frac{1}{8}\right)^p \frac{n!}{(n-2p)!p!}$$

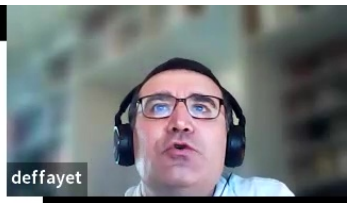


In 4 (and less) dimensions, these covariantized theories are the Horndeski theories,

obtained by Horndeski in 1974 as the most general scalar tensor theory in 4D which has second order field equation for the scalar and the metric

CD, X.Gao, D. Steer, G. Zahariade appared in march 2011

Horndeski paper was exhumed in a nice preprint by Charmousis, Copeland, Padilla, Saffin (« the fab four ») which appeared in june 2011



$$
\begin{aligned}
\mathcal{L}_H \;=\; & -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_1 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3\nu_3} - \frac{4}{3}\kappa_{1,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right) \\
& -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_3 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3}\pi_{\nu_3} - 4\kappa_{3,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3}\pi_{\nu_3}\right) \\
& -\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\left(F R_{\mu_1\mu_2\nu_1\nu_2} - 4F_{,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\right) \\
& -2\kappa_8\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2} \\
& -3\left(2F_{,\pi} + X\kappa_8\right)X + \kappa_9, \qquad F_{,X} = \kappa_{1,\pi} - \kappa_3 - 2X\kappa_{3,X}
\end{aligned}
$$

In fact, one can show that the two sets of theories (Horndeski and covariantized generalized Galileons) are identical in 4D (even though they start from different hypothesis)

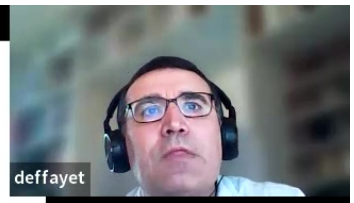
$$
\mathcal{L}_H = \sum_{0}^{3}\mathcal{L}_n^{\rm cov}\{f_n\}
$$

$$
\begin{aligned}
X f_0(\pi, X) &= -\kappa_9(\pi, X) - \frac{X}{2}\int dX\,\left(2\kappa_8 - 4\kappa_{3,\pi}\right)_{,\pi}, \\
X f_1(\pi, X) &= X\left(4\kappa_{3,\pi} + \kappa_8\right) - \frac{1}{2}\int dX\,\left(2\kappa_8 - 4\kappa_{3,\pi}\right) + 6F_{,\pi} \\
X f_2(\pi, X) &= 4\left(F + X\kappa_3\right)_{,X}, \\
X f_3(\pi, X) &= \frac{4}{3}\kappa_{1,X}.
\end{aligned}
$$

CD Gao, Steer, Zahariade
(in the PRD published version)

Kobayashi, Yamaguchi, Yokoyama,
arXiv 1105.5723 [hep-th]
(version 3 which appeared mid-june 2011)

Horndeski 1974 vs CD, Gao, Steer, Zahariade 2011



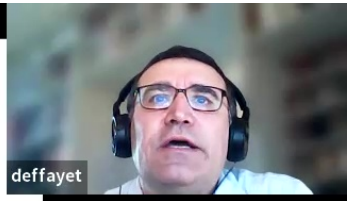
Uniqueness for a scalar-tensor theory with $\leq$ second order EOM in $D \leq 4$ dimensions

Uniqueness for a scalar theory with $\leq$ second order EOM in arbitrary D

Matches in $D \leq 4$

« covariantized » in arbitrary D (keeping $\leq$ second order EOM for the scalar and the metric)

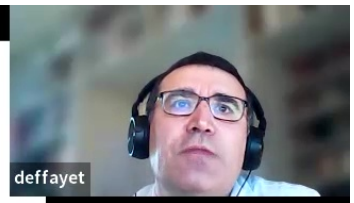
What about arbitrary D ?



- Horndeski gravity has of course been widely used

- In the following, two (abstract) directions:

  - Graphs theory and p-forms



IV. Relation to graphs theory

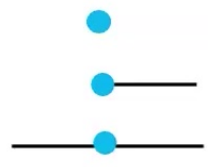
(based on CD and G. Zahariade)

The building block of our flat space-time scalar theories can be represented by graphs

$$\pi \Longrightarrow$$

$$\pi_\mu = \partial_\mu \pi \Longrightarrow$$

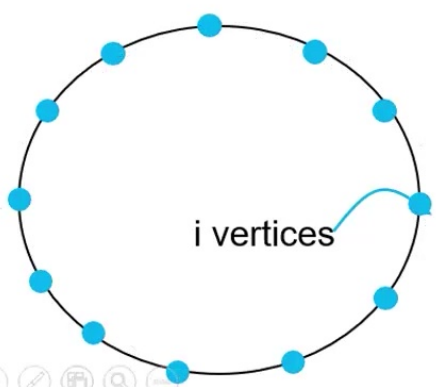
$$\pi_{\mu\nu} = \partial_\mu \partial_\nu \pi \Longrightarrow$$



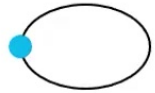
And e.g. « cycles » containing second derivatives

$$[i] \equiv \pi^{\mu_1}{}_{\mu_2} \pi^{\mu_2}{}_{\mu_3} \pi^{\mu_3}{}_{\mu_4} \cdots \pi^{\mu_i}{}_{\mu_1}$$

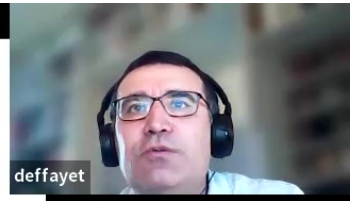
$$\langle i \rangle \equiv \pi_{\mu_1} \pi^{\mu_1}{}_{\mu_2} \pi^{\mu_2}{}_{\mu_3} \pi^{\mu_3}{}_{\mu_4} \cdots \pi^{\mu_i}{}_{\mu_{i+1}} \pi^{\mu_{i+1}}$$



Simple examples

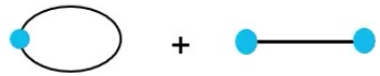
$\Box\pi$ ⟹ 

$\partial_\mu\pi\partial^\mu\pi$ ⟹



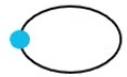
Then adding the algebraic structures on the graphs :

- Addition of graphs

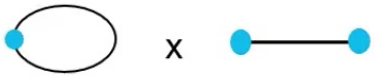


$\Box\pi + \partial_\mu\pi\partial^\mu\pi$

- Multiplication of a graph by a real number

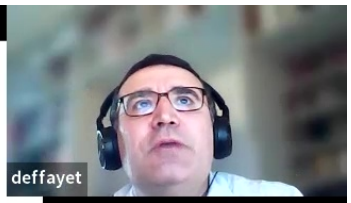
λ  ⟹ $\lambda\Box\pi$

- Internal multiplication of graphs



$\Box\pi\partial_\mu\pi\partial^\mu\pi$

We obtain an $\mathbb{R}$ - algebra of graphs enabling to represent any scalar quantities



Such as the Lagrangians

e.g. $\partial_\mu \pi \partial^\mu \pi$

However, Lagrangians are not unique,
we must quotient by integration by part

$$\partial_\mu \pi \partial^\mu \pi \quad \Leftrightarrow \quad -\pi \Box \pi$$

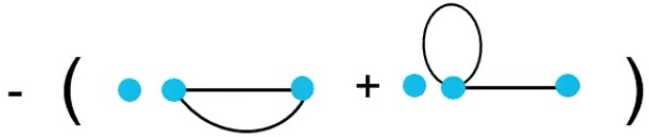
To integrate by part (henceforth **Ibp**) a given graph we follow the following rules

- Given a graph

$$\partial_\mu \pi \partial^\mu \partial^\nu \pi \partial_\nu \pi$$

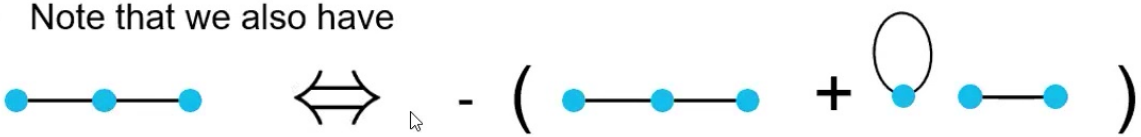
- Choose a vertex and disconnect one leg of the vertex

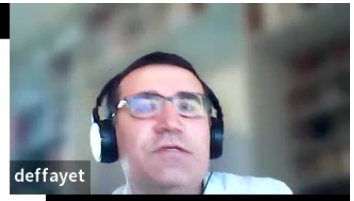
- Reconnect the leg to the other vertices in all possible ways adding the resulting graphs with a global minus sign



$$-\left(\pi \partial^\mu \partial^\nu \pi \partial_\mu \partial_\nu \pi + \pi \partial_\mu \Box \pi \partial^\mu \pi\right)$$

- Note that we also have





So a given scalar theory is defined by

an equivalence class **lpb**(g) of a graph g using

lbp as the equivalence relation

The Euler Lagrange operator $$\frac{\partial \mathcal{L}}{\partial \pi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \pi} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\nu \partial_\mu \pi} \right)$$

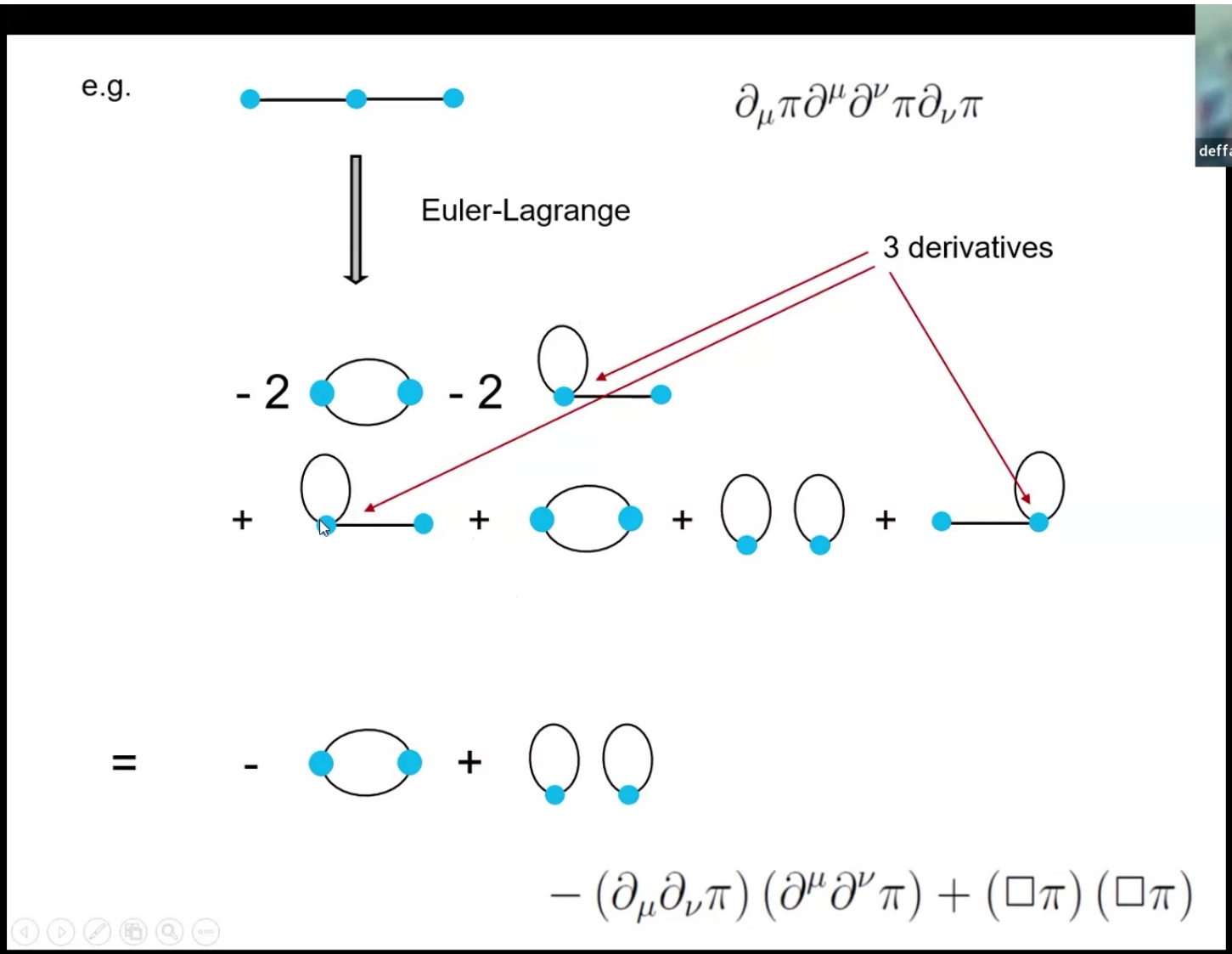
can also be translated as an action on graphs

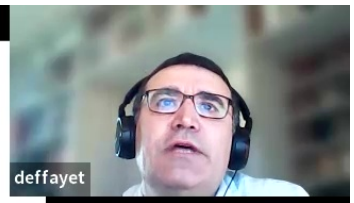
$$\frac{\partial \mathcal{L}}{\partial \pi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \pi} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\nu \partial_\mu \pi} \right)$$

Remove one isolated vertex

Remove successively all the one-legged vertices and reconnect the leg in all possible ways

Remove successively all the two-legged vertices and reconnect each of the legs in all possible ways


e.g.
$\partial_\mu \pi \partial^\mu \partial^\nu \pi \partial_\nu \pi$
Euler-Lagrange
3 derivatives
- 2
- 2
+
+
+
+
=
-
+
$-(\partial_\mu \partial_\nu \pi)(\partial^\mu \partial^\nu \pi) + (\Box \pi)(\Box \pi)$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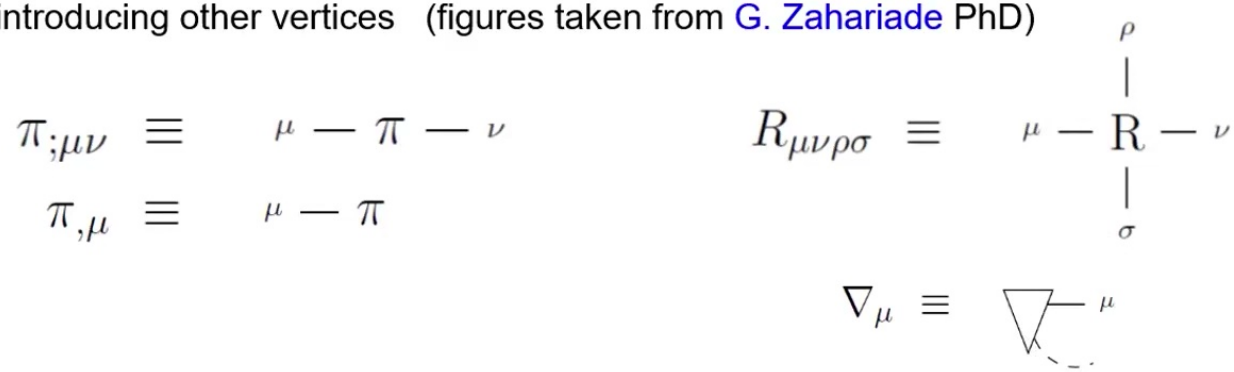
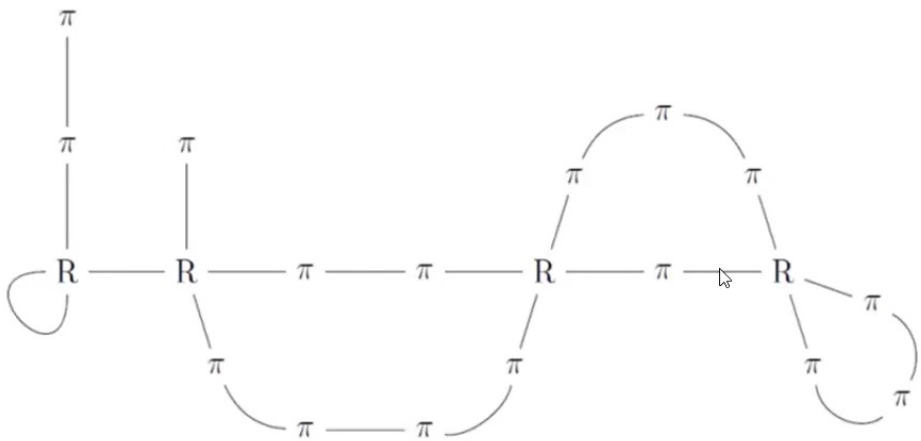
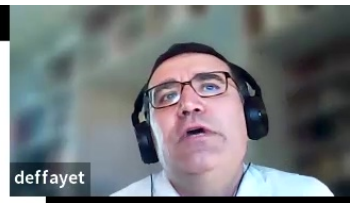
This can be generalized to include a metric (or other fields) introducing other vertices (figures taken from G. Zahariade PhD)



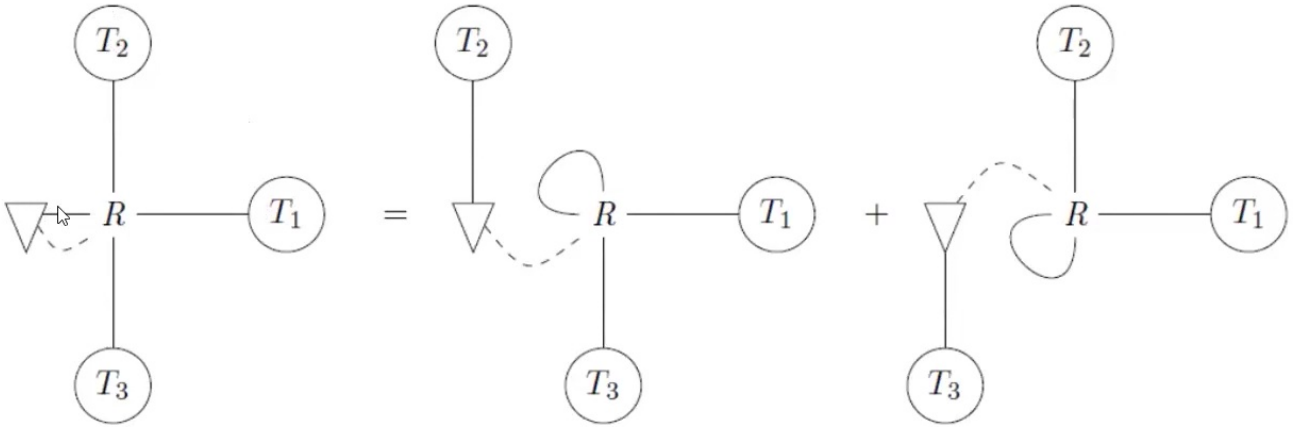
And considering scalars such as

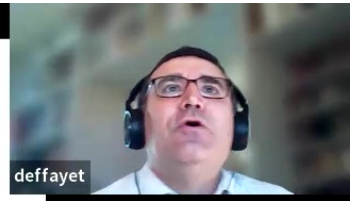




Taking into account symmetries and Bianchi identities …

… implying i.e that





Similarly, Euler Lagrange equations can be translated into the graph theory

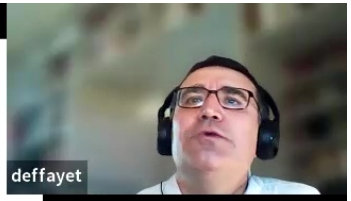
Horndeski and Horndeski-like constructions then imply non trivial results on the graph theories:

- EL operation on certain classes of graphs do not build vertices with more than two legs
- Uniqueness theorem for such graphs theories

Calls for more interactions with graph theorists ?

V. p-forms

Following CD, Deser, Esposito-Farese, 2010
showing the existence of non trivial even-p theories



E.g.

CD, Gümrükçüoglu, Mukohyama, Wang, 2014
CD, Mukohyama, Sivanesan, 2016
CD, Garcia-Saenz, Mukohyama, V.Sivanesan, 2017

Obtained a full classification of single abelian p-form theories with field equations of order less or equal to 2 (and D less or equal to 11) some unseen in the 2010 construction (as well as a no go theorem for p=1)

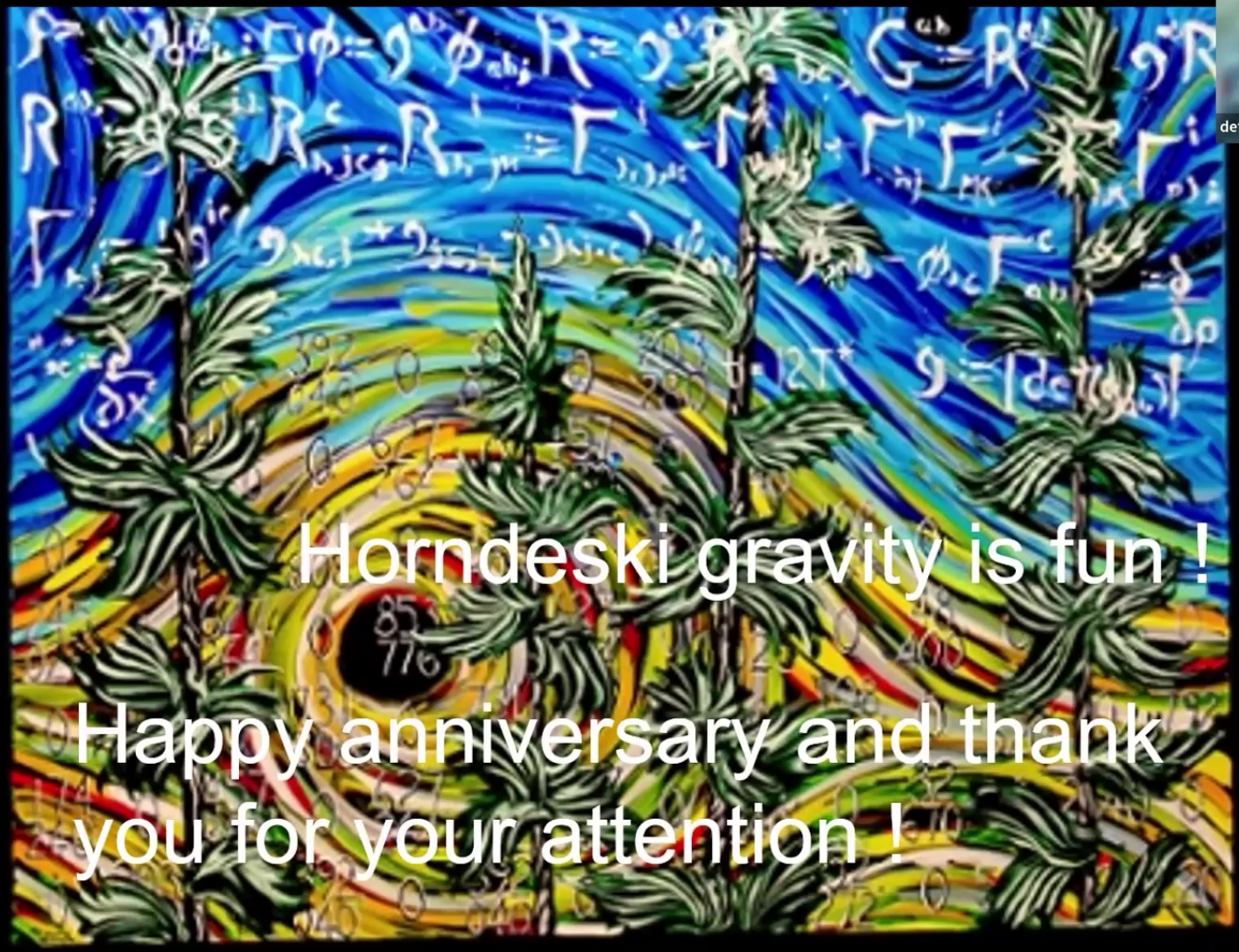
e.g. a 3 form A (with field stength ω) in 9 dimensions

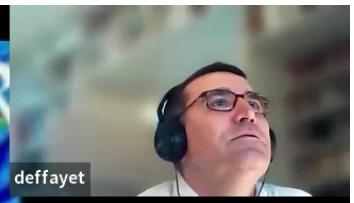
$$\int d^9x \epsilon^{a_1 a_2 \cdots} \epsilon^{b_1 b_2 \cdots} A_{a_1 a_2 b_1, a_3} A_{b_2 b_3 a_4, b_4} \partial_a \omega_B \partial_b \omega_A$$

vs

$$\int d^9x \epsilon^{a_1 a_2 \cdots} \epsilon^{b_1 b_2 \cdots} A_{a_1 a_2 a_3, a_4} A_{b_1 b_2 b_3, b_4} \partial_a \omega_B \partial_b \omega_A$$

More to be understood !



Horndeski gravity is fun !

Happy anniversary and thank you for your attention !