**Title:** Lecture - Carrollian Physics b

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# Carrollian geometry and Symmetries

$x^a = (u, x^A)$

* $ds^2 = \eta_{ab}\, dx^a dx^b = -c^2 du^2 + \delta_{AB}\, dx^A dx^B$

$$\xrightarrow[c \to 0]{} 0\, du^2 + \delta_{AB}\, dx^A dx^B \equiv q_{ab}\, dx^a dx^b$$

* $-c^2 \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \delta^{AB} \end{pmatrix} \xrightarrow{c \to 0} n^a n^b \quad , \quad n^a = \delta^a_u$

* Carrollian geometry $(q_{ab}, n^c)$ emerges naturally in the limit $c \to 0$

↳ Same structure than the one appearing at $\mathscr{I}$. $q_{ab} n^b = 0$.

$$\xrightarrow[c\to 0]{} \quad \ldots \; + \delta_{AB}\, dx^A dx^B = q_{ab}\, dx^a dx^b .$$

* $-c^2 \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \delta^{AB} \end{pmatrix} \xrightarrow{c \to 0} n^a n^b \quad , \; n^a = \delta^a_u$

* Carrollian geometry $(q_{ab}, n^c)$ emerges naturally in the limit $c \to 0$

  ↳ Same structure than the one appearing at $\mathcal{I}$. $\quad q_{ab} n^b = 0$.

Carrollian algebra: $\mathcal{L}_\xi q_{ab} = 0 = \mathcal{L}_\xi n^a$.

$(\,/\!/\; \mathcal{L}_\xi \eta_{ab} = 0$ Poincaré$)$

Conformal Carrollian algebra
CCarr(d)

$\mathcal{L}_\xi q_{ab} = 2\alpha\, q_{ab}$
$\mathcal{L}_\xi n^a = -\alpha\, n^a$.

$(\,/\!/$ conformal algebra
$\mathcal{L}_\xi \eta_{ab} = 2\alpha\, \eta_{ab})$

for $\xi$ $(d=3)$ , $x^a = (u, z, \bar{z})$

$$dx^a dx^b = 2\, dz\, d\bar{z}$$

$$\partial_a = \partial_u$$

$$\xi(T, Y, \bar{Y}) = \left(T + \frac{u}{2}(\partial Y + \bar{\partial}\bar{Y})\right)\partial_u + Y\partial + \bar{Y}\bar{\partial}$$

$$\partial = \partial_z \ , \ \bar{\partial} = \partial_{\bar{z}}$$

Supertranslation: $T = T(z, \bar{z})$

Superrotations: $Y = Y(z)$, $\bar{Y} = \bar{Y}(\bar{z})$

$$* \left[\xi(T_1, Y_1, \bar{Y}_1), \xi(T_2, Y_2, \bar{Y}_2)\right] = \xi(T_{12}, Y_{12}, \bar{Y}_{12})$$

$$\begin{cases} T_{12} = Y_1 \partial T_2 - \frac{1}{2}\partial Y_1 T_2 + c.c. - (1 \leftrightarrow 2) \\ Y_{12} = Y_1 \partial Y_2 - (1 \leftrightarrow 2) \\ \bar{Y}_{12} = \bar{Y}_1 \bar{\partial} \bar{Y}_2 - (1 \leftrightarrow 2) \end{cases}$$

$$CCarr(3) \simeq BMS_4 = \underbrace{\text{Superrotations}}_{(Y, \bar{Y})} \ltimes \underbrace{\text{Supertranslat}^{\circ}}_{T}$$

$x^a dx^b = 2\, dz\, d\bar{z}$

$\partial_a = \partial_\mu$

Supertranslations $\begin{cases} Y = Y(z) \\ \bar{Y} = \bar{Y}(\bar{z}) \end{cases}$

$\partial \equiv \partial_z \;,\; \bar\partial = \partial_{\bar z}$

Superrotations

$\ast \left[ \xi(T_1, Y_1, \bar{Y}_1), \xi(T_2, Y_2, \bar{Y}_2) \right] = \xi(T_{12}, Y_{12}, \bar{Y}_{12})$

$$\begin{cases} T_{12} = Y_1 \partial T_2 - \frac{1}{2} \partial Y_1 T_2 + c.c. - (1 \leftrightarrow 2) \\ Y_{12} = Y_1 \partial Y_2 - (1 \leftrightarrow 2) \\ \bar{Y}_{12} = \bar{Y}_1 \bar\partial \bar{Y}_2 - (1 \leftrightarrow 2) \end{cases}$$

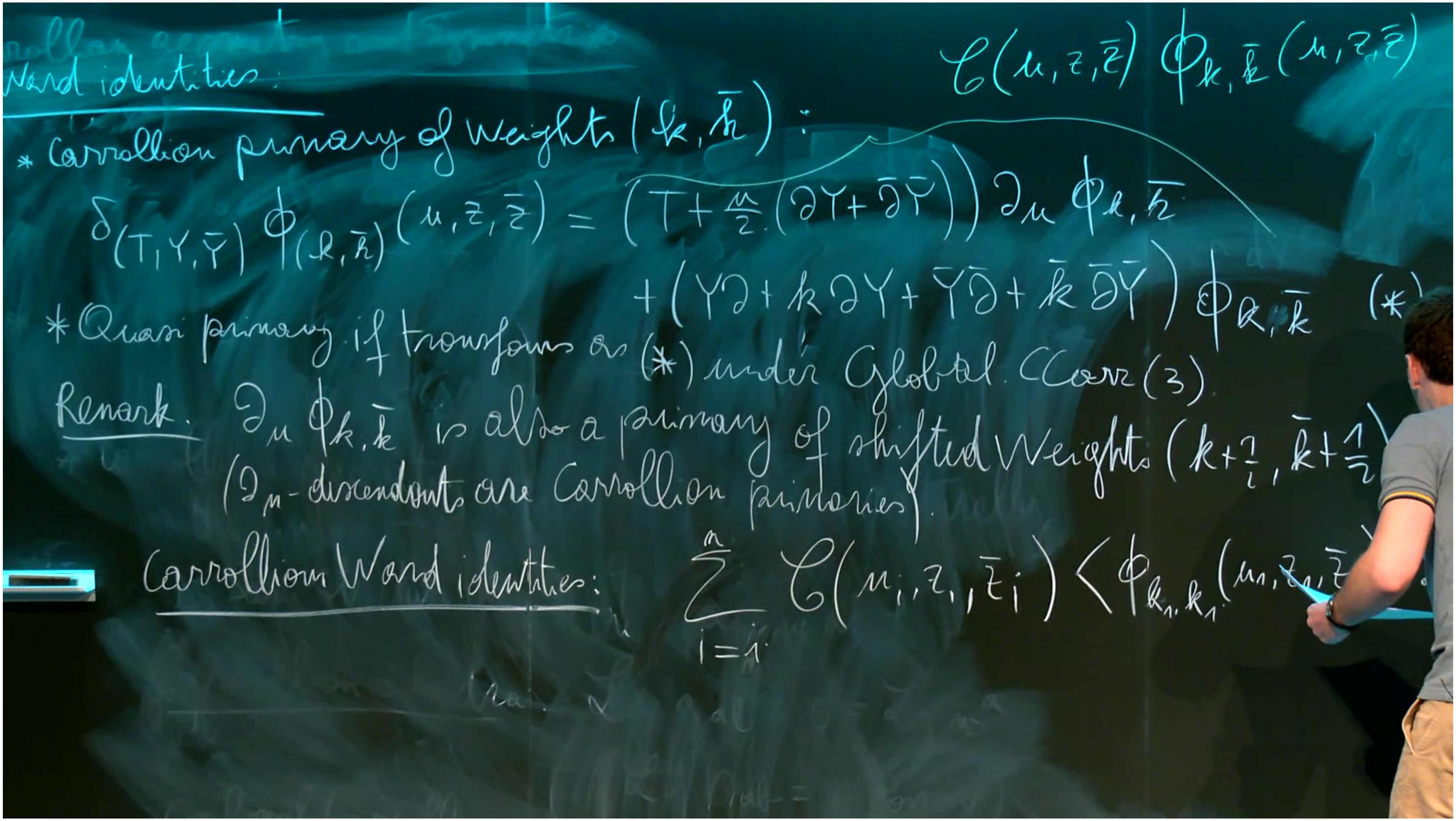
$$SO(d,2) \xrightarrow{c \to 0} \text{Global } CCarr(d)$$

$CCarr(3) \simeq BMS_4$

$= \underset{(Y,\bar{Y})}{\text{Superrotations}} \ltimes \underset{T}{\text{Supertranslat}^\circ}$

Global $CCarr(3) \simeq \text{Poincaré}_4$

$$\begin{cases} T = 1, z, \bar{z}, z\bar{z} \\ Y = 1, z, z^2 \\ \bar{Y} = 1, \bar{z}, \bar{z}^2 \end{cases}$$


Ward identities:
$\mathcal{C}(u,z,\bar z)\,\phi_{k,\bar k}(u,z,\bar z)$
* Carrollian primary of weights $(k,\bar k)$:
$\delta_{(T,Y,\bar Y)}\,\phi_{(k,\bar k)}(u,z,\bar z) = \left(T + \frac{u}{2}(\partial Y + \bar\partial \bar Y)\right)\partial_u \phi_{k,\bar k}$
$+ \left(Y\partial + k\,\partial Y + \bar Y \bar\partial + \bar k\, \bar\partial \bar Y\right)\phi_{k,\bar k}$ (*)
* Quasi primary if transforms as (*) under Global CConf(3)
Remark. $\partial_u \phi_{k,\bar k}$ is also a primary of shifted weights $(k+\frac{1}{2}, \bar k+\frac{1}{2})$
($\partial_u$-descendants are Carrollian primaries).
Carrollian Ward identities: $\sum_{i=1}^{n} \mathcal{C}(u_i,z_i,\bar z_i)\langle \phi_{k_1,\bar k_1}(u_1,z_1,\bar z_1)$

$\mathcal{C}(u,z,\bar z)\ \phi_{k,\bar k}(u,z,\bar z)$ , $X^a=(u,z,\bar z)$

$\bar Y))\, \partial_u \phi_{k,\bar k}$

Supertranslations

$+\bar Y\bar\partial + \bar k\, \bar\partial \bar Y)\, \phi_{k,\bar k}$ $(*)$

Superrotations

obal. CCarr(3)

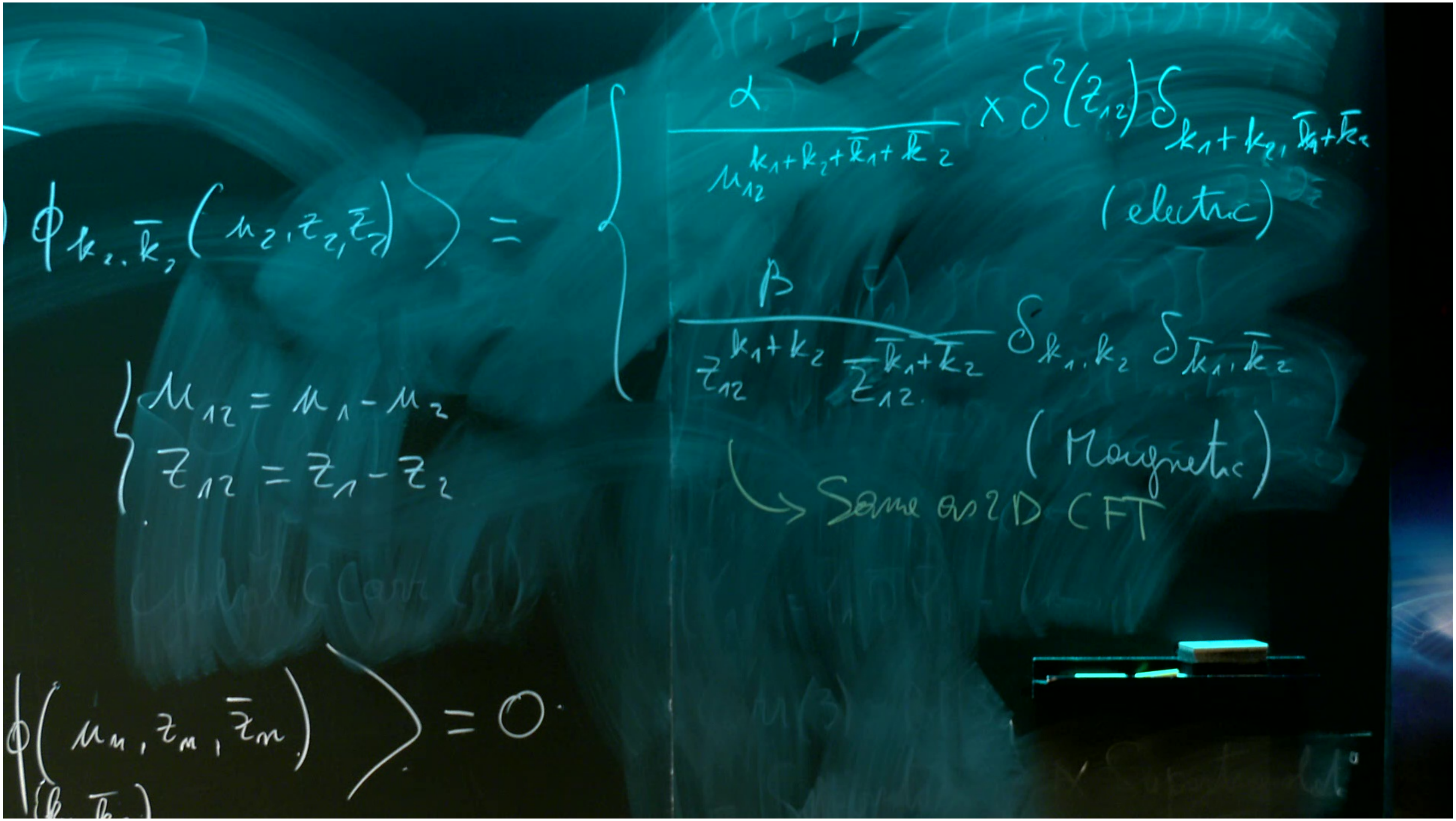
$SO(d,2)$

$\big\downarrow\ c\to 0$

Global CCarr(d)

fted Weights $(k+\frac{1}{2}, \bar k+\frac{1}{2})$

$, z_i, \bar z_i)\ \langle \phi_{k_1,\bar k_1}(u_1,z_1,\bar z_1)\cdots \phi_{(k_i,\bar k_i)}(u_i,z_i,\bar z_i)\cdots \phi_{(k_n,\bar k_n)}(u_n,z_n,\bar z_n)\rangle = 0$


$\phi_{k_2,\bar{k}_2}(u_2, z_2, \bar{z}_2) \rangle =$
$\frac{\alpha}{u_{12}^{k_1+k_2+\bar{k}_1+\bar{k}_2}} \times \delta^2(z_{12})\, \delta_{k_1+k_2,\bar{k}_1+\bar{k}_2}$
(electric)
$\frac{\beta}{z_{12}^{k_1+k_2}\, \bar{z}_{12}^{\bar{k}_1+\bar{k}_2}}\, \delta_{k_1,k_2}\, \delta_{\bar{k}_1,\bar{k}_2}$
(Magnetic)
Same as 2D CFT
$u_{12} = u_1 - u_2$
$z_{12} = z_1 - z_2$
$\phi(u_n, z_n, \bar{z}_n) \rangle = 0.$

t function:

$$\langle \phi_{k_1,\bar{k}_1}(u_1,z_1,\bar{z}_1)\,\phi_{k_2,\bar{k}_2}(u_2,z_2,\bar{z}_2)\rangle = \begin{cases} \dfrac{\alpha}{u_{12}^{k_1+k_2+\bar{k}_1+\bar{k}_2}} \times \delta^2(z_{12})\,\delta_{k_1+k_2,\bar{k}_1+\bar{k}_2} & \text{(electric)} \\[2ex] \dfrac{\beta}{z_{12}^{k_1+k_2}\,\bar{z}_{12}^{\bar{k}_1+\bar{k}_2}}\,\delta_{k_1,k_2}\,\delta_{\bar{k}_1,\bar{k}_2} & \text{(Magnetic)} \end{cases}$$

$\to$ Same as 2D CFT

$$\begin{cases} u_{12} = u_1 - u_2 \\ z_{12} = z_1 - z_2 \end{cases}$$

( Global Carr(3) )

$$\ldots, z_1, \bar{z}_1)\cdots \phi_{(k_n,\bar{k}_n)}(u_n, z_n, \bar{z}_n)\rangle = 0$$

d celestial amplitudes

ttering

$$\omega\, q^\mu(z,\bar z) \quad , \quad q^\mu = \frac{1}{\sqrt{2}}\left(1+z\bar z,\ z+\bar z,\ -i(z-\bar z),\ 1-z\bar z\right)$$

$\to$ energy

$\to \epsilon = \pm 1$ if $\begin{cases} \text{outgoing} \\ \text{incoming} \end{cases}$

mplitudes: $\mathcal{A}_n\left(\{\omega_i, z_i, \bar z_i\}^{\epsilon_i}_{J_i}\right)$

$\to$ # of particles

$$\left(\{u_i, z_i, \bar z_i\}^{\epsilon_i}_{J_i}\right) = \prod_{i=1}^{n}\left(\int_0^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\epsilon_i \omega_i u_i}\right) \mathcal{A}_n\left(\{\omega_i, z_i, \bar z_i\}^{\epsilon_i}_{J_i}\right)$$

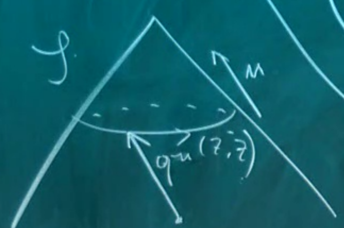
## Carrollian and celestial amplitudes

Massless scattering

$$p^\mu = \epsilon\,\omega\, q^\mu(z,\bar z) \quad , \quad q^\mu = \frac{1}{\sqrt{2}}\left(1+z\bar z,\ z+\bar z,\ -i(z-\bar z),\ 1-z\bar z\right)$$

$\omega \to$ energy

$\epsilon = \pm 1$ if $\begin{cases}\text{outgoing}\\ \text{incoming}\end{cases}$

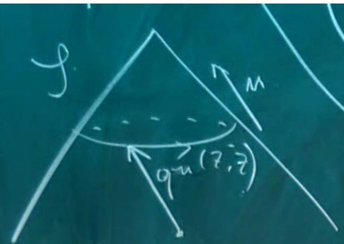



* Amplitudes: $\mathcal{A}_n\left(\{\omega_i, z_i, \bar z_i\}^{\epsilon_i}_{J_i}\right)$.

$n \to$ # of particles

$$C_n\left(\{u_i, z_i, \bar z_i\}^{\epsilon_i}_{J_i}\right) = \prod_{i=1}^{n}\left(\int_0^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\epsilon_i\omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_i, z\right.$$

$=$



→ energy

→ $\epsilon = \pm 1$ if $\begin{cases}\text{outgoing}\\ \text{incoming}\end{cases}$

* Amplitudes: $\mathcal{A}_n\left(\{\omega_i, z_i, \bar{z}_i\}^{\epsilon_i}_{J_i}\right)$

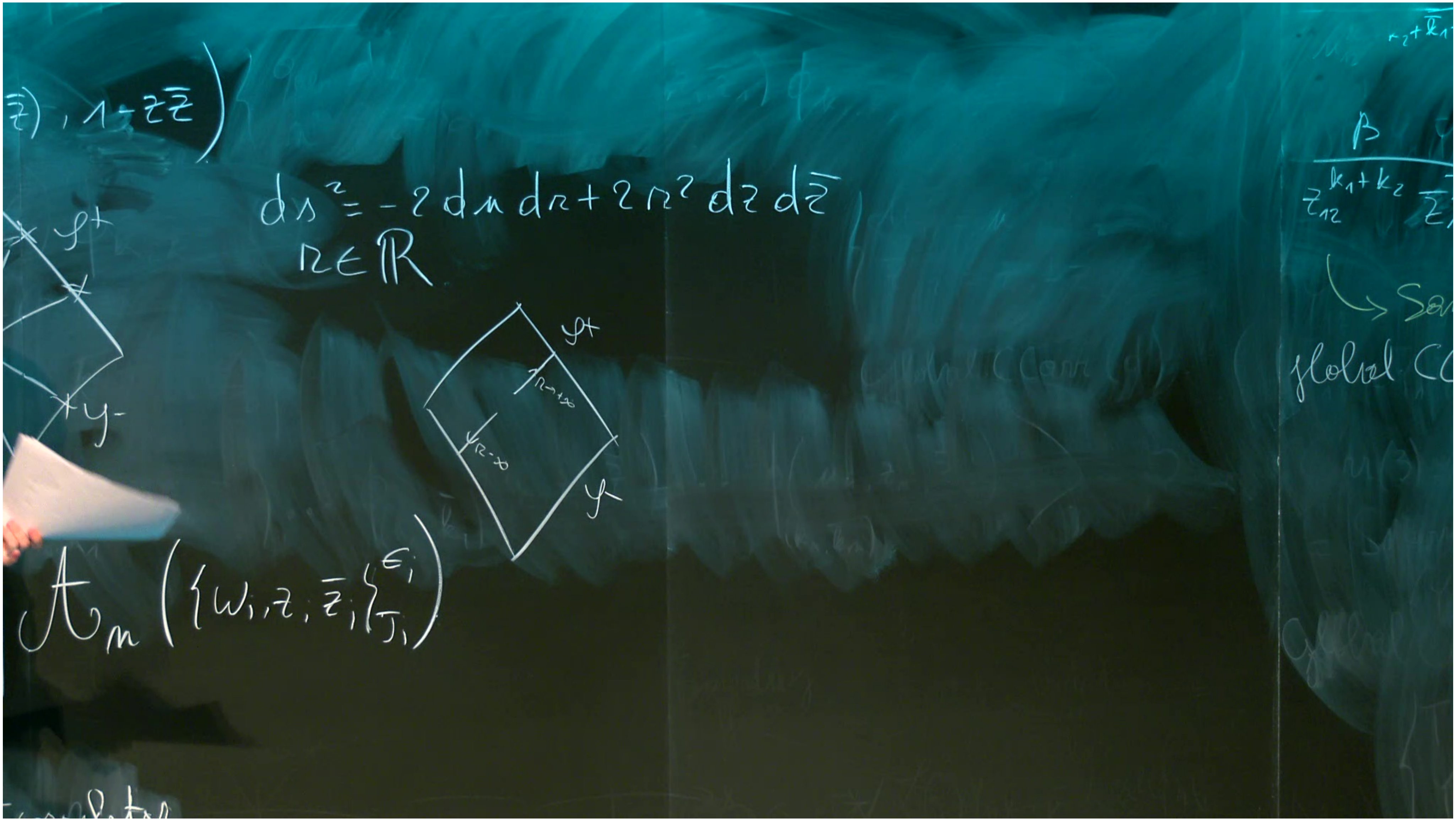
$n$ → # of particles

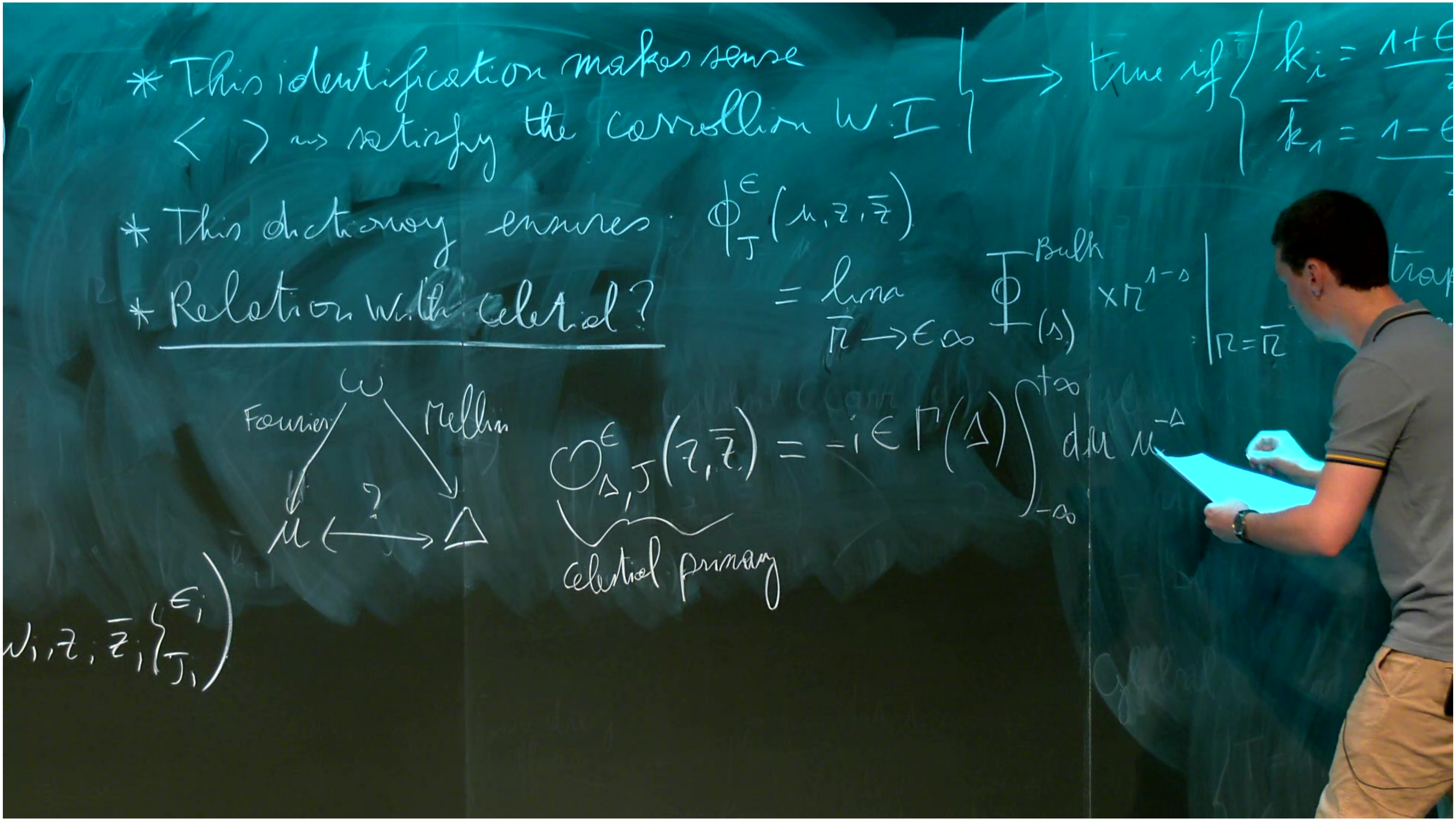
$$C_n\left(\{u_i, z_i, \bar{z}_i\}^{\epsilon_i}_{J_i}\right) = \prod_{i=1}^{n}\left(\int_0^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\epsilon_i \omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_i, z_i,\right.$$

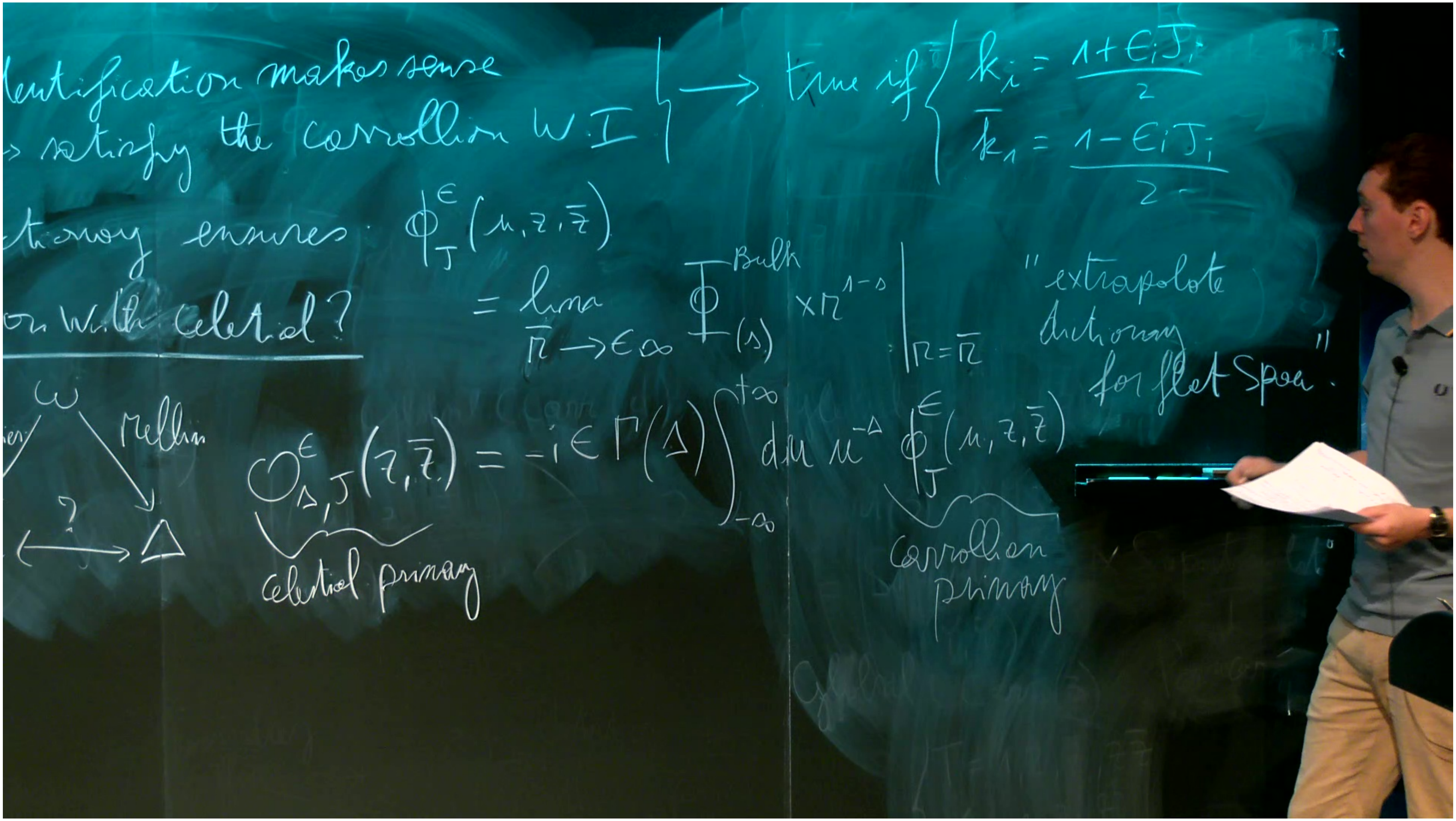
$$\equiv \left\langle \phi^{\epsilon_1}_{k_1,\bar{k}_1}(u_1, z_1, \bar{z}_1) \dots \phi^{\epsilon_n}_{k_n,\bar{k}_n}(u_n, z_n, \bar{z}_n)\right\rangle$$

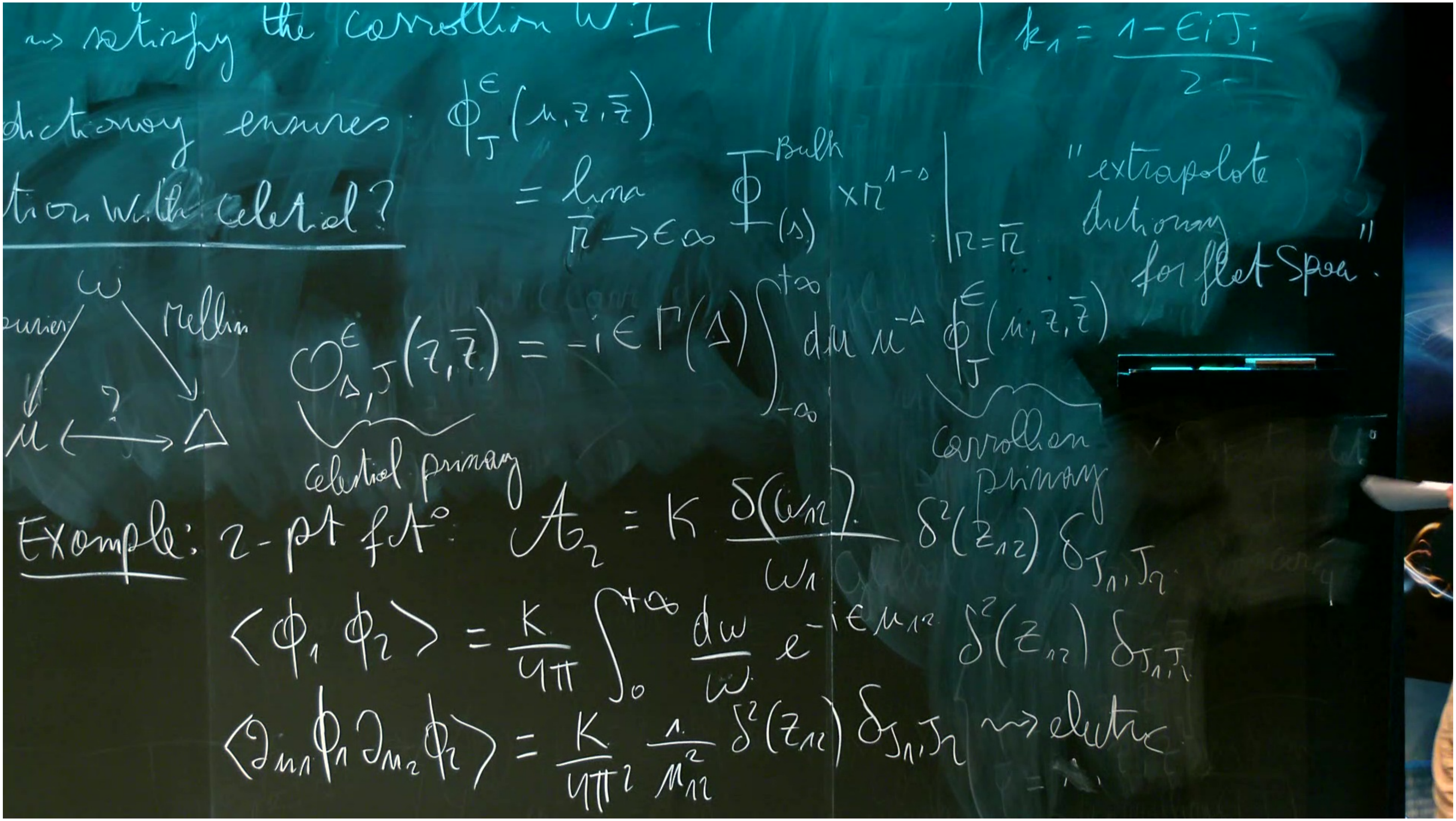
Holographic.

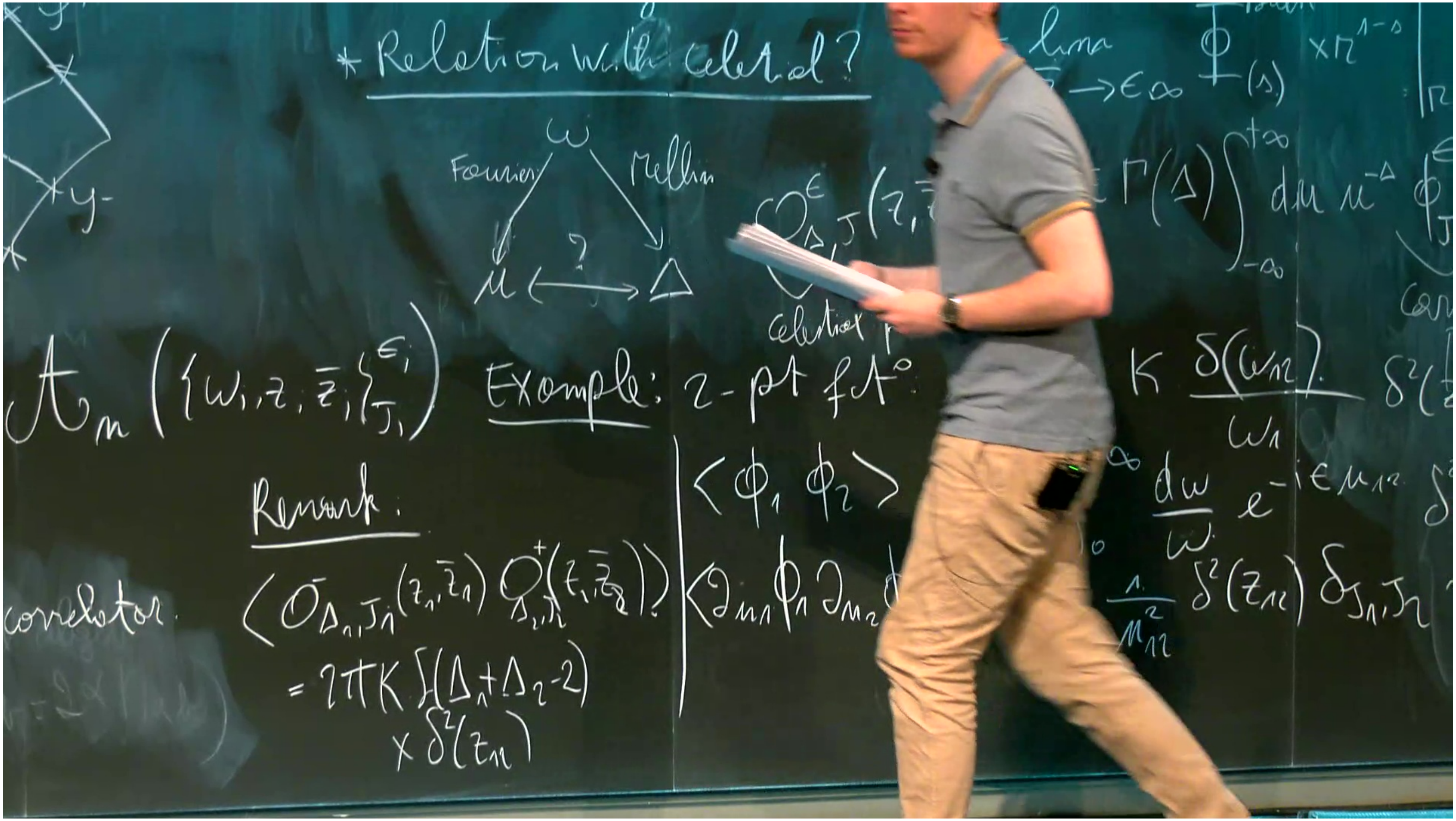
↳ Carrollian CFT correlator.

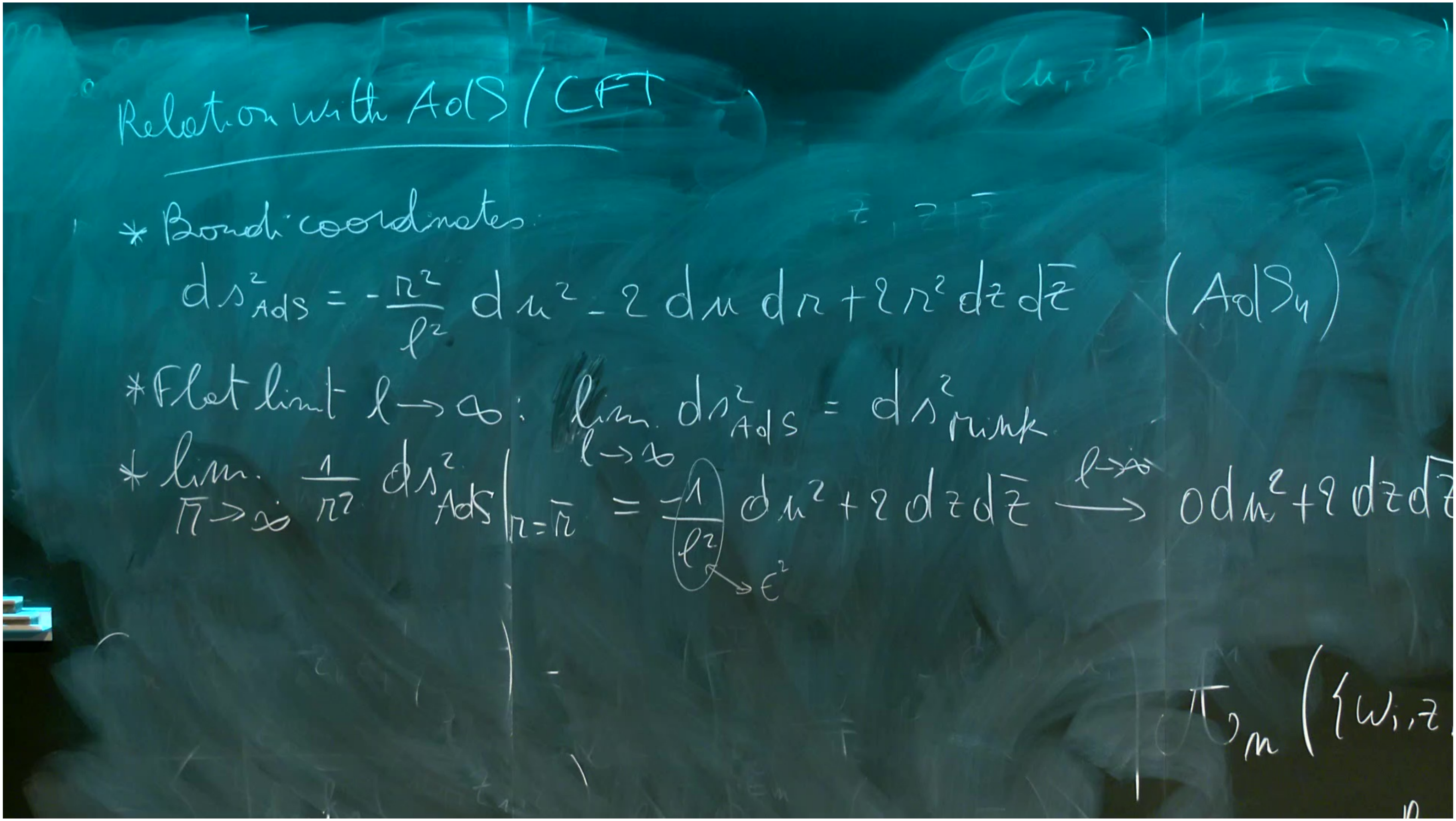

$ds^2 = -2\,du\,dr + 2r^2\,dz\,d\bar{z}$
$r \in \mathbb{R}$
$\mathcal{J}^+$
$r \to +\infty$
$r \to -\infty$
$\mathcal{J}^-$
$\mathcal{A}_n\left(\{\omega_i, z_i, \bar{z}_i\}^{\epsilon_i}_{J_i}\right)$


* This identification makes sense
⟨ ⟩ ⇝ satisfy the Carrollian W.I.
→ true if
$k_i = \frac{1+\epsilon}{\ldots}$
$\bar{k}_i = \frac{1-\epsilon}{\ldots}$
* This dictionary ensures: $\Phi^{\epsilon}_{J}(u,z,\bar{z}) = \lim_{r \to \epsilon\infty} \Phi^{Bulk}_{(s)} \times r^{1-s} \Big|_{r=\bar{r}}$
* Relation with celestial?
ω
Fourier
Mellin
?
$u \longleftrightarrow \Delta$
$\mathcal{O}^{\epsilon}_{\Delta,J}(z,\bar{z}) = -i\epsilon\Gamma(\Delta)\int_{-\infty}^{+\infty} du\, u^{-\Delta}$
celestial primary


lentification makes sense
satisfy the carrollian W.I.
true if
$k_i = \frac{1+\epsilon_i J_i}{2}$
$\bar{k}_1 = \frac{1-\epsilon_i J_i}{2}$
ensures: $\phi^{\epsilon}_{J}(u, z, \bar{z})$
$= \lim_{\bar{r} \to \epsilon\infty} \Phi^{Bulk}_{(s)} \times r^{1-s} \Big|_{r=\bar{r}}$
"extrapolate dictionary for flat Space".
on with celestial?
$\omega$
Mellin
$\Delta$
$\mathcal{O}^{\epsilon}_{\Delta, J}(z, \bar{z}) = -i\epsilon\Gamma(\Delta)\int_{-\infty}^{+\infty} du\, u^{-\Delta} \phi^{\epsilon}_{J}(u, z, \bar{z})$
celestial primary
carrollian primary


⇝ satisfy the Carrollian W.I.
$k_1 = \frac{1-\epsilon_i J_i}{2}$
dictionary ensures: $\phi^{\epsilon}_{J}(u,z,\bar{z})$
$= \lim_{r \to \epsilon\infty} \Phi^{\text{Bulk}}_{(\Delta)} \times r^{1-\Delta} \Big|_{r=\bar{r}}$
"extrapolate dictionary for flat space".
relation with celestial?
$\omega$
Fourier
Mellin
$u \longleftrightarrow \Delta$
?
$\underbrace{\mathcal{O}^{\epsilon}_{\Delta,J}(z,\bar{z})}_{\text{celestial primary}} = -i\epsilon\,\Gamma(\Delta)\int_{-\infty}^{+\infty} du\, u^{-\Delta}\, \underbrace{\phi^{\epsilon}_{J}(u,z,\bar{z})}_{\text{Carrollian primary}}$
Example: 2-pt fct°: $\mathcal{A}_2 = K\,\frac{\delta(\omega_{12})}{\omega_1}\,\delta^2(z_{12})\,\delta_{J_1,J_2}$
$\langle \phi_1 \phi_2 \rangle = \frac{K}{4\pi}\int_0^{+\infty}\frac{d\omega}{\omega}\, e^{-i\epsilon u_{12}}\,\delta^2(z_{12})\,\delta_{J_1,J_2}$
$\langle \partial_{u_1}\phi_1\, \partial_{u_2}\phi_2 \rangle = \frac{K}{4\pi i}\,\frac{1}{u_{12}^2}\,\delta^2(z_{12})\,\delta_{J_1,J_2}$ ⇝ electric


* Relation with Celestial?
Fourier
Mellin
Example: 2-pt fct°:
Remark.
correlator
celestial


Relation with AdS/CFT
* Bondi coordinates:
$ds^2_{AdS} = -\frac{r^2}{\ell^2} du^2 - 2\, du\, dr + 2 r^2 dz\, d\bar{z}$ $(AdS_4)$
* Flat limit $\ell \to \infty$: $\lim_{\ell \to \infty} ds^2_{AdS} = ds^2_{mink}$
* $\lim_{\bar{r} \to \infty} \frac{1}{\bar{r}^2} ds^2_{AdS} \Big|_{r=\bar{r}} = -\frac{1}{\ell^2} du^2 + 2\, dz\, d\bar{z} \xrightarrow{\ell \to \infty} 0\, du^2 + 2\, dz\, d\bar{z}$
$\ell^2 \to \epsilon^2$

* Bondi coordinates:

$$ds^2_{AdS} = -\frac{r^2}{\ell^2}\, du^2 - 2\, du\, dr + 2 r^2\, dz\, d\bar z \qquad (AdS_4)$$

* Flat limit $\ell \to \infty$: $\lim_{\ell\to\infty} ds^2_{AdS} = ds^2_{mink}$

* $$\lim_{r\to\infty} \frac{1}{r^2}\, ds^2_{AdS}\Big|_{r=\bar r} = -\frac{1}{\ell^2}\, du^2 + 2\, dz\, d\bar z \xrightarrow{\ell\to\infty} 0\, du^2 + 2\, dz\, d\bar z$$

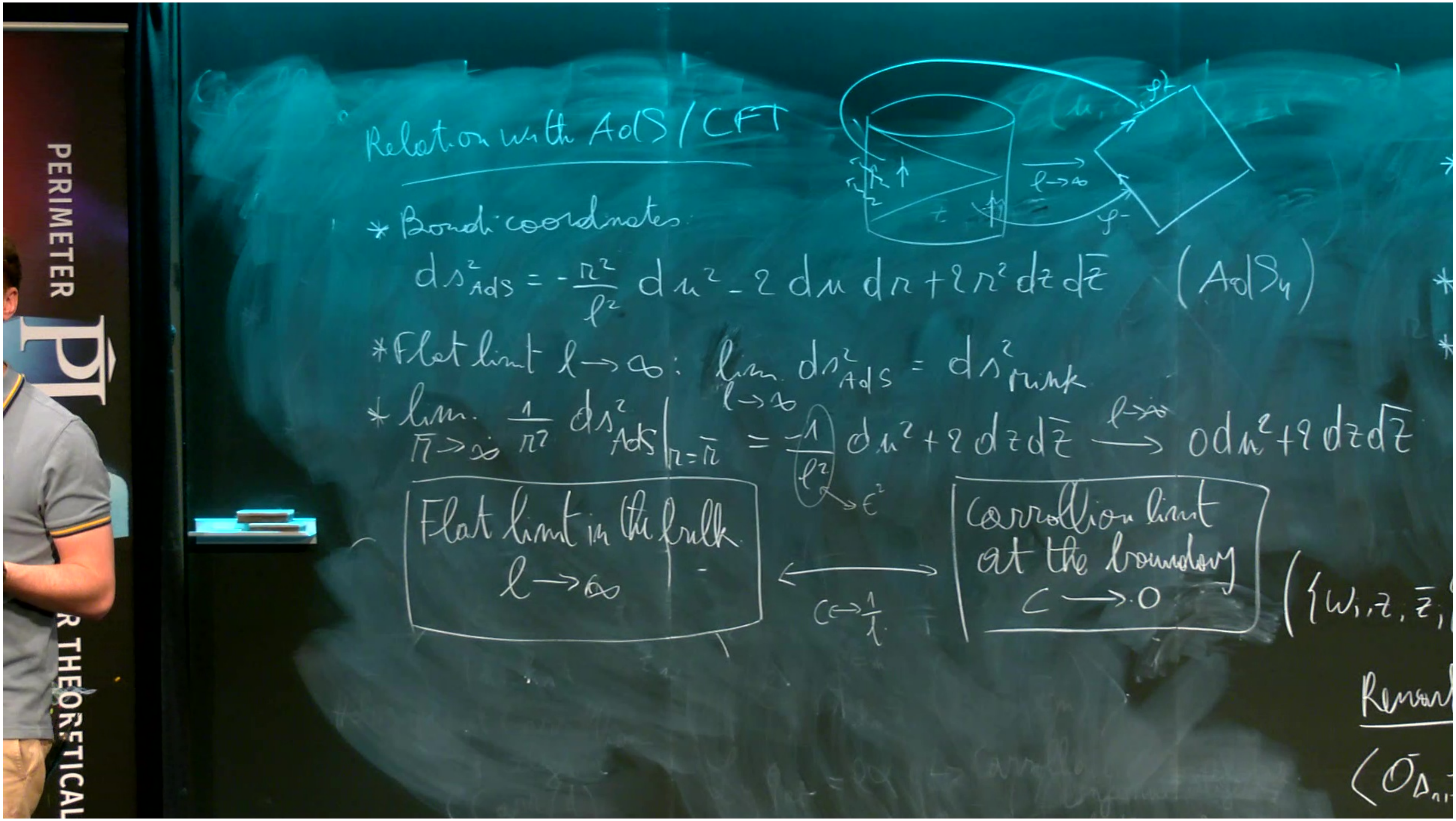
$\ell^2 \to \epsilon^2$

$$\boxed{\text{Flat limit in the bulk } \; \ell \to \infty} \longleftrightarrow \boxed{\text{Carrollian limit at the boundary } \; c \to 0}$$

$(\{w_i, z_i, \bar z_i\}$

Remark

$\langle O_{\Delta, J}$


PERIMETER
Relation with AdS / CFT
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$\ell \to \infty$
$\ell^2 \to \epsilon^2$
Flat limit in the bulk $\ell \to \infty$
Carrollian limit at the boundary $c \to 0$
$c \leftrightarrow \frac{1}{\ell}$