Title: Lecture - Twistors b

Speakers:

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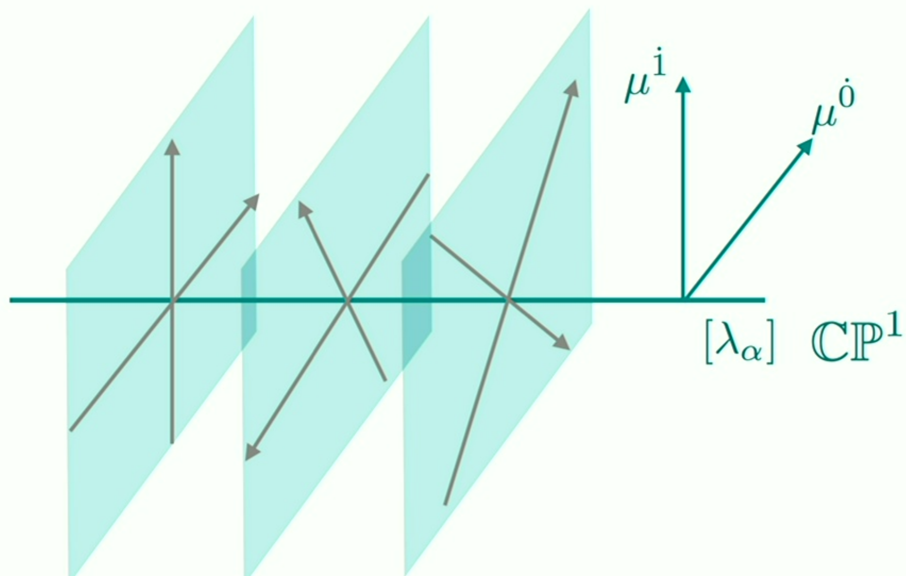
Twistor space of flat $\mathbb{R}^4$ is $\mathbb{PT}' \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$



The fibres are copies of space-time $\mathbb{R}^4$ itself, but how we identify these with $\mathbb{C}^2$ varies as we vary $[\lambda_\alpha] \in \mathbb{CP}^1$

Example: Twistor space of $(\mathbb{R}^4, [\delta])$

- Starting from $\mathbb{PT}'$ we can recover a point of $\mathbb{R}^4$ by taking a holomorphic section

$$\mathbb{CP}^1 \to \mathbb{PT}' \qquad \lambda_\alpha \mapsto (x^{\dot{\alpha}\alpha}\lambda_\alpha, \lambda_\alpha)$$

often called a *twistor line*

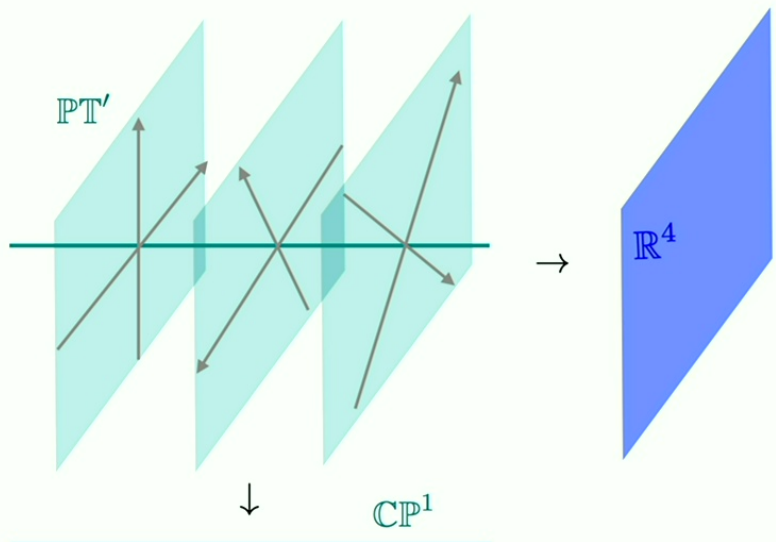
- The space of all such sections is $H^0(\mathbb{CP}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbb{C}^4$
- Points of real Euclidean space correspond to sections fixed by the involution
- We can also place other reality conditions on our section that fix either $\mathbb{R}^{2,2}$ or $\mathbb{R}^{1,3}$ (possible since the whole Weyl tensor vanishes)

Example: Twistor space of $(\mathbb{R}^4, [\delta])$

Note that while the fibration

$$\mathbb{PT}' \to \mathbb{CP}^1 \ , \qquad ([\mu^{\dot\alpha}, \lambda_\alpha]) \mapsto ([\lambda_\alpha])$$

is holomorphic, the fibration $\mathbb{PT}' \to \mathbb{R}^4$ over $\mathbb{R}^4$ is not, because it requires that we first identify the *real* (Euclidean) twistor lines



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- Explicity, given a point $[Z^A] = [\mu^{\dot\alpha}, \lambda_\alpha] \in \mathbb{PT}'$, construct the twistor line that joins Z to its image $\hat{Z}$ under the antiholomorphic involution
- This gives us a point $x = \hat{x} \in \mathbb{R}^4$ whose coordinates are

$$x^{\dot\alpha\alpha} = \frac{\mu^{\dot\alpha}\hat\lambda^\alpha - \hat\mu^{\dot\alpha}\lambda^\alpha}{\langle\hat\lambda\lambda\rangle} \qquad (\text{or } X^{AB} = Z^{[A}\hat{Z}^{B]} \text{ in embedding space})$$

- Thus, knowing where we are in twistor space tells us where we are in $\mathbb{R}^4$, but not in a holomorphic way

From Twistor Space to Space-Time

The non-linear graviton

Building a complex twistor space from $(M, [g])$ requires that we have first solved the asd conditions $W^+ = 0$ on M. The real magic comes from turning the construction around – this is Penrose's *non-linear graviton*

- Let Z be *any* complex 3-fold that contains at least one $\mathbb{CP}^1$ X whose normal bundle ($N \equiv T^{1,0}Z|_X / T^{1,0}X$) obeys

$$N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$$

  so that a nbhd of X in Z looks like a nbhd of a twistor line in $\mathbb{PT}'$

- A theorem of Kodaira implies (since $H^1(X, N) = 0$) that Z actually contains a $\dim_{\mathbb{C}} = 4$ family of such rational curves – this family is our (complexified) space-time $M_{\mathbb{C}}$

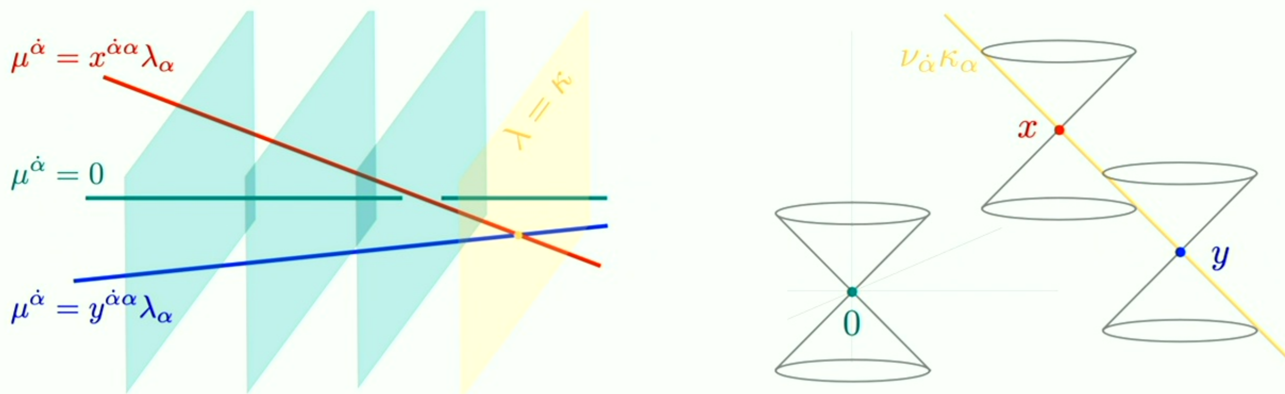
Theorems of Penrose and Atiyah-Hitchin-Singer show that (at least locally) *every* asd 4-mfld $(M, [g])$ arises this way

The conformal structure on M from twistor space

The space $H^0(X, N)$ of holomorphic sections is again $\mathbb{C}^4$, now interpreted as the *tangent* space to $M_\mathbb{C}$ at the point $x \in M_\mathbb{C}$ corresponding to the Riemann sphere X

- We declare that a point $v^{\dot\alpha\alpha}$ in this tangent space defines a *null* vector at x iff the corresponding section intersects X, so $\ker(v) \neq 0$
- This is a quadratic condition on v, so defines a conformal structure $[g]$ on $M_\mathbb{C}$ that is asd by construction
- We obtain a Riemannian real slice $M \subset M_\mathbb{C}$ by endowing Z with an antiholomorphic involution that acts without fixed points, and that restricts to the antipodal map on X

The conformal structure on M from twistor space



Two points $x, y \in M_{\mathbb{C}}$ are null separated iff their corresponding rational curves $X, Y \subset Z[M]$ intersect

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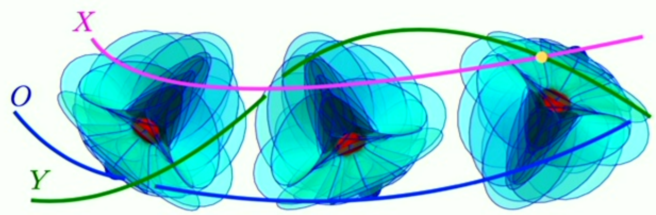
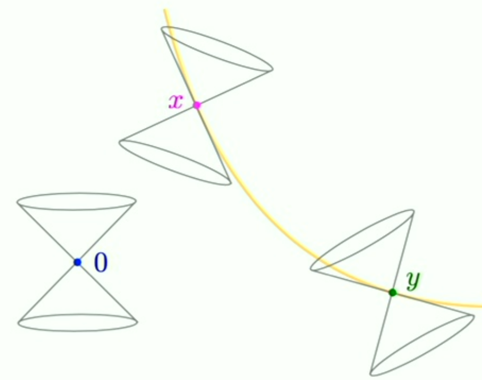


[Image: Andrew Hanson]

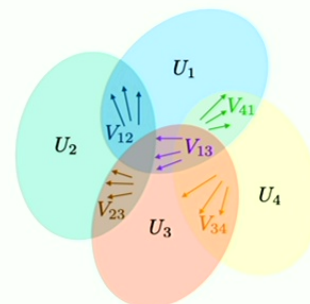


Two points $x, y \in M_{\mathbb{C}}$ are null separated iff their corresponding rational curves $X, Y \subset Z[M]$ intersect

Infinitesimal deformations of $\mathbb{C}$-str

Given a complex Z and its corresponding $(M, [g])$ we obtain a new, nearby conformal structure on M by slightly deforming the complex structure of Z

- Deform $\bar\partial \mapsto \bar D = \bar\partial + V$ for a Beltrami differential
- $\bar D^2 = \bar\partial V + \frac{1}{2}[V, V]$ which must vanish for integrability of the new $\mathbb{C}$-str
- For $V\ \infty^{ml}$, require $\bar\partial V = 0$ with $V \sim V + \bar\partial \chi$ for any smooth vector $\chi \Rightarrow V \in H^{0,1}(Z, T^{1,0})$



This description of $\mathbb{C}$-str deformations plays an important role in the topological B-model & twisted holography

(Re)-Constructing the Metric

ASD Einstein metrics on M from data on Z

Just knowing Z as a complex 3-fold fixes the conformal structure $[g]$ of M, but to get actual metric $g \in [g]$ we must fix a scale, *eg* fix vol_g

- Perhaps surprisingly, metrics on M do *not* come from metrics on Z
- The clue lies in something we've already calculated: if $(M, [g])$ is asd then (calling $\langle \lambda D \lambda \rangle \equiv \tau$ for short) we found

$$\begin{aligned} \tau \wedge d\tau &= \tau \wedge \lambda^\alpha \lambda^\beta R_{\alpha\beta} \\ &= \tau \wedge \left(\frac{2\lambda^\alpha\lambda^\beta}{\langle \lambda \hat{\lambda} \rangle} \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} \, \theta^{(\dot{\alpha}} \wedge \hat{\theta}^{\dot{\beta})} + \frac{s}{12} \theta^{\dot{\alpha}} \wedge \theta_{\dot{\alpha}} \right) \end{aligned}$$

- Hence $\tau \wedge d\tau$ is a (weighted) (3,0)-form iff $\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} = 0$ so that the metric is Einstein ($\mathring{Ric} = 0$ or $Ric = (s/4)g$)
- In this case the differential Bianchi identity says s is constant
- One can check that τ then varies holomorphically ($\mathcal{L}_{\bar{V}_{\dot{\alpha}}} \tau = 0 = \mathcal{L}_{\bar{\partial}_0} \tau$)

ASD Einstein metrics from data on Z

Again, we can turn this around. Starting on twistor space, suppose we pick a choice of $\tau \in H^0(Z, \Lambda^{1,0} \otimes \mathcal{O}(2))$

- Since τ is a holomorphic (1,0)-form

  $$\tau \wedge d\tau \in H^0(Z, \Lambda^{3,0} \otimes \mathcal{O}(4))$$

  so we must have

  $$\tau \wedge d\tau = f\,\Omega$$

  where Ω is an $\mathcal{O}(4)$-valued top holomorphic form and f is some globally holomorphic function

- Z is full of $\mathbb{CP}^1$s, so Liouville's theorem says that any globally holomorphic function on Z must be constant
- This constant is just the scalar curvature! Liouville's theorem does the job of the differential Bianchi identity

ASD Einstein metrics from data on Z

To fix the metric's scale, note that any choice of τ defines a distribution

$$\mathcal{D} \equiv \ker(\tau) \subset T^{1,0} \qquad ie\ V \in \mathcal{D} \ \text{ iff } \ V \lrcorner \tau = 0$$

- On any twistor line X for which $\tau_X \neq 0$ we have $\mathcal{D}|_X \cong N$
- Provided the restriction $\tau_X \neq 0$, we get an $\mathcal{O}(2)$-valued symplectic form ω on X's normal bundle by setting $\omega = (\Omega/\tau_X)|_{N^*}$
- Then our on volume form on M is declared to be

$$vol_g = \frac{\omega \wedge \hat{\omega}}{\langle \lambda \hat{\lambda} \rangle^2}$$

Together with the conformal structure obtained from the $\mathbb{C}$-str of twistor space, this fixes the whole metric on M

Ricci flat asd metrics

If $\tau \wedge d\tau = 0$ then $s = 0$ and we have a Ricci-flat asd metric, solving the vacuum Einstein equations

- Since $\dim_{\mathbb{C}}(Z) = 3$, $\tau \wedge d\tau = 0$ is equivalent to $d\tau = 0 \pmod{\tau}$
- This is the integrability condition for the distribution $\mathcal{D}$
- Frobenius' theorem then says Z is foliated by $\dim_{\mathbb{C}} = 2$ surfaces F whose tangent planes are spanned by $\mathcal{D}$
- Any line for which $\tau_X \neq 0$ must be transverse to this foliation, so (just like for $\mathbb{PT}'$) we have a holomorphic fibration $F \to Z \to \mathbb{CP}^1$ whenever τ defines a Ricci-flat metric

Example: Twistor space of $\mathbb{R}^4$ with the flat metric

We saw that the twistor space of $\mathbb{R}^4$ is $\mathbb{PT}' \equiv \left(\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1\right)$

- With homogeneous coordinates $[\lambda_\alpha]$ on $\mathbb{CP}^1$ and $[\mu^{\dot\alpha}]$ on the fibres, the top $\mathcal{O}(4)$-valued holomorphic form is $\Omega = \frac{1}{2}\langle \lambda d\lambda \rangle \wedge [d\mu \wedge d\mu]$
- Let's pick $\tau = \langle \lambda d\lambda \rangle$ which is non-zero on restriction to any twistor line ($\mu = x\lambda$ with fixed x)
- The distribution $\mathcal{D} = \ker(\tau)$ is spanned by $\{\partial/\partial\mu^{\dot\alpha}\}$ and indeed points along the fibres of $\mathbb{PT}' \to \mathbb{CP}^1$
- The $\mathcal{O}(2)$-valued symplectic form on the normal bundle to X is

$$\omega = (\Omega/\tau_X)_{N^*} = \frac{1}{2}\, dx^{\dot\alpha\alpha} \wedge dx_{\dot\alpha}{}^{\beta}\, \lambda_\alpha \lambda_\beta$$

which gives the usual volume form on $(\mathbb{R}^4, \delta)$ by

$$\mathrm{vol}_\delta = \frac{\omega \wedge \hat\omega}{\langle \hat\lambda \lambda \rangle^2} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^4$$

Example: Twistor space of $\mathbb{R}^4$ with the stereographic metric

On the same twistor space, let's instead choose

$$\tau = \langle \lambda d\lambda \rangle + \Lambda\,[\mu d\mu] \qquad \text{for } \Lambda > 0 \text{ constant}$$

- We're still on $\mathbb{PT}'$, so any metric we construct will have $[g] = [\delta]$
- Now $\tau \wedge d\tau = 2\langle \lambda d\lambda \rangle \wedge [d\mu \wedge d\mu] = 4\Omega$, so this choice will give a metric of constant curvature 4Λ on $\mathbb{R}^4$
- The corresponding distribution is no longer integrable

$$\mathcal{D} = \left\langle \frac{\partial}{\partial \mu^{\dot\alpha}} + \Lambda \frac{\mu_{\dot\alpha}}{\langle \hat\lambda \lambda \rangle} \hat\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} \right\rangle , \qquad [\mathcal{D}, \mathcal{D}] \not\subset \mathcal{D}$$

- On a twistor line $\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha$ now $\tau_X = (1 + \Lambda x^2)\langle \lambda d\lambda \rangle$ so

$$\omega = (\Omega/\tau_X)_{N^*} = \frac{1}{2}\frac{dx^{\dot\alpha\alpha} \wedge dx_{\dot\alpha}{}^{\beta} \lambda_\alpha \lambda_\beta}{(1 + \Lambda x^2)} \quad \Leftrightarrow \quad \mathrm{vol}_g = \frac{dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3}{(1 + \Lambda x^2)^2}$$

corresponding to the stereographic proj^n of the round metric on S^4

Example: Twistor space of the A_{k-1} gravitational instantons

The Gibbons Hawking metrics

$$g = V\,\delta_{\mathbb{R}^3} + V^{-1}(dt + A)^{\odot 2}$$

are Ricci flat if $V = \sum_{i=1}^{k} |\mathbf{x} - \mathbf{a}_i|^{-1}$ for $\mathbf{x} \in \mathbb{R}^3$ and $dA = \star_3 dV$

- Hitchin showed that their twistor spaces are the surface

$$XY = \prod_{i=1}^{k}(Z - a_i(\lambda)) \subset \left(\mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \to \mathbb{CP}^1\right)$$

  where $a_i(\lambda) = a_i^{\alpha\beta}\lambda_\alpha\lambda_\beta$ correspond to the points $\mathbf{a}_i \in \mathbb{R}^3$

- The twistor lines are sections of this fibration obtained by setting $Z = \mathbf{x}^{\alpha\beta}\lambda_\alpha\lambda_\beta$ and factorizing the deg $2k$ polynomial on the rhs

- Again picking $\tau = \langle\lambda d\lambda\rangle$, the adjunction formula gives

$$\omega = \oint \frac{dX \wedge dY \wedge dZ}{P(X,Y,Z)} = \frac{dY \wedge dZ}{Y} \quad \text{on patch } Y \neq 0$$

Hamiltonian deformations of $\mathbb{C}$-str

If we deform the $\mathbb{C}$-str via Beltrami $\bar{\partial} \mapsto \bar{D} = \bar{\partial} + V$, our new twistor space will preserve ω (so new M guaranteed to still be asd Einstein) if we use Hamiltonian deformations

- Choose some $h \in H^{0,1}(Z, \mathcal{O}(2))$ and set $V = \{h, \ \}$, where $\{\ ,\ \}$ is the $\mathcal{O}(-2)$-valued Poisson bracket defined by ω
- This is the role of $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$ on $\mathbb{PT}'$ – it provides a basis for these Hamiltonians h

In fact, h has a simple interpretation on M via the Penrose transform

- Differentiating four times along the fibres of $Z \to \mathbb{CP}^1$ and restricting to a twistor line X gives a fluctuation $\tilde{\psi}$ in W^-

  eg on flat $\mathbb{PT}'$ we have $\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x) = \int \langle \lambda d\lambda \rangle \wedge \left. \frac{\partial^4 h}{\partial \mu^{\dot{\alpha}} \cdots \partial \mu^{\dot{\delta}}} \right|_{\mu = x\lambda}$
- By construction $\tilde{\psi}$ obeys $\nabla^{\dot{\alpha}\alpha} \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0$ wrt the background spin connection – these are the linearised vacuum Einstein eq^{n}s

Gauge Theory

The asd Yang-Mills equations

Let's now switch our attention to gauge theory rather than gravity – much of the story will be very similar

- As usual, we can decompose the Yang-Mills curvature

$$F = F^+ + F^- = F_{\alpha\beta}\,\Sigma^{\alpha\beta} + \tilde{F}_{\dot\alpha\dot\beta}\,\tilde{\Sigma}^{\dot\alpha\dot\beta}$$

  into its self-dual and anti self-dual parts

- This decomposition is conformally invariant
- In terms of spinors, the curvature has components

$$F_{\dot\alpha\dot\beta\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} + \epsilon_{\alpha\beta}\tilde{F}_{\dot\alpha\dot\beta}$$

We'll be interested in the curvatures that are anti self-dual, $F^+ = 0$

- When F is asd we have $D \star F = -DF = 0$ and the Bianchi identity ensures the Yang-Mills equations hold

The asd YM as integrability conditions

Just as $F = 0$ is the integrability condition for the full gauge covariant derivative D, the asd YM eq^{n}s are integrability conditions for $\bar{D}_{\dot\alpha} = \lambda^{\dot\alpha} D_{\dot\alpha\alpha}$

- On flat $\mathbb{R}^4$ we have $D_{\dot\alpha\alpha} = \partial_{\dot\alpha\alpha} + A_{\dot\alpha\alpha}$ so

$$[\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}] = \lambda^\alpha \lambda^\beta [D_{\dot\alpha\alpha}, D_{\dot\beta\beta}]$$
$$= \lambda^\alpha \lambda^\beta \left(\epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta} + \epsilon_{\alpha\beta} \tilde{F}_{\dot\alpha\dot\beta}\right) = \lambda^\alpha \lambda^\beta F_{\alpha\beta}$$

  which vanishes for all λ_α iff $F_{\alpha\beta} = 0$

- Also holds on a general asd mfld after replacing $\lambda^\alpha \partial_{\dot\alpha\alpha} \rightsquigarrow \bar{V}_{\dot\alpha}$
- The $\bar{D}_{\dot\alpha}$ thus form a Lax pair for asd YM – many other integrable systems arise by symmetry reduction of these

Anti self-dual gauge theory on twistor space

Given a gauge bundle $\mathcal{E} \to M$ we construct a bundle $E \to Z[M]$ on its twistor space just by pulling back $\mathcal{E}$ via the twistor fibration $\pi : Z \to M$

- We take $\bar{D}_{\dot\alpha}$ and $\bar{\partial}_0$ to be antiholomorphic covariant derivatives on $E \to Z$
- When the asd YM eq^{n}s hold, $F^{0,2} = \bar{D}^2 = 0$ on Z so $\bar{D}$ is gauge equivalent to $\bar{\partial}$
- In this gauge the transition f^{n}s are holomorphic and we say $E \to Z$ is a holomorphic bundle

The twistor space gauge field is just the pullback of the gauge field A on M, projected onto the antiholomorphic directions

- It has no component along $\mathbb{CP}^1$ ($\bar{D}_0 = \bar{\partial}_0$) so the bundle $E \to Z$ will automatically be holomorphically trivial on each real twistor line

The Penrose-Ward correspondence

Starting from M requires we first solve the non-linear asd YM equations $F_{\alpha\beta} = 0$. Again, the real power comes from going in the other direction

- We construct a holomorphic vector bundle $E \to Z$ by taking $\mathbb{C}^r \times U_i$ over every coordinate patch $U_i \subset Z$ and piece these together using patching functions $\phi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ that are *holomorphic*
- To correspond to a gauge theory on space-time, we need our gauge bundle to be holomorphically trivial on each real twistor line X – this is generic if $c_1(E) = 0$
- We construct a bundle $\mathcal{E} \to M$ by declaring its fibre $\mathcal{E}_x$ of at $x \in M$ to be the space of holomorphic sections $H^0(X, E) \cong \mathbb{C}^r$ of E over X

Reconstructing the gauge field

The space-time gauge field is reconstructed from the patching functions ϕ_{ij}

- Let W_i be the restriction of the patch U_i to X, with $\cup_i W_i = X$
- Holomorphic triviality of $E|_X$ guarantees $\exists$ matrix-valued holomorphic function f_i on W_i such that $\phi_{ij}|_X = f_i(f_j)^{-1}$ on $W_i \cap W_j$
- Since the ϕ_{ij} are holomorphic, $\lambda^\alpha \partial_{\dot\alpha\alpha}(f_i f_j^{-1}) = 0$ which gives

$$\lambda^\alpha A_{\dot\alpha\alpha}(x) = f_i^{-1}\lambda^\alpha \partial_{\dot\alpha\alpha} f_i = f_j^{-1}\lambda^\alpha \partial_{\dot\alpha\alpha} f_j \qquad \text{on the overlap}$$

- The lhs (rhs) is holomorphic on W_i (W_j), so together they define a function on X that's globally holomorphic and hence linear in λ_α
- This defines a space-time gauge field A whose field-strength obeys $F^+ = 0$ by construction

Penrose-Ward & Atiyah-Hitchin-Singer prove that *every* sol^n of the asd YM eq^{n}s on M arises from this correspondence

Example: The Atiyah-Ward approach to 't Hooft's ansatz

- Cover $\mathbb{PT}'$ with the patches $U_0 = \{\lambda_0 \neq 0\}$ and $U_1 = \{\lambda_1 \neq 0\}$
- Suppose we have a rank 2 bundle $E \to \mathbb{PT}'$ defined by the trans^n f^n

$$\phi_{01} = \begin{pmatrix} z & \gamma \\ 0 & z^{-1} \end{pmatrix}$$

on $U_0 \cap U_1$, where $z = \lambda_1/\lambda_0$ and γ is an arbitrary holomorphic function

- Pulling back to the twistor line, we express γ as a Laurent series

$$\gamma(x,z) = \sum_{n\in\mathbb{Z}} z^n \gamma_n(x) = \gamma_+ + \gamma_0 + \gamma_-$$

By matching powers of z, $\lambda^\alpha \partial_{\dot{\alpha}\alpha}\gamma = 0$ implies $\Delta_{\mathbb{R}^4}\gamma_n = 0$ for each n

- On $W_0 \cap W_1$, the transition function splits as $\phi_{01} = f_1^{-1} f_0$ where

$$f_0 = \frac{1}{\sqrt{\gamma_0}} \begin{pmatrix} z & \gamma_0 + \gamma_+ \\ -1 & -z^{-1}\gamma_+ \end{pmatrix}, \qquad f_1 = \frac{1}{\sqrt{\gamma_0}} \begin{pmatrix} 1 & -z\gamma_- \\ -z^{-1} & \gamma_0 + \gamma_- \end{pmatrix}$$

- The gauge field itself takes the form of 't Hooft's ansatz

$$A^a_\mu = \eta^a_{\mu\nu} \frac{1}{\gamma_0} \partial_\nu \gamma_0$$

where $\eta^a_{\mu\nu}$ are the 't Hooft symbols

Example: The spin bundle $\mathbb{S}^-$

We saw earlier that the curvature of the spin connection on $\mathbb{S}^-$ is

$$\tilde{R}_{\dot\alpha\dot\beta} = \tilde{\Psi}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\,\tilde{\Sigma}^{\dot\gamma\dot\delta} + \frac{s}{12}\,\tilde{\Sigma}_{\dot\alpha\dot\beta} + \Phi_{\dot\alpha\dot\beta\gamma\delta}\,\Sigma^{\gamma\delta}$$

so is an anti self-dual 2-form iff the corresponding metric is Einstein

- Given $\tau \neq 0$ as before, the bundle $\mathcal{D} \equiv \ker(\tau) \subset T^{1,0}$ has rank 2 and on restriction to any twistor line X one has $\mathcal{D}|_X \cong N_{X|Z}$
- Hence $\mathcal{D} \otimes K_Z^{1/4}$ is a rank 2 holomorphic bundle that is trival on every twistor line

The Penrose-Ward correspondence then constructs the spin bundle $\mathbb{S}^-$

- The fibre $\mathbb{S}^-_x$ at $x \in M$ is the space $H^0(X, \mathcal{D} \otimes K^{1/4}) \cong \mathbb{C}^2$ of holomorphic sections of this bundle over the twistor line X
- We get an asd spin connection on $\mathbb{S}^-$ (and one can check it is torsion-free)

Infinitesimal deformations of the bundle

As with gravity, if we're given a holomorphic bundle $E \to Z$, we can build a nearby one by slightly deforming its $\mathbb{C}$-str

- We replace $\bar\partial \mapsto \bar D = \bar\partial + a$ for some $a \in \Lambda^{0,1} \otimes \mathfrak{g}$ so that holomorphic sections ψ are now defined by $\bar D \psi = 0$
- For the deformed bundle to still be holomorphic we must have $\bar D^2 = \bar\partial a + \frac{1}{2}[a, a] = 0$. At the infinitesimal level this says $\bar\partial a = 0$ with $a \sim a + \bar\partial \chi$ for χ smooth, so $a \in H^{0,1}(Z, \mathfrak{g})$
- The Penrose transform constructs a fluctuation in the background F^-

  eg around the trivial $E \to \mathbb{PT}'$, $f_{\dot\alpha\dot\beta}(x) = \int_X \langle \lambda d\lambda \rangle \wedge \left.\frac{\partial^2 a}{\partial \mu^{\dot\alpha} \partial \mu^{\dot\beta}}\right|_{\mu = x\lambda}$
- By construction this fluctuation obeys $D^{\dot\alpha\alpha} f_{\dot\alpha\dot\beta} = 0$, which are the YM eq^n*s* linearised around the asd background

Twistor QFTs

Finally, let's briefly consider (Lagrangian) QFT on twistor space. We'll ask that our theories are local on twistor space and that they depend only holomorphically on the twistor data

- One can place plenty of non-holomorphic QFTs on Z as a smooth 6-mfld, but they typically won't correspond to theories on M

A couple of points should be obvious immediately:

- Asking to be local *on twistor space* is very strong! Since $x \in M \Leftrightarrow \mathbb{CP}^1 \subset Z$, most local theories on M are non-local on Z
- Asking to be holomorphic is very dangerous! Chiral QFTs typically suffer from anomalies

When these difficulties can be overcome, the QFTs we obtain have very special properties

BF-type theories

The simplest class of theory one can construct is of BF-type. The prototype is to choose a (complex) bundle $E \to Z$ together with a $\bar{D}$-operator and a further field $B \in \Omega^{0,1}(Z, K_Z \otimes \mathfrak{g})$, with action

$$S_{BF}[B, \bar{D}] = \int_Z \mathrm{tr}(B \wedge F^{0,2})$$

- At least classically, this theory has gauge redundancy

  $$\bar{D} \mapsto g^{-1} \bar{D} g \,, \qquad B \mapsto g^{-1} B g + \bar{D} C$$

  for g a smooth gauge transformation & C a B-field transformation

- Varying B gives equation of motion

  $$F^{0,2} = 0 \quad \Leftrightarrow \quad E \to Z \text{ holomorphic}$$

  corresponding to an asd YM field by the Penrose-Ward construction

BF-type theories

There are various other natural theories of this type:

- An action for deforming the twistor space $\mathbb{C}$-str

$$S[B,N] = \int_Z B \lrcorner N \quad = \int_Z B \lrcorner \left(\bar{\partial} V + \frac{1}{2}[V,V] \right)]$$

  where N is the Nijenhuis tensor and now $B \in \Omega^{0,1}(Z, K_Z \otimes \Lambda^{1,0}_{cl})$

- This reduces to $\int_M B^+ \wedge W$, giving asd conformal gravity on M

- A twistor action for asd Einstein gravity

$$S[g,h] = \int_Z g \wedge \left(\bar{\partial} h + \frac{1}{2}\{h,h\} \right)$$

  for $\{h, \ \}$ a Ham^n def^n of the a$\mathbb{C}$-str and $g \in \Omega^{0,1}(Z, K_Z^{3/2})$

- This reduces to $\int_M \Gamma_{\alpha\beta} \wedge d(e^{\dot{\alpha}\alpha} \wedge e_{\dot{\alpha}}{}^{\beta})$

Reducing to an action on M

The twistor BF-theory has a larger gauge redundancy than on space-time

- We can use this to fix $\bar{D}|_X = g^{-1}\bar{\partial}_X g$ (*ie* gauge field on X is pure gauge) and $B|_X = b^+ \wedge \omega|_X$ for $\omega|_X$ the Kähler form on X
- Solving the mixed horizontal-vertical field equation in this gauge reduces the twistor BF action to an action on M

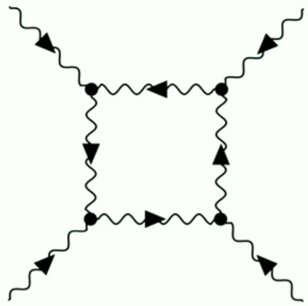
Unsurprisingly, twistor holomorphic BF theory reduces to self-dual BF theory on M

$$S_{asdYM}[b, D] = \int_M \mathrm{tr}(b^+ \wedge F)$$

- Field equations give $F^+ = 0$ (asd YM) and $Db^+ = 0$ as before

Anomalies

Unfortunately, all these twistor theories suffer from a fatal gauge anomaly



- In 6d anomaly comes from a box diagram
- Measures failure of 1-loop partition fn $\mathcal{Z} = \det(\bar{D})$ to be gauge invariant
- Theory on M still exists, but no longer integrable (*eg* 1-loop all $+$ amplitudes on $\mathbb{R}^4$)