Title: A Novel Perspective on the Continuum Limit in Quantum Gravity

Speakers: Susanne Schander

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Abstract: Some of the most fundamental challenges in quantum gravity involve determining how to take the continuum limit of the underlying regularized theory and how to preserve the causal structure of space-time. Several approaches to quantum gravity attempt to address these questions, but the technical challenges are substantial.

In this talk, we present a novel approach to a lattice-regularized theory of quantum gravity, using techniques from standard lattice quantum field theories to overcome these challenges. Our approach is inspired by quantum geometrodynamics, the earliest attempt at quantizing gravity. While the original approach suffered from the usual shortcomings pertaining to the multiplication of distributions and consequently failed, we propose a novel lattice regularization that is especially well suited for studying the continuum limit. First, we examine the lattice corrections to the theory and quantize these lattice theories in a manner that ensures the manifest causal structure of space-time. Next, we discuss the constructions involved in obtaining a well-defined continuum limit and explain how our approach can overcome some conceptually unappealing aspects.

Zoom link

A Novel Perspective on the Continuum Limit in Quantum Gravity

Susanne Schander

PERIMETER INSTITUTE

PI Quantum Gravity Seminar
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in collaboration with Thorsten Lang

arXiv:2305:09650, arXiv:2305.10097, arXiv:2311.00245
and forthcoming publications.

What is Quantum Geometrodynamics?

Classical Basics

- Earliest approach to the quantization of general relativity
(DeWitt '67, Arnowitt et al. '62)
- Start from classical Hamiltonian formulation
- Canonical variables:
Spatial metric $q_{ab}(x)$ and conjugate momentum $p^{ab}(x)$
- First class system of Hamiltonian and diffeomorphism constraints:

$$\mathcal{H} = \frac{1}{\sqrt{q}}\left(q_{ac}q_{bd} - \frac{1}{n-1}q_{ab}q_{cd}\right)p^{ab}p^{cd} - \sqrt{q}R,$$

$$\mathcal{D}_a = -2D_b p^b{}_a$$

- Hamiltonian fully constrained

What is Quantum Geometrodynamics?

Quantization

- Naive canonical quantization:

$$\hat{q}_{ab}(x)\Psi[q_{ab}] = q_{ab}(x)\Psi[q_{ab}], \qquad \hat{p}^{ab}(x)\Psi[q_{ab}] = -\mathrm{i}\frac{\delta\Psi[q_{ab}]}{\delta q_{ab}(x)}$$

- Implementation of constraints in the quantum theory:

$$\mathcal{H}(\hat{q},\hat{p})\Psi = 0 \qquad \mathcal{D}_a(\hat{q},\hat{p})\Psi = 0$$

What is Quantum Geometrodynamics?

Open Questions

- How can we make sense of non–linear functions such as $\mathcal{H}(\hat{q}, \hat{p})$ of operator–valued distributions $\hat{q}_{ab}(x)$ and $\hat{p}^{ab}(x)$?
- What Hilbert space do the wave–functionals $\Psi[q_{ab}]$ belong to?
- How can we enforce that $\hat{q}_{ab}(x)s^a s^b$ is a positive operator for all s?

Failure to address these and other issues led to the abandonment of quantum geometrodynamics
(Kiefer '07, Isham '91)

Other approaches

... and led to the birth of alternative approaches:

- Canonical LQG (Ashtekar, Dittrich, Lewandowski, Pullin, Rovelli, Sahlmann, Smolin, Thiemann, Varadarajan,...)
- Spin foams (Bianchi, Dittrich, Dupuis, Engle, Freidel, Girelli, Han, Livine, Perez, Rovelli, Speziale,...)
- Causal dynamical triangulations (Ambjorn, Loll, Jurkiewicz,...)
- ...

Common theme: Reformulate the theory and then adopt lattice regularizations in order to gain non–perturbative control

A lattice regularization in the original ADM variables has never been investigated!

Overview

2. Forward Solutions

2.1. A Regularization Scheme

2.1 Forward Solutions – A Regularization Scheme

General Idea

Regularization

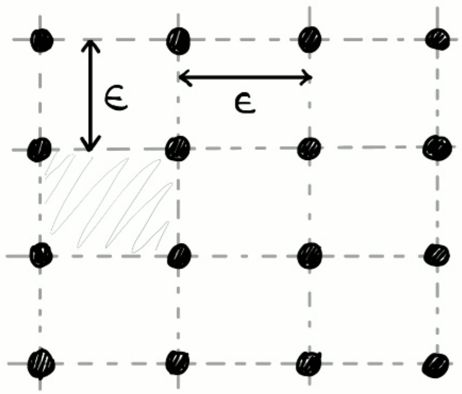
- IR: Torus as spatial manifold
- UV: Restrict phase space of classical geometrodynamics to piecewise constant fields on a cubic lattice
- Replace derivatives ∂_a by finite differences Δ_a

Implementation

- Evaluate constraints on restricted phase space
- Compute lattice corrections to constraint algebra
- Quantize and study continuum limit

2.1 Forward Solutions – A Regularization Scheme

Example in two spatial dimensions



Restrict phase space of field variables $q_{ab}(x,y), p^{cd}(x,y)$ to piecewise constant fields, e.g.:

$$q_{ab}(x) = \sum_{X,Y=1}^{N} q_{ab}^{XY} \chi_{XY}(x)$$

- Associate lattice degrees of freedom q_{ab}^{XY} to the lattice site (X, Y)
- Lattice degrees of freedom inherit Poisson bracket algebra from continuum fields:

$$\left\{ q_{ab}^{X_1 Y_1}, p^{cd}_{X_2 Y_2} \right\} = \frac{1}{\epsilon^2} \delta_a^{(c} \delta_b^{d)} \delta_{X_1}^{X_2} \delta_{Y_1}^{Y_2}$$

- Torus regularization implies periodic boundary conditions

2.1 Forward Solutions – A Regularization Scheme

Evaluation of the constraints on the restricted phase space yields lattice regularized constraints:

$$\mathcal{H}[N] = \epsilon^2 \sum_{XY} N^{XY} \left(\frac{1}{\sqrt{q}} \left(q_{ac} q_{bd} - \frac{1}{n-1} q_{ab} q_{cd} \right) p^{ab} p^{cd} - \sqrt{q} R \right)^{XY}$$

$$\mathcal{D}_a[N^a] = \epsilon^2 \sum_{XY} N^a_{XY} \left(-2\Delta_b (q_{ac} p^{cb}) + (\Delta_a q_{bc}) p^{bc} \right)^{XY}$$

Note: Chain rule for finite differences acquires extra term proportional to lattice constant

$\Rightarrow$ necessity of a choice regarding the order of execution

2.1 Forward Solutions – A Regularization Scheme

Constraint algebra on the lattice acquires extra terms:

$$\left\{\mathcal{D}[\vec{M}],\mathcal{D}[\vec{N}]\right\}=\mathcal{D}[\mathcal{L}_{\vec{M}}\vec{N}]+\epsilon\,A_{\mathcal{D}\mathcal{D}}(\vec{M},\vec{N}),$$
$$\left\{\mathcal{D}(\vec{N}),\mathcal{H}[N]\right\}=\mathcal{H}(\mathcal{L}_{\vec{N}}N)+\epsilon\,A_{\mathcal{D}\mathcal{H}}(\vec{N},N),$$
$$\{\mathcal{H}[M],\mathcal{H}[N]\}=\mathcal{D}[F(q,M,N)]+\epsilon\,A_{\mathcal{H}\mathcal{H}}(M,N)$$

- First class property broken
- Unphysical degrees of freedom
- Suppressed on fine lattices $\epsilon \to 0$

Hint for continuum limit: Tune the limit such that long time evolutions are matched with sufficiently fine lattice spacings in order to control the deviation from the constraint surface

2. Forward Solutions

2.2. Quantum Theory with Pos.–Def. Metric

2.2 Forward Solutions – Quantum Theory

Standard Schrödinger Representation

$$(\hat{q}^{XY}_{ab}\psi)(q) = q^{XY}_{ab}\psi(q)$$
$$(\hat{p}^{ab}_{XY}\psi)(q) = -i\frac{\partial}{\partial q^{XY}_{ab}}\psi(q)$$

with $\psi(q) \in \mathcal{H} = L^2\left(\mathbb{R}^3, \mathrm{d}q_{ab}\right)$ for each lattice site (X, Y)

Satisfy standard commutation relations

$$\left[\hat{q}^{X_1Y_1}_{ab}, \hat{p}^{cd}_{X_2Y_2}\right] = \frac{i}{\epsilon^2}\delta^{(c}_{a}\delta^{d)}_{b}\delta^{X_2}_{X_1}\delta^{Y_2}_{Y_1},$$
$$\left[\hat{q}^{X_1Y_1}_{ab}, \hat{q}^{X_2Y_2}_{cd}\right] = \left[\hat{p}^{ab}_{X_1Y_1}, \hat{p}^{cd}_{X_2Y_2}\right] = 0$$

States can have support on non–positive definite metrics – causal structure lost!

2.2 Forward Solutions – Quantum Theory

Our idea of using a different representation

- ... that ensures positive–definiteness (Isham, Kakas '84, Klauder '99)
- but keeps the standard canonical commutation relations

Cholesky Decomposition

Every positive definite matrix q can be decomposed into the product

$$q = u^{\mathsf{T}} u,$$

where u is an upper triangular matrix with positive diagonal elements. This decomposition is unique.

Note that $\mathrm{UT}_{+}(2, \mathbb{R})$ is a Lie group.

2.2 Forward Solutions – Quantum Theory

Use this Lie group $\mathrm{UT}_+(2,\mathbb{R})$ to construct a Hilbert space:

$$\mathcal{H} = L^2(\mathrm{UT}_+(2,\mathbb{R}), \rho(u)\mathrm{d}u)$$

where $\rho(u)\mathrm{d}u$ is the left Haar measure associated with $\mathrm{UT}_+(2,\mathbb{R})$

Representation of $\hat{q}_{ab}^{XY}$ on $\mathcal{H}$

$$\begin{aligned}(\hat{q}_{11}\psi)(u) &= u_{11}^2\psi(u),\\ (\hat{q}_{12}\psi)(u) &= u_{11}u_{12}\psi(u),\\ (\hat{q}_{22}\psi)(u) &= \left(u_{12}^2 + u_{22}^2\right)\psi(u).\end{aligned}$$

Realizes positive–definiteness of the spatial metric

How to represent the momentum operator?

2.2 Forward Solutions – Quantum Theory

First, define generators of shifts in positive q–directions

$$\hat{U}(s)\hat{q}_{ab}\hat{U}(s)^{\mathsf{T}} = \hat{q}_{ab} + s_{ab},$$

where $s_{ab} > 0$. The following $\hat{U}(s)$ does the job

$$(\hat{U}(s)\psi)(u) = \sqrt{\frac{\det J_q(u)}{\det J_q(g_s(u))}\frac{\rho(g_s(u))}{\rho(u)}}\psi(g_s(u)),$$

where g_s is a diffeo on $UT_+(2,\mathbb{R})$ with $g_s(u) = q^{-1}(q(u) + s)$.

One can show that $\left\{\hat{U}(s) \in B(\mathcal{H}), s\,\text{pos. def.}\right\}$ forms a strongly continuous contraction semigroup.

2.2 Forward Solutions – Quantum Theory

To define the momentum operators, we use that the contraction semigroup $\left\{ \hat{U}(s) \in B(\mathcal{H}), s \,\text{pos. def.} \right\}$ admits the infinitesimal generators

$$i\hat{p}^{ab}\,\psi = \left(\frac{\mathrm{d}}{\mathrm{d}s_{ab}} \hat{U}(s)\psi \right)_{s=0} .$$

This yields

$$i\hat{p}^{11} = \frac{1}{2u_{11}}\frac{\partial}{\partial u_{11}} - \frac{u_{12}}{2u_{11}^2}\frac{\partial}{\partial u_{12}} + \frac{u_{12}^2}{2u_{11}^2 u_{22}}\frac{\partial}{\partial u_{22}} - \frac{2u_{22}^2 + u_{12}^2}{2u_{11}^2 u_{22}^2},$$

$$i\hat{p}^{12} = \frac{1}{u_{11}}\frac{\partial}{\partial u_{12}} - \frac{u_{12}}{u_{11}u_{22}}\frac{\partial}{\partial u_{22}} + \frac{u_{12}}{u_{11}u_{22}^2},$$

$$i\hat{p}^{22} = \frac{1}{2u_{22}}\frac{\partial}{\partial u_{22}} - \frac{1}{2u_{22}^2}.$$

2.2 Forward Solutions – Quantum Theory

With this representation, $\hat{q}_{ab}^{XY}$ and $\hat{p}^{cd}_{XY}$ satisfy the standard commutation relations

$$\left[\hat{q}_{ab}^{X_1 Y_1}, \hat{p}^{cd}_{X_2 Y_2}\right] = \frac{i}{\epsilon^2}\delta_a^{(c}\delta_b^{d)}\delta_{X_1}^{X_2}\delta_{Y_1}^{Y_2},$$
$$\left[\hat{q}_{ab}^{X_1 Y_1}, \hat{q}_{cd}^{X_2 Y_2}\right] = \left[\hat{p}^{ab}_{X_1 Y_1}, \hat{p}^{cd}_{X_2 Y_2}\right] = 0.$$

At the same time, $\hat{q}_{ab}^{XY}$ is positive definite in the sense that

$$\hat{q}_{ab}s^a s^b$$

is a positive operator for any s.

3. Representation of Gauge Transformations

3. Representation of Gauge Transformations

- Restrict to theories whose constraints form a Lie algebra (e.g., the diffeo constraints)
- For illustrative purposes consider a scalar field theory

Classical continuum theory

General form of continuum constraint:

$$D[f] = \int_{\mathbb{T}} \mathcal{D}(\phi(x), \partial\phi(x), \pi(x), \partial\pi(x)) f(x) \mathrm{d}x$$

Satisfies first class Poisson bracket algebra:

$$\{D[f], D[g]\} = D[F(f, \partial f, g, \partial g)]$$

3. Representation of Gauge Transformations

Classical lattice theory

Use lattice discretization $\phi_n(x) = \sum_{k=1}^{N_n} \phi_{nk} \chi_{X_k}(x)$. Lattice constraints are given by:

$$D_n[f_n] = \sum_{k=1}^{N_n} \mathcal{D}(\phi_{nk}, \Delta^n \phi_{nk}, \pi_{nk}, \Delta^n \pi_{nk}) f_{nk} \epsilon_n$$

Algebra on the lattice:

$$\{D_n[f_n], D_n[g_n]\} = D_n[F_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)] + \epsilon_n G_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)$$

3. Representation of Gauge Transformations

Solve Hamilton's equations of motion on the lattice:

$$\frac{\mathrm{d}\phi_n[g_n]}{\mathrm{d}s} = \{\phi_n[g_n], D_n[f_n]\}$$

- Solution only depends on inital data for ϕ_n if $D_n[f_n]$ is of first order in π_n (diffeo constraints in GR).
- The Hamiltonian flow $\varphi_s^{D_n[f_n]}$ can be interpreted as an approximate gauge transformation.

3. Representation of Gauge Transformations

Quantum lattice theory

Define approximate gauge transformations in the quantum theory on the lattice:

$$\left(\hat{U}\left(\varphi_s^{D_n[f_n]}\right)\psi_n\right)((\phi_{nk})_k) = \sqrt{\det\left(J_{\varphi_s^{D_n[f_n]}}((\phi_{nk})_k)\right)}\,\psi_n(\varphi_s^{D_n[f_n]}((\phi_{nk})_k))$$

Forms a unitary one–parameter group $\Rightarrow$ generator exists.
See Thiemann '22 for related approach

Provides a quantum representation of the lattice constraint:

$$i\frac{\mathrm{d}}{\mathrm{d}s}\left(\hat{U}\left(\varphi_s^{D_n[f_n]}\right)\psi_n\right)((\phi_{nk})_k)\Big|_{s=0} = \left(\hat{D}_n[f_n]\psi\right)((\phi_{nk})_k)$$

3. Representation of Gauge Transformations

What about the Hamiltonian Constraint?

Weyl quantization can be generalized to our new representation of the CCR:

$$Q[f] = \iint_{\mathbb{R}\times\mathbb{R}_+} \tilde{f}(\xi,\kappa) e^{\frac{1}{2}i\xi\kappa} e^{i\xi\hat{q}} U(\kappa) \mathrm{d}\xi \mathrm{d}\kappa + \text{h.c.}$$

This ensures

$$Q\left[(aq+bp)^n\right] = \frac{1}{2}\left((a\hat{q}+b\hat{p})^n + (a\hat{q}+b\hat{p}^\dagger)^n\right).$$

Can be used to quantize lattice constraints involving difficult expressions, such as inverse square roots:

$$\hat{H}_n[N_n] = Q[H_n[N_n]]$$

3. Representation of Gauge Transformations

Can there exist any diffeo-invariant states?

(Ashtekar '09): A diffeo-invariant state Ψ must be annihilated by the canonical variables: $\hat{q}_{ab}(x)\Psi = \hat{p}^{ab}(y)\Psi = 0$.

In conflict with the canonical commutation relations

$$0 = \left[\hat{q}_{ab}(x), \hat{p}^{ab}(y)\right] \Psi \stackrel{!}{=} i\hbar\delta(x, y)\Psi \neq 0.$$

Abhay concludes that there **can be no diffeomorphism invariant state** in the kinematical Hilbert space.

3. Representation of Gauge Transformations

Can there exist any diffeo-invariant states?

Underlying assumption: Algebra of canonical variables is bounded.

This is not the case in our representation!

Diffeomorphism invariant states may therefore exist in the (continuum) Hilbert space, but they will not be in the domain of the canonical operators.

Reasonable as the canonical operators are not Dirac observables and thus expectation values with respect to diffeo-invariant states need never be taken.

4. Continuum Limit

4. Continuum Limit

The Weyl algebra on the lattice is spanned by the exponentiated canonical variables:

$$W_n = \overline{\mathrm{span}\{e^{i\hat{\phi}_n[f_n]+i\hat{\pi}_n[g_n]}\}}$$

Let $W = \varprojlim W_n$ be the inverse limit with identifications

$$\hat{\phi}_{n+1,2k} f_{n+1,2k} + \hat{\phi}_{n+1,2k+1} f_{n+1,2k+1} \equiv \hat{\phi}_{nk}(f_{n+1,2k} + f_{n+1,2k+1})$$

Choose a sequence ψ_n of states on every lattice. Define

$$\omega_n\left(e^{i\hat{\phi}_n[f_n]+i\hat{\pi}_n[g_n]}\right) := \left\langle \psi_n, e^{i\hat{\phi}_n[f_n]+i\hat{\pi}_n[g_n]}\psi_n \right\rangle.$$

If ω_n forms Cauchy sequence, define

$$\omega\left(\lim_{n\to\infty} e^{i\hat{\phi}_n[f_n]+i\hat{\pi}_n[g_n]}\right) := \lim_{n\to\infty} \omega_n\left(e^{i\hat{\phi}_n[f_n]+i\hat{\pi}_n[g_n]}\right).$$

Use GNS–construction to obtain continuum Hilbert space.

4. Gauge Transformations in the Continuum

The approximately gauge transformed version of an element $w_n \in W_n$ is given by

$$\hat{w}_n^{D_n[f_n]}(s) = \hat{U}_{D_n[f_n]}(s)\hat{w}_n\hat{U}_{D_n[f_n]}(-s).$$

Under mild conditions, it can again be expanded in terms of lattice Weyl algebra elements:

$$\hat{w}_n^{D_n[f_n]}(s) = \sum_k c_{nk} e^{i\hat{\phi}_n[f_{nk}]+i\hat{\pi}_n[g_{nk}]}.$$

4. Gauge Transformations in the Continuum

This allows us to define approximately gauge transformed lattice states $\omega_n^{D_n[f_n]}(s)$ in terms of the states ω_n:

$$\left(\omega_n^{D_n[f_n]}(s)\right)(\hat{w}_n) := \omega_n(\hat{w}_n^{D_n[f_n]}(-s)) = \sum_k c_{nk}\omega_n\left(e^{i\hat{\phi}_n[f_{nk}]+i\hat{\pi}_n[g_{nk}]}\right).$$

We can now again take the continuum limit of these states in order to define the gauge transformed version of ω:

$$\left(\omega^{D[f]}(s)\right)(\hat{w}) := \lim_{n\to\infty}\left(\omega_n^{D_n[f_n]}(s)\right)(\hat{w}).$$

5. Summary and Outlook

Summary

- Start: Classical lattice regularization of geometrodynamics.
- Quantization of lattice theory: Inherently pos. def. metric plus standard canonical commutation relations.
- Representation of approximate gauge transformations (spatial diffeos) on the lattice.
- Criterion for existence of continuum limit.

5. Summary and Outlook

Outlook

- Convergence proofs of difference schemes.
- Explore converging sequences of lattice states.
- Study continuum limit of approximate gauge transformations.
- Prove strong continuity of representations of gauge groups.

- Use generalized Weyl quantization to represent lattice Hamiltonian constraint.
- Study continuum limit of Hamiltonian constraint (probably involves renormalization techniques).

Thank you for your attention!

arXiv:2305:09650, arXiv:2305.10097, arXiv:2311.00245