Title: Lecture - IR S-matrix a

Speakers:

Collection: Celestial Holography Summer School 2024

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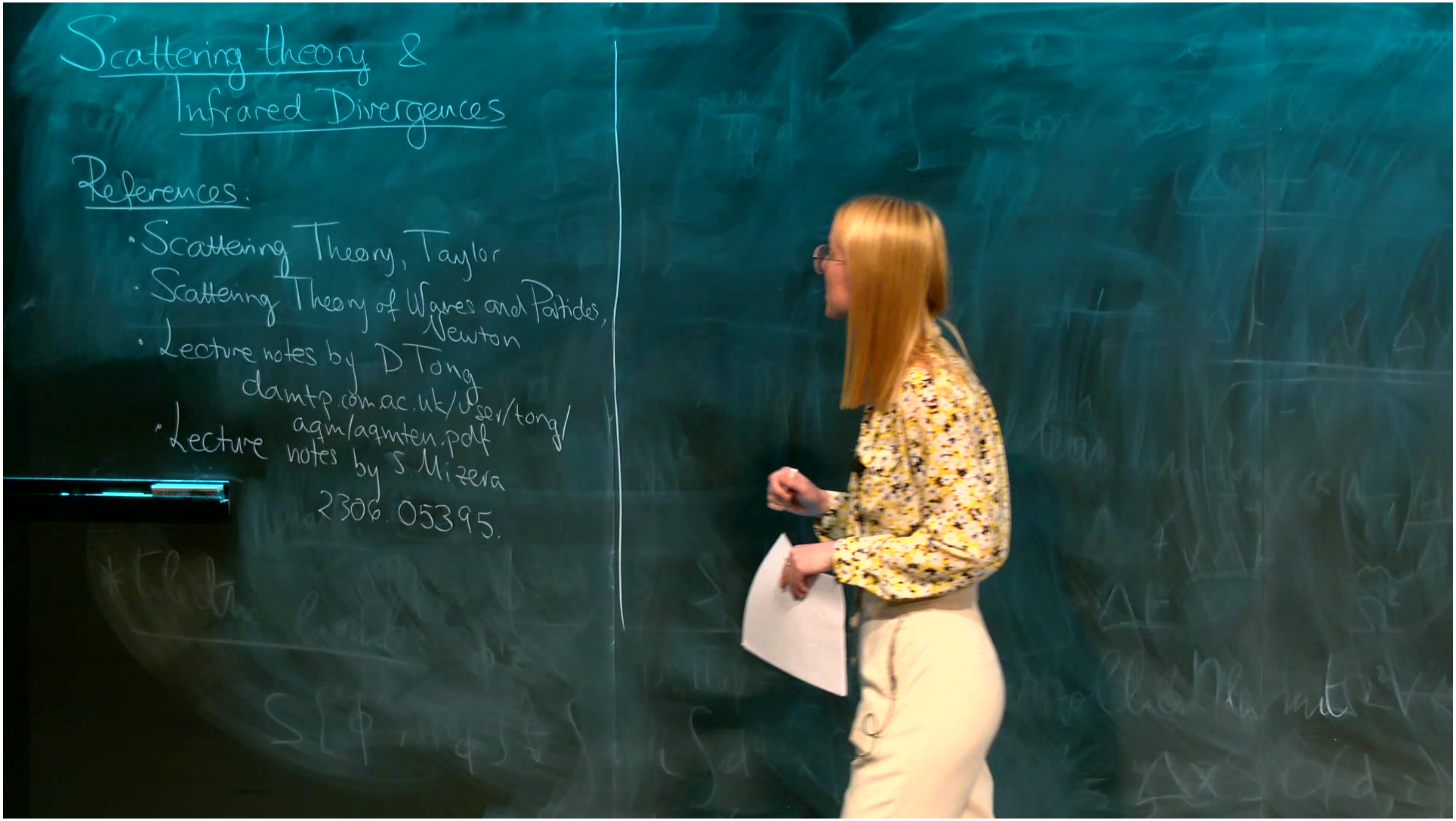

Scattering theory &
Infrared Divergences
References:
· Scattering Theory, Taylor
· Scattering Theory of Waves and Particles, Newton
· Lecture notes by D.Tong
damtp.cam.ac.uk/user/tong/
aqm/aqmten.pdf
· Lecture notes by S. Mizera
2306.05395.

PLAN:

1. IR divergences in Quantum Mechanics
2. — " " — in QFT.

Goal: Understand the physics.

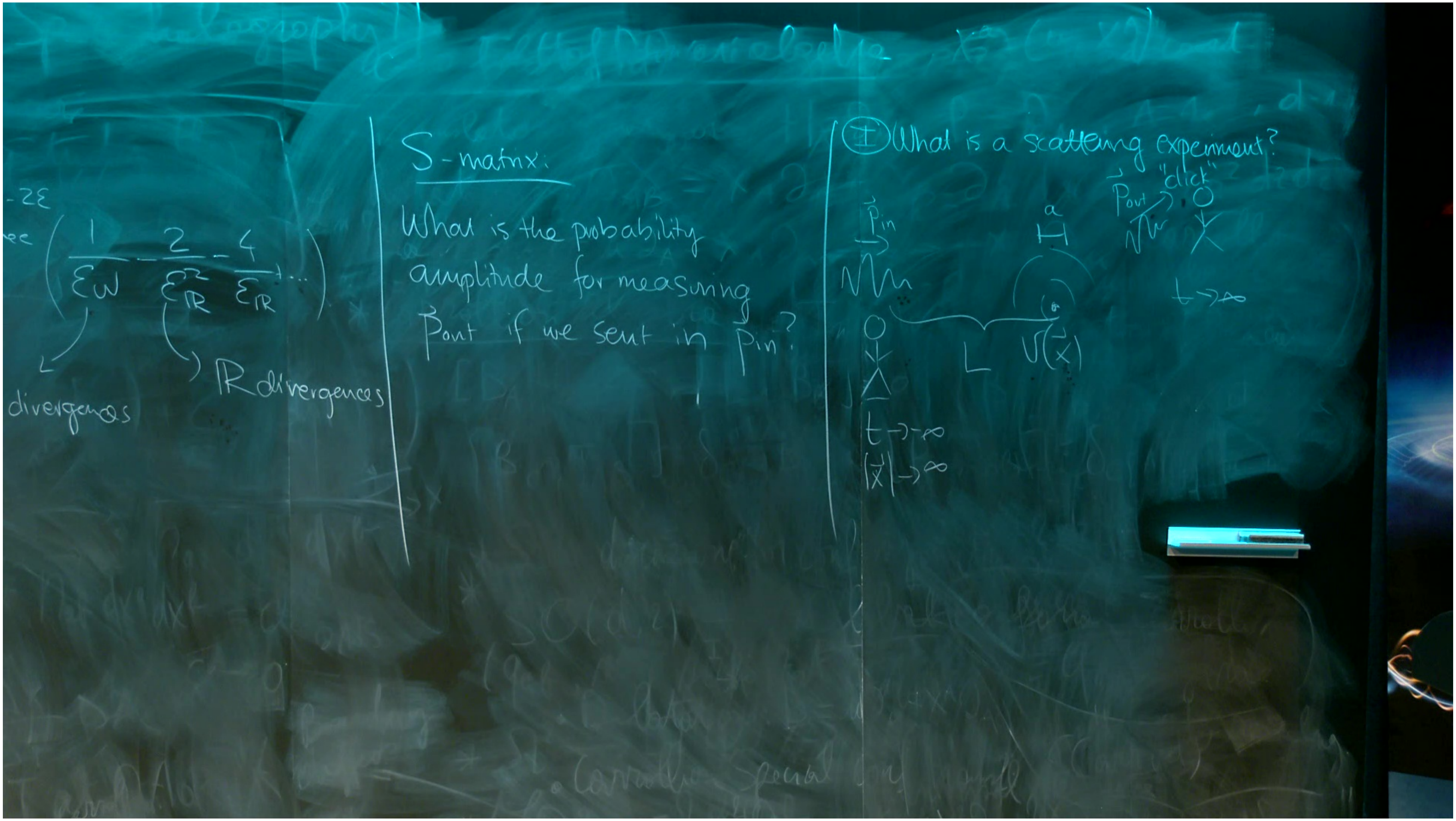
Motivation:


$D = 4 - 2\varepsilon$

$$\propto \mathcal{M}^{\text{tree}} \left( \frac{1}{\varepsilon_{UV}} - \frac{2}{\varepsilon_{IR}^2} - \frac{4}{\varepsilon_{IR}} + \ldots \right)$$

UV divergences

IR divergences



$-2\varepsilon$
$\left(\frac{1}{\varepsilon_{UV}} - \frac{2}{\varepsilon_{IR}^2} - \frac{4}{\varepsilon_{IR}} + \dots\right)$
divergences
IR divergences
S-matrix:
What is the probability amplitude for measuring $\vec{p}_{out}$ if we sent in $\vec{p}_{in}$?
(I) What is a scattering experiment?
$\vec{p}_{in}$
$t \to -\infty$
$|\vec{x}| \to \infty$
a
L
$V(\vec{x})$
$\vec{p}_{out}$
"click"
$t \to \infty$

How do we formalize this?

If $V(\vec{x},t) = V(\vec{x})$ then

$$\Psi(\vec{x},t) = e^{-iEt}\,\psi(\vec{x})$$

$$\left(-\frac{\vec{\nabla}^2}{2m} + V(\vec{x})\right)\psi(\vec{x}) = E\,\psi(\vec{x}).$$

If the potential $V(\vec{x})$ is short ranged then asymptotically $|\vec{x}| \to \infty$:

$$\psi(\vec{x}) \sim e^{\pm i\vec{q}\cdot\vec{x}}, \quad \vec{q} \in \mathbb{C}^3$$

$$E = \frac{\vec{q}^{\,2}}{2m}$$

- Bound states: $\psi(\vec{x}) \sim \dfrac{e^{-\lambda r}}{r}$, $\lambda > 0$, $E = -\dfrac{\lambda^2}{2m} < 0$
- Scattering states: $\psi(\vec{x}) \sim e^{i\vec{p}\cdot\vec{x}} \sim \dfrac{e^{\pm ipr}}{r}$, $\vec{p} \in \mathbb{R}^3$, $p = |\vec{p}|$. $E = \dfrac{p^2}{2m} > 0.$

If the potential $V(\vec{x})$ is short ranged then asymptotically $|\vec{x}| \to \infty$

$$\psi(\vec{x}) \sim e^{\pm i \vec{q} \cdot \vec{x}}, \quad \vec{q} \in \mathbb{C}^3, \quad E = \frac{\vec{q}^2}{2m}$$

- Bound states: $\psi(\vec{x}) \sim \frac{e^{-\lambda r}}{r}$, $r = |\vec{x}|$, $\lambda > 0$, $E = -\frac{\lambda^2}{2m} < 0$

- Scattering states: $\psi(\vec{x}) \sim e^{i \vec{p} \cdot \vec{x}} \sim \frac{e^{\pm ipr}}{r}$, $\vec{p} \in \mathbb{R}^3$, $p = |\vec{p}|$, $E = \frac{p^2}{2m} > 0$.

Define:

- "In" wavefunctions:

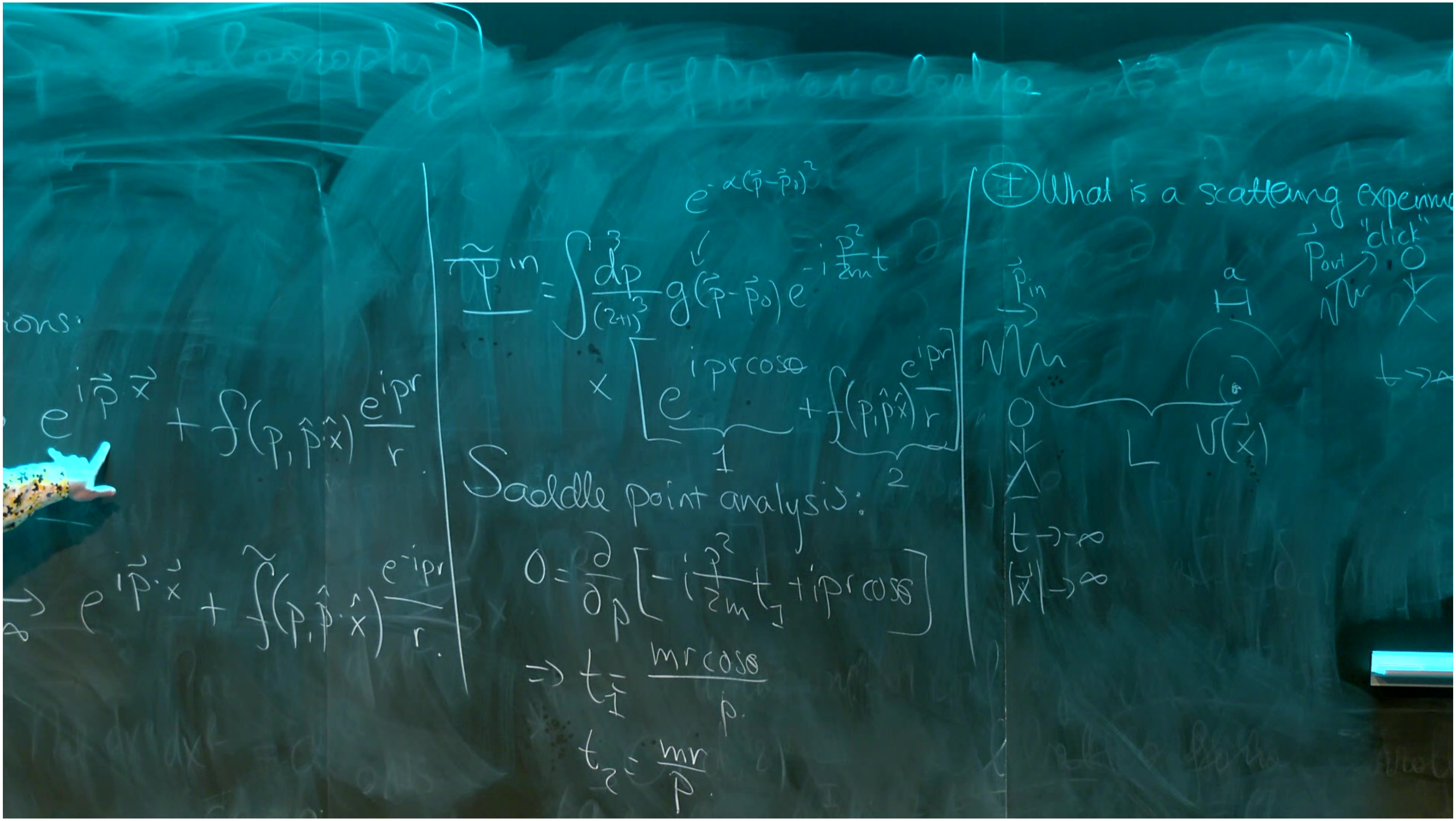
$$\psi^{in} \xrightarrow[r = |\vec{x}| \to \infty]{} e^{i \vec{p} \cdot \vec{x}} + f(p, \hat{p} \cdot \hat{x}) \frac{e^{ipr}}{r}$$

- "Out" - " -

$$\psi^{out} \xrightarrow[r \to \infty]{} e^{i \vec{p} \cdot \vec{x}} + \tilde{f}(p, \hat{p} \cdot \hat{x}) \frac{e^{-ipr}}{r}.$$

S-matrix:

What is the amplitude $\vec{p}_{out}$ if we


ions:
$e^{i\vec{p}\cdot\vec{x}} + f(p,\hat{p}\cdot\hat{x})\frac{e^{ipr}}{r}$
$\rightarrow e^{i\vec{p}\cdot\vec{x}} + \tilde{f}(p,\hat{p}\cdot\hat{x})\frac{e^{ipr}}{r}$
$e^{-\alpha(\vec{p}-\vec{p}_0)^2}$
$\tilde{\Psi}^{in} = \int \frac{d^3p}{(2\pi)^3} g(\vec{p}-\vec{p}_0) e^{-i\frac{p^2}{2m}t}$
$\times \left[ \underbrace{e^{iprcos\theta}}_{1} + \underbrace{f(p,\hat{p}\cdot\hat{x})\frac{e^{ipr}}{r}}_{2} \right]$
Saddle point analysis:
$0 = \frac{\partial}{\partial p}\left[ -i\frac{p^2}{2m}t_I + iprcos\theta \right]$
$\Rightarrow t_I = \frac{mrcos\theta}{p}$
$t_2 = \frac{mr}{p}$
I What is a scattering experim
$\vec{p}_{in}$
a
$\vec{p}_{out}$
"click"
L
$V(\vec{x})$
$t \to -\infty$
$|\vec{x}| \to \infty$
$t \to \infty$

$$+ f(p, \hat{p}\cdot\hat{x}) \frac{e^{ipr}}{r}$$

$$\vec{p}\cdot\vec{x} + \tilde{f}(p, \hat{p}\cdot\hat{x}) \frac{e^{-ipr}}{r}$$

$$\tilde{\Psi}^{in} = \int \frac{d^3p}{(2\pi)^3}\, g(\vec{p}-\vec{p}_0)\, e^{-i\frac{p^2}{2m}t}$$

$$g = e^{-\alpha(\vec{p}-\vec{p}_0)^2}$$

$$\times \Big[ \underbrace{e^{iprcos\theta}}_{1} + \underbrace{f(p, \hat{p}\cdot\hat{x}) \frac{e^{ipr}}{r}}_{2} \Big]$$

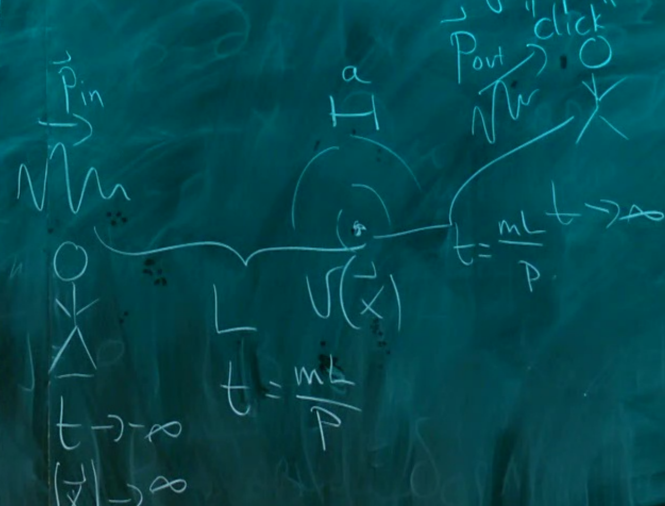
Saddle point analysis:

$$0 = \frac{\partial}{\partial p}\left[-i\frac{p^2}{2m}t_1 + iprcos\theta\right]$$

$$\Rightarrow t_1 = \frac{mr\cos\theta}{p}$$

$$t_2 = \frac{mr}{p}$$

(I) What is a scattering experiment?



$$S_{\vec{p}_{in},\vec{p}_{out}} = \langle \psi^{out} | \psi^{in} \rangle.$$

What if we have a Coulomb potential?

Define:

- "In" wavefunctions:

$$\psi^{in} \xrightarrow[r=|\vec{x}|\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + f(p, \hat{p}\cdot\hat{x}$$

- "Out" – " " –

$$\psi^{out} \xrightarrow[r\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + \tilde{f}(p, \hat{p}\cdot$$

$$S_{\vec{p}_{in},\vec{p}_{out}} = \langle \psi^{out} | \psi^{in} \rangle.$$

What if we have a Coulomb potential?

$$\left( -\frac{\vec{\nabla}^2}{2m} + \frac{Z\alpha}{r} \right) \psi(\vec{x}) = E\,\psi(\vec{x}) \qquad \alpha = e^2/4\pi$$

$$\Rightarrow \left( \vec{\nabla}^2 + 2mE - \frac{2\gamma}{r} \right) \psi(\vec{x}) = 0.$$

Define:

- "In" wavefunctions:

$$\psi^{in} \xrightarrow[r=|\vec{x}|\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + f(p, \hat{p}\cdot\hat{x}$$

- "Out" — " " —

$$\psi^{out} \xrightarrow[r\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + \tilde{f}(p, \hat{p}\cdot$$

tential?

$\psi(\vec{x})$

0,

Zmα.

Define:

· "In" wavefunctions:

$$\psi^{in} \xrightarrow[r=|\vec{x}|\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + f(p,\hat{p}\cdot\hat{x})\frac{e^{ipr}}{r}$$

· "Out" - " " —

$$\psi^{out} \xrightarrow[r\to\infty]{} e^{i\vec{p}\cdot\vec{x}} + \tilde{f}(p,\hat{p}\cdot\hat{x})\frac{e^{-ipr}}{r}.$$

Asymptotically:

$$\psi(\vec{x}) \sim \frac{e^{\pm ipr + g(r)}}{r}$$

Schr. eq

$$\Rightarrow \frac{d^2g}{dr^2} + \left(\frac{dg}{dr}\right)^2 \pm 2ip\frac{dg}{dr} = \frac{2\gamma}{r}$$

$$\Rightarrow g(r) \sim \mp i\frac{\gamma}{p}\log(pr).$$

$$\psi(\vec{x}) \sim \frac{e^{\pm ipr \mp i\frac{\gamma}{p}\log(pr)}}{r}$$

Full solution:

$$\psi^{in}(\vec{x}) = e^{i\vec{p}\cdot\vec{x}}\, e^{-\pi\frac{\gamma}{2p}}\, \Gamma\left(1+\frac{i\gamma}{p}\right)\, {}_1F_1\left(-i\gamma, 1, i(pr-\vec{p}\cdot\vec{x})\right)$$

Define:

- "In" wavefunctions:

$$\psi^{in} \xrightarrow[r=|\vec{x}|\to\infty]{} e^{i\vec{p}\cdot\vec{x}+i\frac{\gamma}{p}\log(pr)} + f(p,\hat{p}\cdot\hat{x})\frac{e^{ipr-i\frac{\gamma}{p}\log(pr)}}{r}$$

- "Out" – " "

$$\psi^{out} \xrightarrow[r\to\infty]{} e^{i\vec{p}\cdot\vec{x}\cdots} + \tilde{f}(p,\hat{p}\cdot\hat{x})\frac{e^{-ipr\cdots}}{r}$$

# Define

• "In" wavefunctions:

$$\psi^{in} \xrightarrow[r=|\vec{x}|\to\infty]{} e^{i\vec{p}\cdot\vec{x}+i\frac{\gamma}{p}\log(pr)} + f(p,\hat{p}\cdot\hat{x})\,\frac{e^{ipr-\frac{i\gamma}{p}\log(pr)}}{r}$$

• "Out" - " "

$$\psi^{out} \xrightarrow[r\to\infty]{} e^{i\vec{p}\cdot\vec{x}\cdots} + \tilde{f}(p,\hat{p}\cdot\hat{x})\,\frac{e^{-ipr\cdots}}{r}$$

$$\langle\psi^{out}|\psi^{in}\rangle = -\frac{\gamma}{p^2}\,\frac{\Gamma\left(1+\frac{i\gamma}{p}\right)}{\Gamma\left(1-\frac{i\gamma}{p}\right)}\,\frac{1}{2\sin^2\frac{\theta}{2}} \times e^{-i\frac{\gamma}{p}\log\left(\sin^2\theta/2\right)}$$

## Key point:

We can envision two types of experiments

- probe long-range effects
- agnostic to long-range effects
  e.g. total cross section $\propto |f|^2$

(I) What is ...

$\vec{p}_{in}$

L

$t\to-\infty$

$|\vec{x}|\to\infty$

t

## Time delay:

$$0 = \frac{\partial}{\partial p}\left[ipr - \frac{i\gamma}{p}\log\left[pr(1-\cos\theta) - i\frac{p^2}{2m}t\right]\right]$$

$$\Rightarrow t = \frac{m}{p}r + \frac{m\gamma}{p^3}\log\left[pr(1-\cos\theta)\right]$$

## What if we don't have the exact solution?

Perturbation theory:

$\vec{p}$ $\vec{p}\,'$ $= \ \downarrow\vec{k} + \ \ldots + \ldots$

$f(\vec{p},\vec{p}\,')$ $\quad V(\vec{k})$

$\downarrow\vec{k}$ $\quad V(\vec{k}) = \frac{4\pi\gamma/m}{k^2}$

$\vec{k}$ $\quad G(p,k) = \frac{1}{p^2-k^2+i\epsilon}$

## Define:

- "In" wavefunc

$\psi^{in} \xrightarrow{r=|\vec{x}|\to\infty}$

- "Out" −

$\psi^{out} \xrightarrow{r\to}$

$\langle\psi^{out}|\psi^{in}\rangle$

## Born series

$$f^{LO} \propto \frac{-2\gamma}{(\vec{p}-\vec{p}\,')^2} = -\frac{\gamma}{p^2(1-\cos\theta)}$$

$$f^{NLO} \propto \int \frac{d^3\vec{k}}{(2\pi)^3}\, \frac{1}{\left[(\vec{k}-\vec{p})^2\right]\left[p^2-k^2+i\varepsilon\right]\left[(\vec{p}\,'-\vec{k})^2\right]}$$

$$\tilde{k} \sim \vec{p} \qquad = \infty$$

$$\Psi(\vec{x}) \sim e^{i\vec{p}\cdot\vec{x} + \frac{i\gamma}{p}\log(pr\ldots)} + \hat{f}(p,\ldots)\,\frac{e^{ipr + \frac{i\gamma}{p}\log(pr\ldots)}}{r}$$

If we blindly expand in $\gamma$

$$f(p,\cos\theta) \stackrel{?}{=} \frac{-\gamma}{p^2(1-\cos\theta)} + \frac{i\gamma^2}{p^3(1-\cos\theta)}\left(\ldots + \log(pr)\right)$$

① What is a s

$\vec{p}_{in}$

$t \to -\infty$ $\quad t = \frac{m}{p}$

$|\vec{x}| \to \infty$

L

## We have to regulate!

0) $\frac{i\gamma^2}{p^2(1-\cos\theta)}\log(pr)$ as $r\to\infty$

1)

2)

## What if we don't have the exact solution?

Perturbation theory: $\boxed{V(r) \propto \frac{\gamma}{r}e^{-mr}}$

$\vec{p} \to \vec{p}\,'$ $= \quad \vec{k} \quad + \quad + \dots$

$f(\vec{p},\vec{p}\,') \qquad V(\vec{k})$

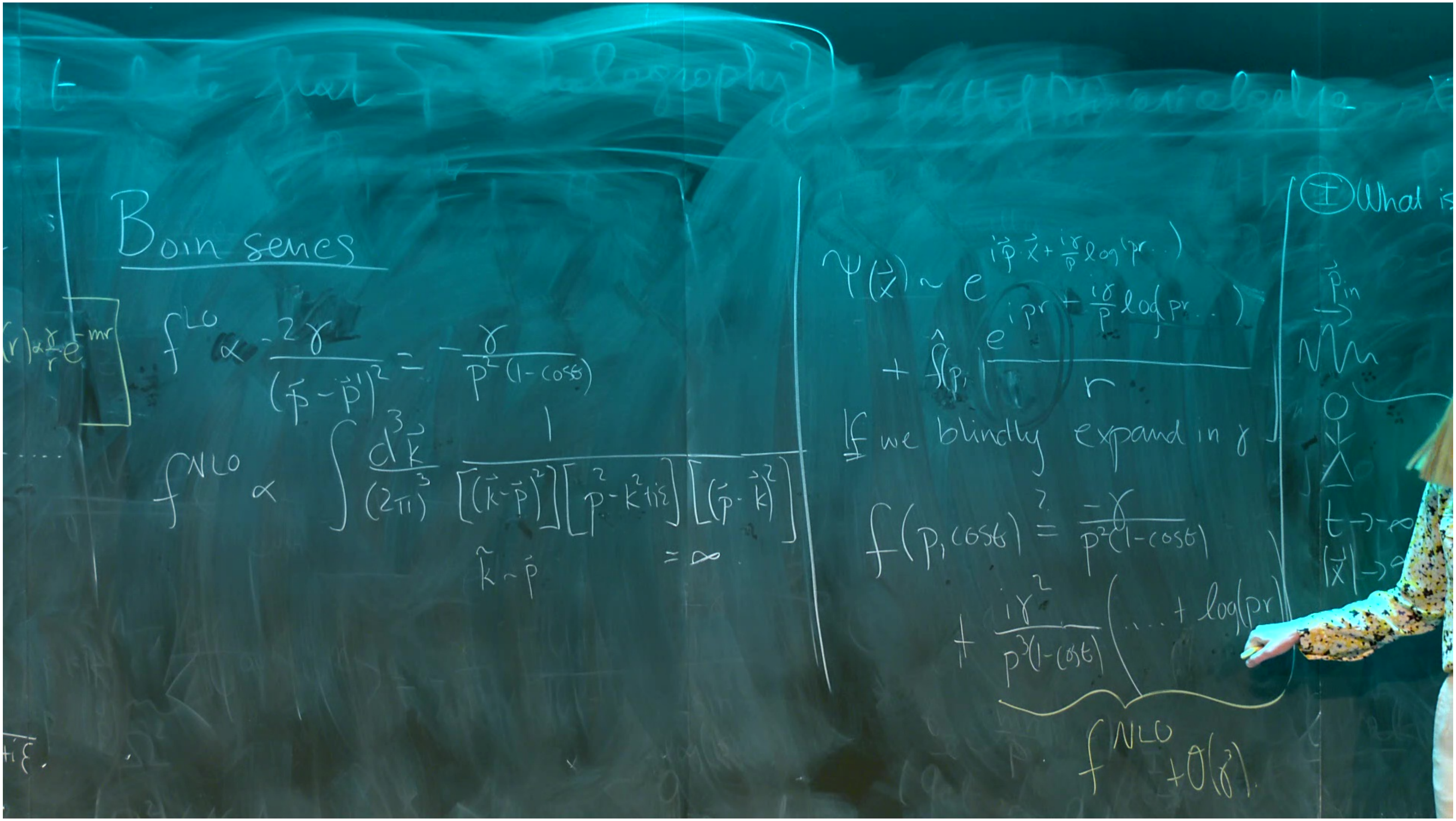
$\vec{k} \qquad V(\vec{k}) = \frac{4\pi\gamma/m}{\vec{k}^2+m^2}$

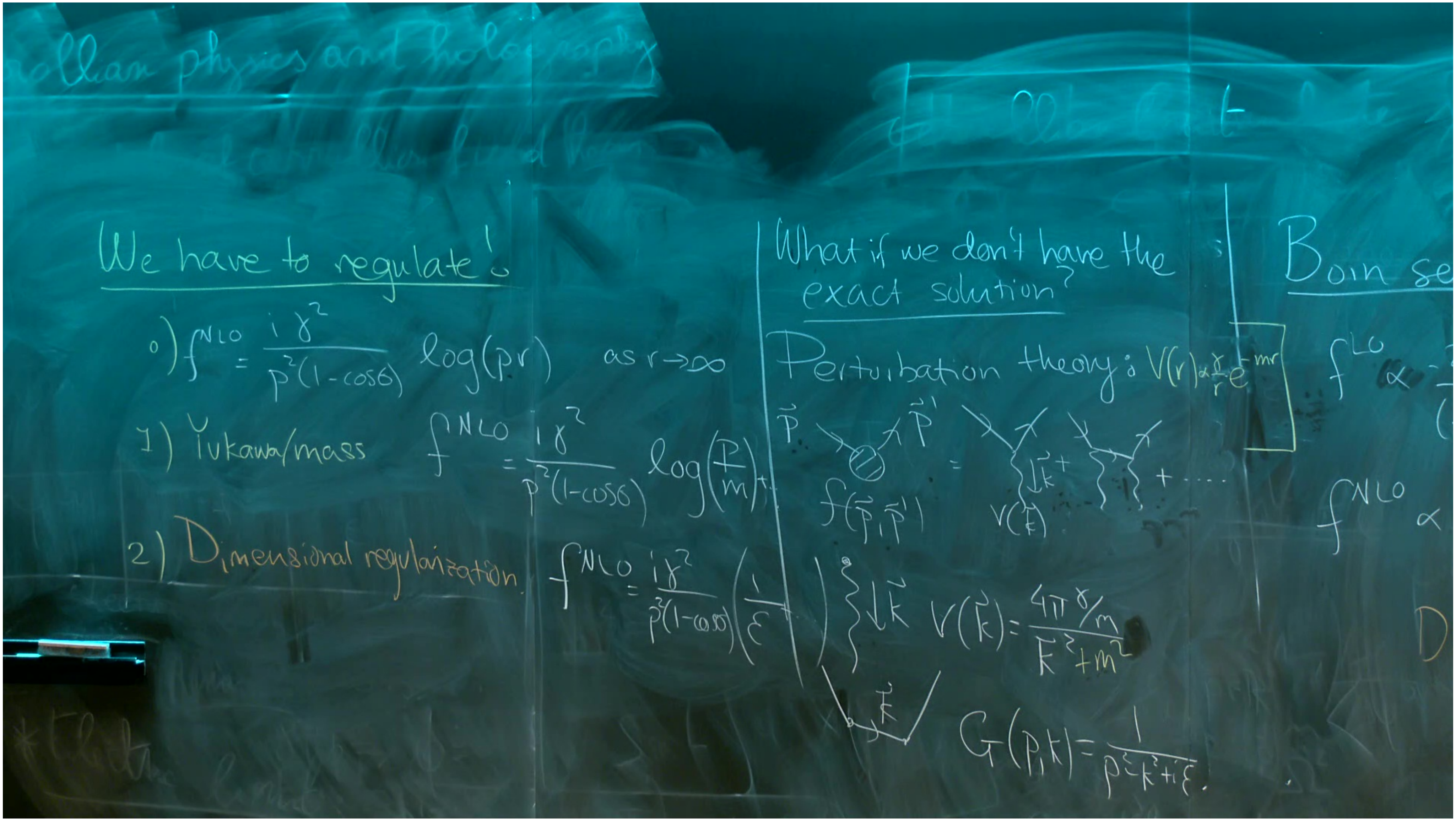
$\vec{k} \qquad G(p,k) = \frac{1}{p^2-k^2+i\epsilon}$

## Born series

$f^{LO} \propto -\frac{2\gamma}{(\vec{p}-}$

$f^{NLO} \propto$

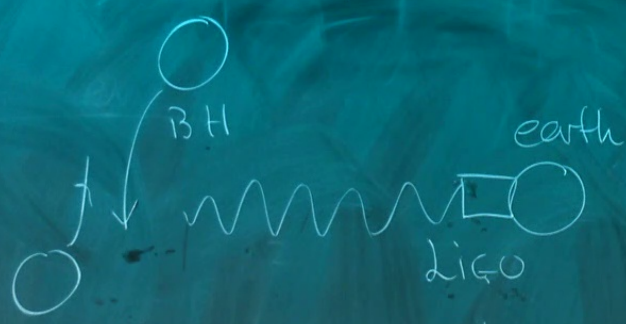

Born series
$V(r) \propto \frac{\gamma}{r} e^{-mr}$
$f^{LO} \propto -\frac{2\gamma}{(\vec{p}-\vec{p}')^2} = -\frac{\gamma}{p^2(1-\cos\theta)}$
$f^{NLO} \propto \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{[(\vec{k}-\vec{p})^2][p^2-k^2+i\varepsilon][(\vec{p}'-\vec{k})^2]} = \infty$
$\vec{k} \sim \vec{p}$
$\psi(\vec{x}) \sim e^{i\vec{p}\cdot\vec{x} + \frac{i\gamma}{p}\log(pr \ldots)}$
$+ \hat{f}(p, \ldots) \frac{e^{ipr + \frac{i\gamma}{p}\log(pr \ldots)}}{r}$
If we blindly expand in $\gamma$
$f(p, \cos\theta) \overset{?}{=} \frac{-\gamma}{p^2(1-\cos\theta)}$
$+ \underbrace{\frac{i\gamma^2}{p^3(1-\cos\theta)}(\ldots + \log(pr))}_{f^{NLO}} + O(\gamma^3)$
① What is
$\vec{p}_{in}$
$t \to -\infty$
$|\vec{x}| \to \infty$


We have to regulate!
0) $f^{NLO} = \frac{i\gamma^2}{p^2(1-\cos\theta)} \log(pr)$ as $r\to\infty$
1) Yukawa/mass $f^{NLO} = \frac{i\gamma^2}{p^2(1-\cos\theta)} \log\left(\frac{p}{m}\right)+$
2) Dimensional regularization $f^{NLO} = \frac{i\gamma^2}{p^2(1-\cos\theta)}\left(\frac{1}{\varepsilon}+\right)$
What if we don't have the exact solution?
Perturbation theory: $V(r) \propto \frac{\gamma}{r} e^{-mr}$
$\vec{p}$ $\vec{p}'$
$f(\vec{p},\vec{p}')$
$\vec{k}$
$V(\vec{k})$
$V(\vec{k}) = \frac{4\pi\gamma/m}{\vec{k}^2+m^2}$
$G(p,k) = \frac{1}{p^2-k^2+i\varepsilon}$
Born se
$f^{LO} \propto$
$f^{NLO} \propto$

$\boxed{V(r) \propto \frac{1}{r} e^{-mr}}$

$+ \cdots$

$\xi + i\xi$




BH $\langle \Psi_{BH} | S^\dagger a^\dagger S | \Psi_{BH} \rangle$

Two sources of Shapiro time delay:

① Graviton escaping the BH potential

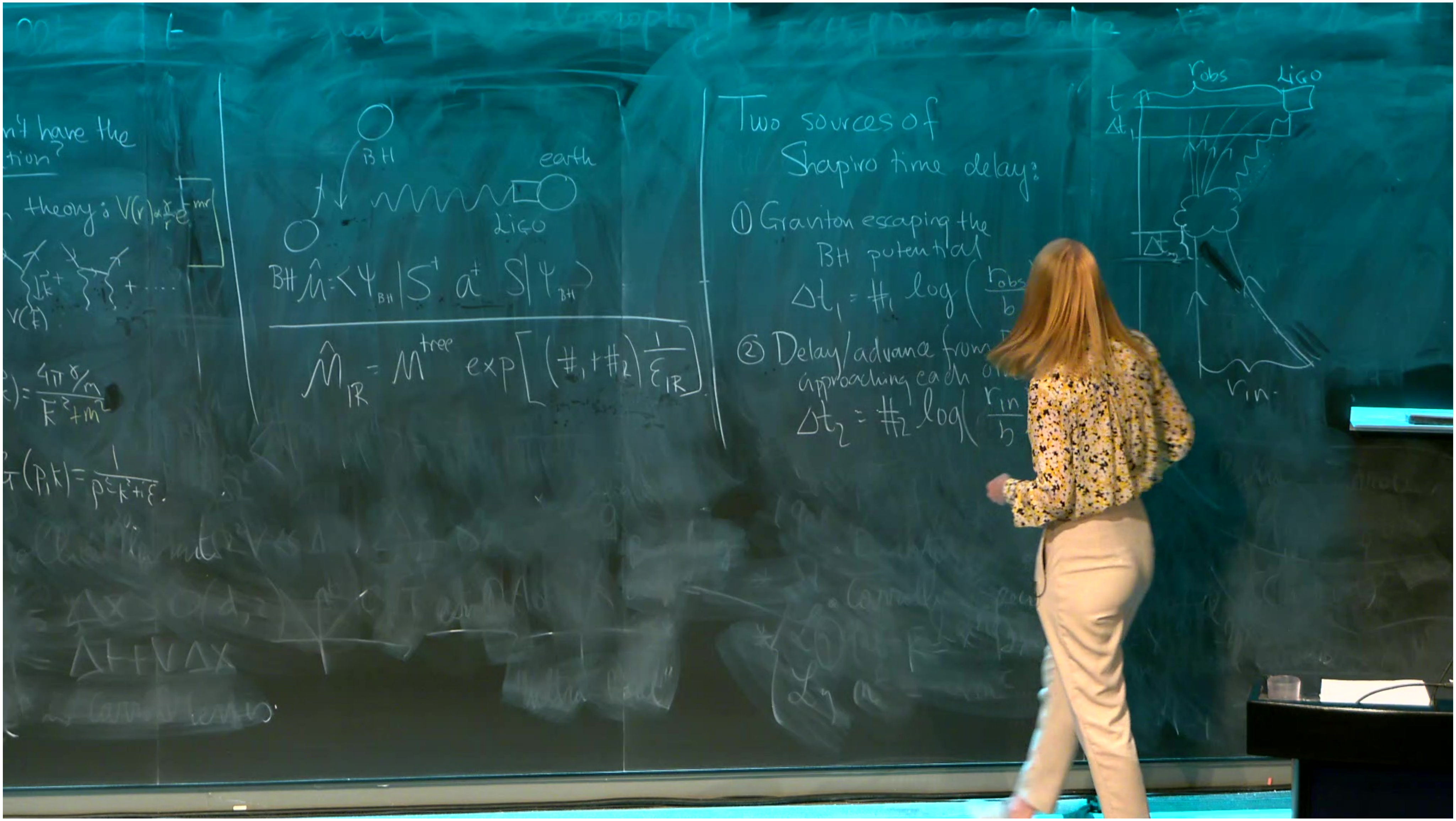
$$\Delta t_1 = \#_1 \log\left(\frac{r_{obs}}{b}\right)$$

(I) What

$\vec{p}_{in}$

$t \to -\infty$

$|\vec{x}| \to \infty$


Two sources of
Shapiro time delay:
① Graviton escaping the
BH potential
② Delay/advance from
approaching each
BH
earth
LIGO
tree
$r_{obs}$
$r_{in}$