Title: Lecture - Twistors a

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Introduction to Twistor Theory

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Motivation

In two dimensions, it's often useful to think of $\mathbb{R}^2$ as the complex plane $\mathbb{C}$

- d'Alembert's general solution of $\Box\phi = 0 \quad \Leftrightarrow \quad \phi = f(z) + \tilde{f}(\bar{z})$
- Riemann showed that a conformal class $[g]$ of metrics on a compact, oriented surface Σ is the same thing as a choice of $\mathbb{C}$-str on Σ
- Allowing for poles, we get an ∞-dim^{nl} enhancement of the group of conformal isometries $SL(2;\mathbb{C}) \rightsquigarrow Vir$
- Holomorphic factorisation of CFT partition functions

$$\mathcal{Z}(m, \bar{m}) = \|s(m)\|^2$$

Motivation

We may hope to extend this to $2n$ dimensions (especially $d = 4$). However

- Not every smooth 4-mfld admits a $\mathbb{C}$-str (*eg* S^4)
- There's no unique $\mathbb{C}$-str on $\mathbb{R}^4$ (or on the tangent space)

Twistor theory is a way of 'working with all $\mathbb{C}$-structures at once'

- The twistor space of an oriented Riemannian 4-mfld (M, g) is an auxiliary 6-mfld $Z[M]$
- Over any coordinate patch $U \subset M$, as a smooth mfld twistor space is just $Z[U] \cong S^2 \times U$
- When $(M, [g])$ is anti self-dual, twistor space has a natural $\mathbb{C}$-str even if M itself does not (...but being asd is a very restrictive condition)

Motivation

When twistor space is a $\mathbb{C}$ 3-fold, many of the previous benefits extend:

- The Penrose transform states that

$$\left\{ \begin{array}{c} \text{solutions to massless helicity } s \\ \text{free field equations on } U \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{holomorphic functions on } Z[U] \\ \text{of homogeneity } 2s-2 \end{array} \right\}$$

- A choice of asd conformal class $[g]$ on M is equivalent to a choice of $\mathbb{C}$-str on $Z[M]$
- Allowing for poles (on divisors), we get an ∞-dim^{nl} enhancement of conformal isometries, now associated with integrability
- QFTs on $Z[M]$ that depend only holomorphically on the twistor data have many beautiful properties

Twistor space gives us new ways to think about geometric objects in 4d

Geometry in Four Dimensions

Self-dual 2-forms

Let (M, g) be an oriented, Riemannian 4-mfld and let Λ^p be the bundle of p-forms

- The metric & orientation give us a Hodge star operator

$$\star : \Lambda^p \to \Lambda^{4-p} \qquad \text{obeying} \qquad \star^2 = 1$$

  which is defined for any $\alpha, \beta \in \Lambda^p$ by

$$\alpha \wedge *\beta = (\alpha, \beta)\, \mathrm{vol}_g$$

- In particular $\star$ maps 2-forms to 2-forms, so we can decompose

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

  into the $\pm$ eigenspaces of $\star$, called self-dual and anti self-dual forms

- This decomposition is conformally invariant, as we see from

$$(\star\omega)_{\mu\nu} = \sqrt{g}\, \epsilon_{\mu\nu\kappa\lambda}\, g^{\kappa\rho} g^{\lambda\sigma}\, \omega_{\rho\sigma}$$

Decomposition of the curvature

We could similarly decompose $(n/2)$-forms on any even dimensional mfld, but 4 dimensions is special because curvatures / field-strengths are 2-forms

- The Riemann curvature provides another map $Riem : \Lambda^2 \to \Lambda^2$ defined by

$$Riem \ : \ \omega \mapsto R_{\mu\nu}{}^{\kappa\lambda}\,\omega_{\kappa\lambda}\, dx^\mu \wedge dx^\nu$$

- Decomposing 2-forms into their sd / asd parts, $Riem$ decomposes as

$$Riem = \begin{array}{c} \\ \end{array} \begin{array}{cc} \Lambda^+ & \Lambda^- \\ \left[\begin{array}{c|c} W^+ + s/12 & \overset{\circ}{Ric} \\ \hline \overset{\circ}{Ric} & W^- + s/12 \end{array}\right] & \end{array} \begin{array}{c} \Lambda^+ \\ \Lambda^- \end{array}$$

where s is the scalar curvature, $\overset{\circ}{Ric} = Ric - (s/4)g$ is the trace-free Ricci tensor and $W^\pm$ the sd / asd Weyl tensors

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$$\begin{array}{cccc} & \Lambda^+ & \Lambda^- & \\ Riem = & \left[\begin{array}{c|c} W^+ + s/12 & \mathring{Ric} \\ \hline \mathring{Ric} & W^- + s/12 \end{array}\right] & & \begin{array}{c}\Lambda^+ \\ \Lambda^-\end{array} \end{array}$$

  where s is the scalar curvature, $\mathring{Ric} = Ric - (s/4)g$ is the trace-free Ricci tensor and $W^\pm$ the sd / asd Weyl tensors

(M, g) is called *anti self-dual* if $W^+ = 0$ (or self-dual if $W^- = 0$)

Spinors and forms

The decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ is closely related to the isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of Lie algebras

- Any 2-form ω itself provides a transformation

$$\omega : \Lambda^1 \to \Lambda^1 \qquad \text{acting as} \qquad \alpha_\mu dx^\mu \mapsto \omega_\mu{}^\nu \alpha_\nu \, dx^\mu$$

  where the index is raised with the metric

- This transformation is skew-adjoint, so on the cotangent space at any point it can be thought of as an element of $\mathfrak{so}(4) \cong su(2) \oplus \mathfrak{su}(2)$

- We also have

$\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	real sd/asd 2-forms on $\mathbb{R}^{2,2}$
$\mathfrak{so}(4)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	complex sd/asd (2,0)-forms on $\mathbb{C}^4$
$\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$	no analogue on $\mathbb{R}^{1,3}$

Spinors and forms

It's useful to introduce the bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ of complex, 2-component left & right spinors (if M is spin, else only locally)

- These are the bundles of the fundamental representations of $SU(2)_\pm$, or left & right Weyl spinors of $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$
- $SU(2)$ preserves a symplectic form on the fibres of $\mathbb{S}^\pm$
$$\langle \chi, \psi \rangle = \epsilon^{\beta\alpha} \chi_\alpha \psi_\beta \qquad [\tilde{\chi}, \tilde{\psi}] = \tilde{\chi}^{\dot{\alpha}} \tilde{\psi}^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}$$
where $\chi, \psi \in \mathbb{S}^+$ while $\tilde{\chi}, \tilde{\psi} \in \mathbb{S}^-$. We can thus identify $(\mathbb{S}^\pm)^* \cong \mathbb{S}^\pm$
- It also preserves conjugations $\hat{\ }: \mathbb{S}^\pm \to \mathbb{S}^\pm$ acting as
$$\chi_\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \hat{\chi}_\alpha = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}, \qquad \tilde{\chi}^{\dot{\alpha}} = \begin{pmatrix} c \\ d \end{pmatrix} \mapsto \hat{\tilde{\chi}}^{\dot{\alpha}} = \begin{pmatrix} -\bar{d} \\ \bar{c} \end{pmatrix}$$
These conjugations have no non-trivial fixed points
- Together these give the fibres of $\mathbb{S}^\pm$ an Hermitian inner product
$$\textit{eg} \text{ on } \mathbb{S}^+ \text{ we have } \langle \hat{\chi} \chi \rangle = |a|^2 + |b|^2 \geq 0$$

Spinors and forms

Let $e^a = e^a{}_\mu dx^\mu$ be a basis of vierbein 1-forms (*ie* a coframe) so that

$$g = g_{\mu\nu}(x)\, dx^\mu \odot dx^\nu = \delta_{ab}\, e^a \odot e^b$$

- We can identify the complexified cotangent bundle $\Lambda^1_{\mathbb{C}} := \Lambda^1 \otimes \mathbb{C}$ with $\mathbb{S}^+ \otimes \mathbb{S}^-$ by taking

$$e^{\dot\alpha\alpha} = \frac{1}{\sqrt{2}} e^a \sigma_a^{\dot\alpha\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^0 + ie^3 & ie^1 + e^2 \\ ie^1 - e^2 & e^0 - ie^3 \end{pmatrix}$$

  where $\sigma_a^{\dot\alpha\alpha} = (1^{\dot\alpha\alpha}, i\boldsymbol{\sigma}^{\dot\alpha\alpha})$ are the unit quaternions

- The components of $e^{\dot\alpha\alpha}$ are complex, but $\hat{e}^{\dot\alpha\alpha} = e^{\dot\alpha\alpha}$ if the e^a are real
- The metric becomes $g = 2\det(e^{\dot\alpha\alpha}) = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\, e^{\dot\alpha\alpha}\, e^{\dot\beta\beta} = \delta_{ab}\, e^a e^b$

Spinors and forms

The coframe provides a basis of 2-forms $\Sigma^{ab} = e^a \wedge e^b$ which we can also write in terms of spinors

- By antisymmetry

$$\Sigma^{\dot\alpha\alpha\dot\beta\beta} = e^{\dot\alpha\alpha} \wedge e^{\dot\beta\beta} = \frac{1}{2}\left(\epsilon^{\dot\alpha\dot\beta}\, e^{\dot\gamma\alpha} \wedge e_{\dot\gamma}^{\ \beta} + \epsilon^{\alpha\beta}\, e^{\dot\alpha\gamma} \wedge e^{\dot\beta}_{\ \gamma}\right)$$
$$= \epsilon^{\dot\alpha\dot\beta}\,\Sigma^{\alpha\beta} + \epsilon^{\alpha\beta}\,\tilde\Sigma^{\dot\alpha\dot\beta}$$

  where $\Sigma^{\alpha\beta} = \Sigma^{(\alpha\beta)}$ and $\tilde\Sigma^{\dot\alpha\dot\beta} = \tilde\Sigma^{(\dot\alpha\dot\beta)}$

- The $\Sigma^{\alpha\beta}$ form a basis of Λ^+ while the $\tilde\Sigma^{\dot\alpha\dot\beta}$ form a basis of Λ^-

The spin connection

Since $\Lambda^1_{\mathbb{C}} \cong \mathbb{S}^+ \otimes \mathbb{S}^-$, the Levi-Civita connection can be written as a pair of connections $(\Gamma^{\alpha}{}_{\beta}, \tilde{\Gamma}^{\dot\alpha}{}_{\dot\beta})$, each acting on separate spin bundles $\mathbb{S}^{\pm}$

- Metric compatibility implies that

$$De^{\dot\alpha\alpha} = de^{\dot\alpha\alpha} + \Gamma^{\alpha}{}_{\beta} \wedge e^{\dot\alpha\beta} + \tilde{\Gamma}^{\dot\alpha}{}_{\dot\beta} \wedge e^{\dot\beta\alpha} = 0$$

  and that $\Gamma^{\alpha}{}_{\beta}$ and $\tilde{\Gamma}^{\dot\alpha}{}_{\dot\beta}$ are $\mathfrak{su}(2)$ connections (*ie* trace free)

- The curvature of the connection on $\mathbb{S}^+$

$$R^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\gamma} \wedge \Gamma^{\gamma}{}_{\beta} \ \in \Gamma(\Lambda^2 \otimes \mathfrak{su}(2))$$

- Lowering indices using $\epsilon_{\alpha\beta}$, $R_{\alpha\beta}$ thus maps self-dual 2-forms to general 2-forms, and is the first column of the map *Riem*

The spin connection

We can expand this curvature in our basis of 2-forms as

$$R_{\alpha\beta} = \Psi_{\alpha\beta\gamma\delta}\,\Sigma^{\gamma\delta} + \frac{s}{12}\,\Sigma_{\alpha\beta} + \Phi_{\alpha\beta\dot\alpha\dot\beta}\,\tilde\Sigma^{\dot\alpha\dot\beta}$$

$\Psi_{\alpha\beta\gamma\delta} = \Psi_{(\alpha\beta\gamma\delta)}$ is the spinor form of W^+, while $\Phi_{\alpha\beta\dot\alpha\dot\beta} = \Phi_{(\alpha\beta)(\dot\alpha\dot\beta)}$ is the spinor form of $\mathring{Ric}$

- (M, g) is an Einstein mfld (*ie* $\mathring{Ric} = 0$, or $\Phi_{\alpha\beta\dot\alpha\dot\beta} = 0$) iff $R^\alpha{}_\beta$ is itself self-dual as a 2-form
- Recalling that $[g]$ is anti self-dual if $W^+ = 0$ shows that (M, g) is an asd sol^n of the vacuum Einstein eq^{n}s $Ric = 0$ iff $R^\alpha{}_\beta = 0$

Spinors and almost $\mathbb{C}$-structures

Now let's try to think of M as a complex manifold

- We first pick an almost complex structure (*ie* a choice of isomorphism $T_x^*M \simeq \mathbb{C}^2$ at each point $x \in M$)
- Do this by picking a fixed spinor ξ_α and declaring that $e^{\dot\alpha} = e^{\dot\alpha\alpha}\xi_\alpha$ are a basis of (1,0)-forms on T_x^*M

For example, if $\xi_\alpha = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ then

$$e^{\dot\alpha} = e^{\dot\alpha\alpha}\xi_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} e^0 + ie^3 & ie^1 + e^2 \\ ie^1 - e^2 & e^0 - ie^3 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^0 + ie^3 \\ i(e^1 + ie^2) \end{pmatrix}$$

so that we treat $e^0 + ie^3$ and $e^1 + ie^2$ as our (1,0)-forms

Instead choosing the conjugate spinor $\hat\xi_\alpha = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ picks $e^0 - ie^3$ and $e^1 - ie^2$ to be our (1,0)-forms, complex conjugate of those before

Spinors and almost $\mathbb{C}$-structures

An overall rescaling $\xi_\alpha \mapsto r\xi_\alpha$ by any non-zero $r \in \mathbb{C}$ just rescales the same basis, so doesn't change the space of (1,0)-forms

- The space of almost $\mathbb{C}$-structures on $T_x^* M$ is the projective space $(P\mathbb{S}^+)_x$ of spinors at x

The a$\mathbb{C}$-str defined by any ξ_α is compatible with the metric (& orientation) because (using the scaling freedom to normalise $\langle \hat{\xi} \xi \rangle = 1$)

$$g = \delta_{ab}\, e^a \odot e^b = e^{\dot\alpha} \odot \hat{e}^{\dot\beta}\, \epsilon_{\dot\beta\dot\alpha}$$

The final expression is an Hermitian metric on $\mathbb{C}^2$

- Metric compatibility means $(P\mathbb{S}^+)_x \cong SO(4)/U(2) \cong S^2$

From Space-Time to Twistor Space

Twistor space and its almost $\mathbb{C}$-structure

As a smooth 6-mfld, twistor space is just the total space $Z[M] \cong P\mathbb{S}^+$ of this projective spin bundle

- Over any open coordinate patch $U \subset M$, $Z[U] \cong S^2 \times U$ with coordinates $(x^{\dot{\alpha}\alpha}, [\lambda_\alpha])$
- The conjugation $(x^{\dot{\alpha}\alpha}, [\lambda_\alpha]) \mapsto (\hat{x}^{\dot{\alpha}\alpha}, [\hat{\lambda}_\alpha])$ fixes U pointwise (*ie* $\hat{x} = x$), but acts as the antipodal map on S^2

$Z[M]$ itself now acquires a preferred a$\mathbb{C}$-str defined by combining the $\mathbb{C}$-str on $S^2 \cong \mathbb{CP}^1$ with the a$\mathbb{C}$-str on M defined by the point $[\lambda_\alpha] \in \mathbb{CP}^1$

- At each point $p = (x, [\lambda]) \in Z[U]$, up to scaling the 1-forms

$$\theta^{\dot{\alpha}} = e^{\dot{\alpha}\alpha}\lambda_\alpha\,, \qquad \langle \lambda D \lambda \rangle = \lambda^\alpha d\lambda_\alpha + \lambda^\alpha \lambda^\beta\, \Gamma_{\alpha\beta}$$

form a basis of the holomorphic cotangent space $\Lambda^{1,0}_p$

Twistor space and its almost $\mathbb{C}$-structure

Dually, we can define the a$\mathbb{C}$-str by choosing a $\dim_{\mathbb{C}} = 3$ subspace $T_p^{0,1} \subset T_pZ \otimes \mathbb{C}$ at each point p

- Vectors in $T^{0,1}$ 'point in antiholomorphic directions', so should annihilate $\Lambda^{1,0}$ (*ie* $\bar{V} \lrcorner\, \omega = 0$ for all $\bar{V} \in T^{0,1}$ and all $\omega \in \Lambda^{1,0}$)
- In our case, first introduce a set of basis vectors (frame) $V_{\dot\alpha\alpha}$ of $TM \otimes \mathbb{C}$ dual to $e^{\dot\beta\beta}$ in the sense that $V_{\dot\alpha\alpha} \lrcorner\, e^{\dot\beta\beta} = \delta^{\dot\beta}{}_{\dot\alpha}\, \delta^{\beta}{}_{\alpha}$
- Then a basis of $T_p^{0,1}$ at each $p = (x, [\lambda])$ is given by

$$\bar{V}_{\dot\alpha} = \lambda^\alpha V_{\dot\alpha\alpha} - \lambda^\alpha \lambda^\beta\, \Gamma_{\dot\alpha\alpha\beta\gamma} \frac{\partial}{\partial \lambda_\gamma}\,, \qquad \bar{\partial}_0 = \langle \hat{\lambda}\lambda \rangle\, \lambda_\alpha \frac{\partial}{\partial \hat{\lambda}_\alpha}$$

  where $\bar{\partial}_0$ is the usual antiholomorphic vector on $\mathbb{CP}^1$ and we defined $\Gamma_{\dot\alpha\alpha\beta\gamma} = (V_{\dot\alpha\alpha} \lrcorner\, \Gamma_{\beta\gamma})$

Twistor space as a complex 3-fold

In general this a$\mathbb{C}$-str will not be integrable – our choice can fail to be consistent as we move around following different paths to the same point

- If the a$\mathbb{C}$-str is integrable, then the exterior derivative should map

$$d : \Lambda^{1,0} \to \Lambda^{2,0} \oplus \Lambda^{1,1}$$

  and the image is a (form-valued) linear combination of (1,0)-forms

- Equivalently, in terms of vector fields the a$\mathbb{C}$-str is integrable iff

$$[\bar{V}, \bar{W}] \in T^{0,1} \quad \text{for all} \quad \bar{V}, \bar{W} \in T^{0,1}$$

When these conditions hold $Z[M]$ is a complex 3-fold, meaning we can cover it with coordinate patches $U_i \subset \mathbb{C}^3$ such that the transition functions ϕ_{ij} are *holomorphic* on each $U_i \cap U_j$

Twistor space as a complex 3-fold

Now let's check for integrability of the twistor a$\mathbb{C}$-str

- We have

$$
\begin{aligned}
d(\langle\lambda D\lambda\rangle) &= D\lambda^\alpha \wedge D\lambda_\alpha + \lambda^\alpha\lambda^\beta R_{\alpha\beta} \\
&= D\lambda^\alpha \wedge D\lambda_\alpha + \lambda^\alpha\lambda^\beta\left(\Psi_{\alpha\beta\gamma\delta}\,\Sigma^{\gamma\delta} + \Phi_{\alpha\beta\dot\alpha\dot\beta}\,\tilde\Sigma^{\dot\alpha\dot\beta} + \frac{s}{12}\,\Sigma_{\alpha\beta}\right)
\end{aligned}
$$

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&= \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\,\Psi_{\alpha\beta\gamma\delta}\,\frac{\hat\theta^{\dot\alpha}\wedge\hat\theta_{\dot\alpha}}{\langle\lambda\hat\lambda\rangle^2} \qquad (\text{mod } \Lambda^{1,0})
\end{aligned}
$$

Twistor space as a complex 3-fold

Now let's check for integrability of the twistor a$\mathbb{C}$-str

- We have
$$d(\langle\lambda D\lambda\rangle) = D\lambda^\alpha \wedge D\lambda_\alpha + \lambda^\alpha\lambda^\beta R_{\alpha\beta}$$
$$= D\lambda^\alpha \wedge D\lambda_\alpha + \lambda^\alpha\lambda^\beta\Big(\Psi_{\alpha\beta\gamma\delta}\,\Sigma^{\gamma\delta} + \Phi_{\alpha\beta\dot\alpha\dot\beta}\,\tilde\Sigma^{\dot\alpha\dot\beta} + \frac{s}{12}\,\Sigma_{\alpha\beta}\Big)$$

- The identity $\xi^\alpha\langle\lambda\hat\lambda\rangle = \lambda^\alpha\langle\xi\hat\lambda\rangle + \hat\lambda^\alpha\langle\lambda\xi\rangle$ allows us to write
$$D\lambda^\alpha \wedge D\lambda_\alpha \propto \langle D\lambda\,\hat\lambda\rangle \wedge \langle\lambda D\lambda\rangle + \langle\lambda D\lambda\rangle \wedge \langle\hat\lambda D\lambda\rangle = 2\langle\lambda D\lambda\rangle \wedge \langle\hat\lambda D\lambda\rangle$$

- Similarly $\langle\lambda\hat\lambda\rangle\, e^{\dot\alpha\alpha} = \hat\theta^{\dot\alpha}\lambda^\alpha - \theta^{\dot\alpha}\hat\lambda^\alpha$ so that
$$\tilde\Sigma^{\dot\alpha\dot\beta} = e^{\dot\alpha\alpha} \wedge e^{\dot\beta}_{\ \alpha} \ \propto\ 2\theta^{(\dot\alpha} \wedge \hat\theta^{\dot\beta)}$$
$$\Sigma^{\alpha\beta} = e^{\dot\alpha\alpha} \wedge e_{\dot\alpha}^{\ \beta} \ \propto\ \lambda^\alpha\lambda^\beta\,\hat\theta^{\dot\alpha} \wedge \hat\theta_{\dot\alpha} - 2\lambda^{(\alpha}\hat\lambda^{\beta)}\,\theta^{\dot\alpha} \wedge \hat\theta_{\dot\alpha} + \hat\lambda^\alpha\hat\lambda^\beta\,\theta^{\dot\alpha} \wedge \theta_{\dot\alpha}$$

Twistor space as a complex 3-fold

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&= D\lambda^\alpha \wedge D\lambda_\alpha + \lambda^\alpha\lambda^\beta \Big(\Psi_{\alpha\beta\gamma\delta}\,\Sigma^{\gamma\delta} + \Phi_{\alpha\beta\dot\alpha\dot\beta}\,\tilde\Sigma^{\dot\alpha\dot\beta} + \frac{s}{12}\,\Sigma_{\alpha\beta}\Big) \\
&= \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\,\Psi_{\alpha\beta\gamma\delta}\,\frac{\hat\theta^{\dot\alpha}\wedge\hat\theta_{\dot\alpha}}{\langle\lambda\hat\lambda\rangle^2} \qquad (\text{mod } \Lambda^{1,0})
\end{aligned}
$$

- A similar computation shows that $d\theta^{\dot\alpha} = 0$ (mod $\Lambda^{1,0}$) always

Thus twistor space $Z[U]$ has an integrable a$\mathbb{C}$-str provided $\Psi_{\alpha\beta\gamma\delta} = 0$ (*ie* $W^+ = 0$) so that $[g]$ is anti self-dual

- There may be further topological obstructions to extending this globally over $Z[M]$

Example: Twistor space of $(\mathbb{R}^4, [\delta])$

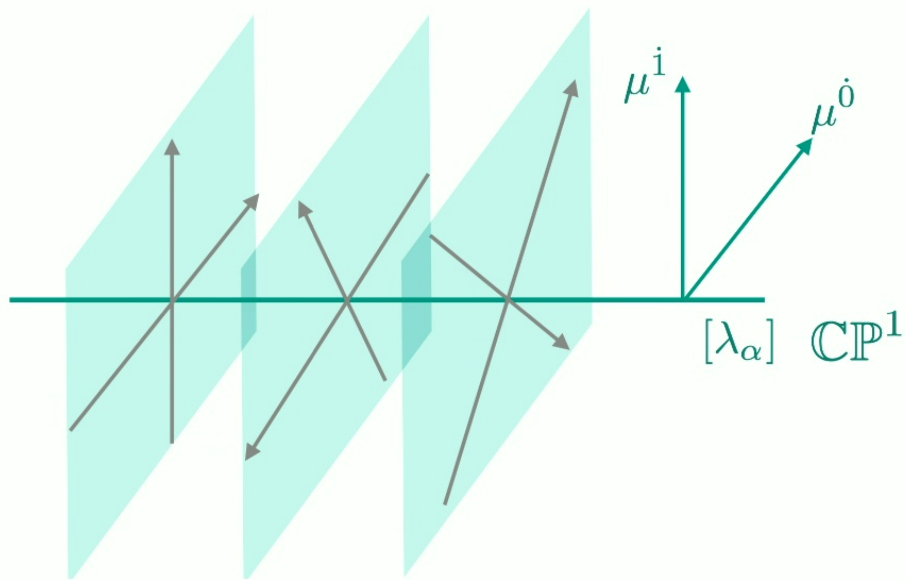
On flat $\mathbb{R}^4$ we have $e^{\dot\alpha\alpha} = dx^{\dot\alpha\alpha}$ and $\Gamma^\alpha{}_\beta = 0 = \tilde\Gamma^{\dot\alpha}{}_{\dot\beta}$

- $\theta^{\dot\alpha} = dx^{\dot\alpha\alpha}\lambda_\alpha$ and $\langle\lambda d\lambda\rangle$ form a basis of $\Lambda^{1,0}$ while $\lambda^\alpha(\partial/\partial x^{\dot\alpha\alpha})$ and $\bar\partial_0$ form a basis of $T^{0,1}$
- The integrability conditions are trivially satisfied
- We take the holomorphic coordinates to be $(\mu^{\dot\alpha}, \lambda_\alpha)$ defined up to scaling $(\mu^{\dot\alpha}, \lambda_\alpha) \sim (r\mu^{\dot\alpha}, r\lambda_\alpha)$ and with $\lambda_\alpha \neq 0$
- As a complex 3-fold

$$Z[\mathbb{R}^4] \cong \begin{array}{c} \mathcal{O}(1)\oplus\mathcal{O}(1) \\ \downarrow \\ \mathbb{CP}^1 \end{array} \qquad \text{and is often called } \mathbb{PT}'$$

- The spinor conjugation gives an antiholomorphic involution of $\mathbb{PT}'$ defined by $(\mu^{\dot\alpha}, \lambda_\alpha) \mapsto (\hat\mu^{\dot\alpha}, \hat\lambda_\alpha)$. This is the antipodal map on the $\mathbb{CP}^1$ described by the λ_αs

Twistor space of flat $\mathbb{R}^4$ is $\mathbb{PT}' \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$



The fibres are copies of space-time $\mathbb{R}^4$ itself, but how we identify these with $\mathbb{C}^2$ varies as we vary $[\lambda_\alpha] \in \mathbb{CP}^1$