Title: Opening Talk

Speakers:

Collection: 50 Years of Horndeski Gravity: Exploring Modified Gravity

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50 Years of Horndeski Gravity:The Beginning

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$$\mathcal{L} = \sqrt{-g}\left\{\mathcal{U}_1\, \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\, \varphi_\alpha{}^\lambda R_{\beta\gamma}{}^{\mu\nu} + \tfrac{2}{3}\mathcal{U}_{1X}\, \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\, \varphi_\alpha{}^\lambda \varphi_\beta{}^\mu \varphi_\gamma{}^\nu\right\} +$$

$$+ \sqrt{-g}\left\{\mathcal{U}_3\, \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\, \varphi_\alpha \varphi^\lambda R_{\beta\gamma}{}^{\mu\nu} + 2\mathcal{U}_{3X}\, \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\, \varphi_\alpha \varphi^\lambda \varphi_\beta{}^\mu \varphi_\gamma{}^\nu\right\} +$$

$$+ \sqrt{-g}\left\{[F+2W]\, \delta^{\alpha\beta}_{\lambda\mu}\, R_{\alpha\beta}{}^{\lambda\mu} + 2F_X\, \delta^{\alpha\beta}_{\lambda\mu}\, \varphi_\alpha{}^\lambda \varphi_\beta{}^\mu\right\} +$$

$$+ \sqrt{-g}\left\{2\mathcal{U}_8\, \delta^{\alpha\beta}_{\lambda\mu}\, \varphi_\alpha \varphi^\lambda \varphi_\beta{}^\mu - 6\left[F_\varphi + 2W_\varphi - X\mathcal{U}_8\right]\Box\varphi\right\} + \sqrt{-g}\, \mathcal{U}_9$$

$X := -\frac{1}{2} g^{\alpha\beta}\varphi_\alpha\varphi_\beta$, $\varphi_\alpha := \nabla_\alpha \varphi$; $\varphi_{\alpha\beta} := \nabla_\beta \nabla_\alpha \varphi$; $F = F(\varphi, X)$

$F_X := 2(\mathcal{U}_3 + 2X\mathcal{U}_{3X} - \mathcal{U}_{1\varphi})$; $F_X := \frac{\partial F}{\partial X}$; $F_\varphi := \frac{\partial F}{\partial \varphi}$; $W = W(\varphi)$

$g = \det(g_{\alpha\beta})$; $\mathcal{U}_1, \mathcal{U}_3, \mathcal{U}_8, \mathcal{U}_9$ ARE ARBITRARY FUNCTIONS of φ & X. Originally $\rho := -2X = \varphi^\alpha \varphi_\alpha$

$$R_\alpha{}^\beta{}_{\mu\nu} = \Gamma_\alpha{}^\beta{}_{\mu,\nu} - \Gamma^\beta_{\alpha\nu,\mu} + \Gamma^\lambda_{\alpha\mu}\Gamma^\beta_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\beta_{\lambda\mu} \;;\; R_{\alpha\mu} := R_\alpha{}^\beta{}_{\mu\beta} \;;\; R := g^{\alpha\beta}R_{\alpha\beta} \quad (1)$$

5x5 GENERALIZED KRONECKER DELTA

$$\delta^{abcde}_{pqrst} := \det \begin{vmatrix} \delta^a_p & \delta^a_q & \delta^a_r & \delta^a_s & \delta^a_t \\ \delta^b_p & \delta^b_q & \delta^b_r & \delta^b_s & \delta^b_t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta^e_p & \delta^e_q & \delta^e_r & \delta^e_s & \delta^e_t \end{vmatrix} =$$

WHAT IN A SPACE OF 4-DIMENSIONS?

LOVELOCK (1969)

USE $\delta^{\cdots}_{\cdots}$ TO PROVE DIMENSIONALLY DEPENDENT IDENTITIES, SUCH AS THE LANCZOS IDENTITY: IF $n=4$

$$2R^{ih}R_{jh} + 2R^{hk}R_h{}^i{}_{kj} - RR^i_j - R^{ihkl}R_{jhkl} =$$

$$= \frac{1}{4}\delta^i_j \left(4R^{hk}R_{hk} - R^2 - R^{hklm}R_{hklm}\right)$$

Pf: $0 = \delta^{ibcde}_{jqrst} R_{bc}{}^{qr} R_{de}{}^{st}$ $\underset{(2)}{\text{in}}$ A 4-SPACE

PETER BERGMANN (1968)

$$L = L(g_{\alpha\beta}; g_{\alpha\beta,\mu}; g_{\alpha\beta,\mu\nu}; \varphi; \varphi_{,\mu}) \qquad (1)$$

$$L_{BD} := \sqrt{g}\,(\varphi R - \omega X/\varphi); \quad L_B := \sqrt{g}\,(f_1(\varphi)R + f_2(\varphi)X + f_3(\varphi))$$

REPLACE $f_2(\varphi)X + f_3(\varphi)$ IN L_B BY $\eta = \eta(\varphi, X)$ TO GET

$$\tilde{L}_B := \sqrt{g}\,(f_1(\varphi)R + \eta(\varphi, X))$$

<u>PROBLEM</u>: DETERMINE IN A SPACE OF 4-DIMENSIONS ALL LAGRANGE SCALAR DENSITIES OF THE FORM GIVEN IN (1) WHICH ARE $\ni$ $E^{\alpha\beta}(L)$ & $E(L)$ ARE AT MOST OF 2ND ORDER, WHERE $E^{\alpha\beta}(L) := \frac{\delta L}{\delta g_{\alpha\beta}}$ & $E(L) := \frac{\delta L}{\delta \varphi}$.

(3)

MASTER'S THESIS WAS DEVOTED TO:

(i) DEVELOPING THE MATHEMATICAL MACHINERY REQUIRED TO SOLVE THE ABOVE PROBLEM; +

(ii) INVESTIGATING BRANS-DICKE THEORY.

(i) INVOLVES DEVELOPING THE INVARIANCE IDENTITIES THAT $L = L(g_{\alpha\beta}; g_{\alpha\beta,\mu}; g_{\alpha\beta,\mu\nu}; \varphi; \varphi_{,\mu})$ SATISFIES + SHOWING THAT

$$E^{\alpha\beta}(L)_{|\beta} = \frac{1}{2}\varphi^{\alpha}E(L)$$

MASTER'S THESIS WAS FINISHED IN FEB. 1971

(4)

USING LOVELOCK'S (1969) RESULT ON THE FORM OF THE MOST GENERAL 2^{ND} ORDER METRIC LAGRANGIAN THAT YIELDS 2^{ND} ORDER FIELD EQ'S WE PROVED

<u>THEOREM (HORNDESKI & LOVELOCK (1972))</u>: IN A SPACE OF 4-DIMENSIONS THE MOST GENERAL LAGRANGIAN OF THE FORM $L = L(g_{\alpha\beta}; g_{\alpha\beta,\mu}; g_{\alpha\beta,\mu\nu}; \varphi; \varphi_{,\mu})$ WHICH IS $\ni$ $E^{\alpha\beta}(L)$ & $E(L)$ ARE AT MOST 2^{ND} ORDER IS

$$L_{HL} := \sqrt{g}\,\{ f_1(\varphi)[R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}] + f_2(\varphi)\varphi_\alpha\varphi_\beta G^{\alpha\beta} + \\ + f_3(\varphi)R + \underset{(5)}{\eta(\varphi, X)}\} + K\epsilon^{\alpha\beta\mu\nu}R^{\lambda\sigma}{}_{\alpha\beta}R_{\lambda\sigma\mu\nu}.$$

PH.D. PROBLEM: CONSTRUCT IN A SPACE OF 4-DIMENSIONS THE MOST GENERAL 2ND ORDER SCALAR-TENSOR FIELD EQS. DERIVABLE FROM A LAGRANGIAN OF ARBITRARY DIFFERENTIAL ORDER IN THE DERIVATIVES OF $g_{\alpha\beta}$ & φ.

KEY TO SOLVING THIS PROBLEM IS THE IDENTITY

$$E^{\alpha\beta}(L)_{|\beta} = \frac{1}{2}\varphi^{\alpha}E(L).$$

PROBLEM: CONSTRUCT IN A SPACE OF 4-DIMENSIONS ALL PAIRS $(A^{\alpha\beta}, A)$ WHICH ARE ∋: (i) $A^{\alpha\beta}$ & A ARE AT MOST OF 2ND ORDER IN $g_{\alpha\beta}$ & φ; (ii) $A^{\alpha\beta}=A^{\beta\alpha}$; & (iii) $A^{\alpha\beta}{}_{|\beta} = \frac{1}{2}\varphi^{\alpha}A$.

(6)

TO FIND A LAGRANGIAN THAT CAN YIELD ALL SUITABLE $(A^{\alpha\beta}, A)$ PAIRS I TRIED $g_{\alpha\beta}A^{\alpha\beta}$.

$$\mathcal{L} = \sqrt{g}\{\varkappa_1 \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\varphi_\alpha{}^\lambda R_{\beta\gamma}{}^{\mu\nu} + \tfrac{2}{3}\varkappa_{1X}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\varphi_\alpha{}^\lambda\varphi_\beta{}^\mu\varphi_\gamma{}^\nu\} +$$

$$+\sqrt{g}\{\varkappa_3\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\varphi_\alpha\varphi^\lambda R_{\beta\gamma}{}^{\mu\nu} + 2\varkappa_{3X}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\varphi_\alpha\varphi^\lambda\varphi_\beta{}^\mu\varphi_\gamma{}^\nu\}+$$

$$+\sqrt{g}\{[F+2W]\delta^{\alpha\beta}_{\lambda\mu}R_{\alpha\beta}{}^{\lambda\mu} + 2F_X\delta^{\alpha\beta}_{\lambda\mu}\varphi_\alpha{}^\lambda\varphi_\beta{}^\mu\}+$$

$$+\sqrt{g}\{2\varkappa_8\delta^{\alpha\beta}_{\lambda\mu}\varphi_\alpha\varphi^\lambda\varphi_\beta{}^\mu - 6[F_\varphi + 2W_\varphi - X\varkappa_8]\Box\varphi\} + \sqrt{g}\,\varkappa_9$$

$X := -\frac{1}{2}g^{\alpha\beta}\varphi_\alpha\varphi_\beta$; $\varphi_\alpha = \nabla_\alpha\varphi$; $\varphi_{\alpha\beta} := \nabla_\beta\nabla_\alpha\varphi$; $F = F(\varphi, X)$

$F_X := 2[\varkappa_3 + 2X\varkappa_{3X} - \varkappa_{1\varphi}]$; $F_X = \frac{\partial F}{\partial X}$; $F_\varphi := \frac{\partial F}{\partial \varphi}$; $W = W(\varphi)$.

IF $(A^{\alpha\beta}, A)$ ARE REQUIRED TO BE QUASI-LINEAR IN 2$^{\underline{ND}}$ DERIVATIVES OF $g_{\alpha\beta}$ & φ THEN $\mathcal{L}$ REDUCES TO L_B.

(7)