Title: Infinitesimal structure of BunG

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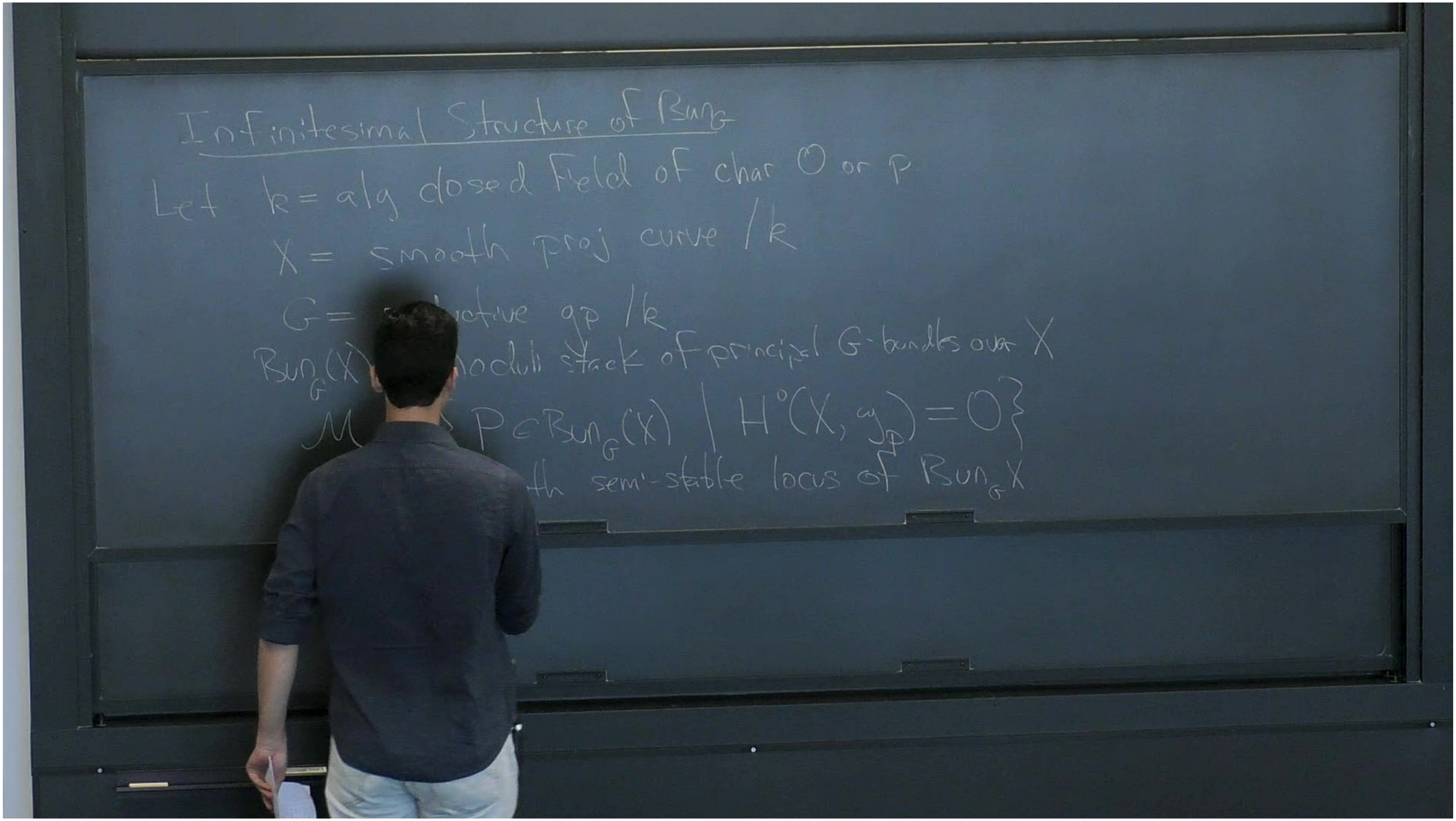
Abstract: Given a semisimple group G and a smooth projective curve X over an algebraically closed field of arbitrary characteristic, let Bun_G(X) denote the moduli space of principal G-bundles over X. For a bundle P without infinitesimal symmetries, we describe the n^th order divided-power infinitesimal jet spaces of Bun_G(X) at P for each n. The description is in terms of differential forms on X^n with logarithmic singularities along the diagonals. Furthermore, we show the pullback of these differential forms to the Fulton-Macpherson compactification space is an isomorphism, thus illustrating a connection between infinitesimal jet spaces of Bun_G(X) and the Lie operad.

Zoom link


Infinitesimal Structure of Bun_G
Let k = alg closed field of char 0 or p
X = smooth proj curve $/k$
G = ctive gp $/k$
$Bun_G(X)$ oduli stack of principal G-bundles over X
$P \in Bun_G(X) \mid H^0(X, g_P) = 0 \}$
th semi-stable locus of $Bun_G X$

$$\mathcal{M}^{G} = \{ P \in Bun_G(X) \mid H^0(X, \mathfrak{g}_P) = 0 \}$$

= smooth semi-stable locus of $Bun_G X$.

Goal: Describe all $J_P^n(\mathcal{M})$ $\forall P \in Bun_G$, $\forall n \geq 1$

§ 1st order jets

Fix $x \in X$, $\mathcal{O}_x = \hat{\mathcal{O}}_{X,x} \simeq k[[t]] \rightsquigarrow D_x = \mathrm{Spec}\, \mathcal{O}_x$

$$K_x = \mathrm{Frac}(\hat{\mathcal{O}}_{X,x}) = k((t)) \rightsquigarrow D_x^\times = \mathrm{Spec}\, K_x$$

$$\mathcal{O}_{out} = \mathcal{O}_X \big|_{X \setminus x}$$

__Thm__ [Beauville-Laszlo] There's an isom of stacks

$$\pi : G(O_{out}) \backslash G(K_x) / G(O_x) \xrightarrow{\sim} Bun_G X$$

$$\Longrightarrow T_P \, Bun_G X = \mathfrak{g}_{K_x} \Big/ \left(Ad_\varphi \mathfrak{g}_O + \mathfrak{g}_{out} \right) \underset{\text{Čech}}{\simeq}$$

where φ is lift of P under π

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$$\Longrightarrow T_P \mathrm{Bun}_G X = \mathfrak{g}_{K_x} \big/ (\mathrm{Ad}_\varphi \mathfrak{g}_0 + \mathfrak{g}_{out}) \underset{\text{Čech}}{\cong} H^1(X, \mathfrak{g}_P) \underset{\text{Serre}}{\cong} H^0(X, \mathfrak{g}_P^* \otimes \Omega_X)^*$$

where φ is lift of P under π

§ Higher Order Jets

Suppose $V = k^n$ Fin dim vec space

$$\mathrm{Sym}(V) = \bigoplus_{i \geq 0} V^{\otimes i} / S_i$$

$$\mathrm{Sym}^{PD}(V) = \bigoplus_{i \geq 0} \left(V^{\otimes i}\right)^{S_i}$$

Ex $$\mathrm{Sym}^{PD}(k) = k[x^{(i)}]_{i \geq 1} \Big/ \left(x^{(n)} x^{(m)} = \binom{n+m}{m} x^{(n+m)} \right)$$

$i \geq 0$

Ex $\mathrm{Sym}^{PD}(k) = k[x^{(i)}]_{i \geq 1} \Big/ \left(x^{(n)} x^{(m)} = \binom{n+m}{m} x^{(n+m)} \right)$

Lem $\mathrm{Sym}^{PD}(V) \times \mathrm{Sym}(V^*) \longrightarrow k$ is perfect pairing

Thm [Beauville-Laszlo] There's an isom of stacks

$$\pi : G(\mathcal{O}_{out}) \backslash G(K_x) / G(\mathcal{O}_x) \xrightarrow{\sim} \mathrm{Bun}_G X$$

$$\Longrightarrow T_P \mathrm{Bun}_G X = \mathfrak{g}_{K_x} \Big/ \left(\mathrm{Ad}_\varphi \mathfrak{g}_0 + \mathfrak{g}_{out} \right) \underset{\substack{\text{Čech} \\ \{D_x, X \setminus x\}}}{\simeq} H^1(X, \mathfrak{g}_P) \underset{\text{Serre}}{\simeq} H^0(X, \mathfrak{g}_P^* \otimes \Omega_X)^*$$

where φ is lift of P under π

Sheafify $\mathcal{M}$ = fin type scheme, smooth, $/k$

$\mathcal{O}^e := \mathcal{O}_{\mathcal{M}} \otimes \mathcal{O}_{\mathcal{M}}$; $I = \mathrm{Ker}\left[\mathcal{O}^e \xrightarrow{mult} \mathcal{O}_{\mathcal{M}}\right] = (f\otimes 1 - 1\otimes f)$

Def (1) $J^n(\mathcal{M}) = \mathcal{O}^e / I^{n+1}$ = sheaf of n-jets

(2) $J^{n,PD}(\mathcal{M}) = \Gamma_I(\mathcal{O}^e) / \overline{I}^{n+1}$ = n^{th} PD nbhd of $\Delta : \mathcal{M} \hookrightarrow \mathcal{M}\times\mathcal{M}$

Sheafify $\quad M =$ fin type scheme, smooth, $/k$

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Def

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Duality $\quad D^{crys}(M)_n = \mathcal{H}om_{\mathcal{O}_M}(J^{n,PD}(M), \mathcal{O}_M)$

$$K_x = \mathrm{Frac}(\hat{\mathcal{O}}_{X,x}) = k((t)) \rightsquigarrow D_x^\times = \mathrm{Spec}\, K_x$$

$$\mathcal{O}_{out} = \mathcal{O}_X|$$

$$\Rightarrow U(T_\varphi \mathcal{M}) = U\tilde{g}_K \Big/ \left((\mathrm{Ad}_\varphi \tilde{g}_0) U\tilde{g}_K + U\tilde{g}_K \cdot \tilde{g}_{out} \right)$$

$$=: \mathbb{M}^{out}_{\varphi} = \text{space of coinvariants of}$$

$$\boxed{\begin{array}{l} \underline{\text{Cor}} \quad \text{There's a perfect pairing} \\ \qquad \varphi : \mathbb{M}^{out}_{\varphi, n} \times J^{n, PD}_{p}(\mathcal{M}) \longrightarrow k \end{array}}$$

§ Log differential forms

Recall Fulton-Macpherson compactification:

$p: \hat{X}^n \longrightarrow X^n$ such that: $D = \bigcup \{x_i = x_j\}$

(1) $p^{-1}(D) =: \hat{D} = \bigcup_{|I| \geq 2} \hat{D}_I$ is normal crossing divisor

(2) $\hat{D}_I \simeq \hat{\mathbb{P}}^I \times \hat{X}^{[n]/I}$, where $[n]/I = [n] \setminus I \cup \{I\}$

(3) $\hat{X}^n = \coprod_{T = [n]\text{-trees}} S_T$, where $\overline{S_T} = \hat{\mathbb{P}}_I \times \hat{X}^J$, $J = \#$ connected comp of T

$(3)\ \lambda = \ldots$ $T=[n]$-trees

<u>Def</u> $\Omega_{\hat{X}^n, \mathring{X}^n}$ = sheaf of <u>top-degree</u> diff. forms on $\hat{X}^n$ which are regular $\mathring{X}^n = \hat{X}^n \setminus \hat{D}$, & log sing on $\hat{D}$.

$$\Omega(\hat{X}^n, \mathring{X}^n) = \Gamma(\hat{X}^n, \Omega_{\hat{X}^n, \mathring{X}^n})$$

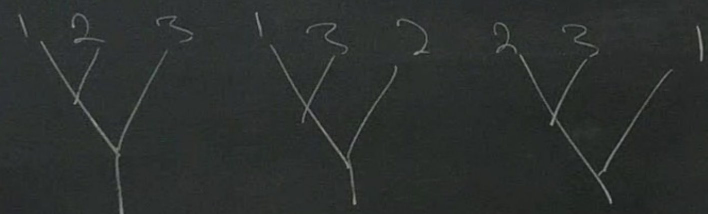
<u>Ex</u> $$\Omega(\hat{\mathbb{P}}^{[3]}, \mathring{\mathbb{P}}^{[3]}) = \left\{ \lambda_{12}\, d\log z_{12} + \lambda_{23}\, d\log z_{23} + \lambda_{31}\, d\log z_{31} \;\middle|\; \lambda_{12}+\lambda_{23}+\lambda_{31}=0,\ \lambda_{ij}\in k,\ z_{ij}=z_i-z_j \right\}$$

Lem There's a perfect pairing:

Lie-operad

$$\text{Residue}: \Omega(\hat{\mathbb{P}}^{I}, \mathring{\mathbb{P}}^{I}) \times \mathrm{Lie}(I) \longrightarrow k$$

$\underline{n=3}$



$$\downarrow$$

D_{12}, D_{123}

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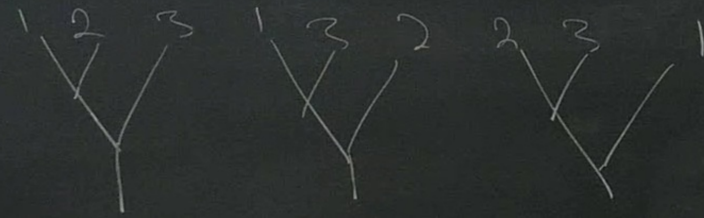

D_{12}, D_{123}

$$I = [n]$$

$$\varphi_I^* : \mathfrak{g}^* \longrightarrow (\mathfrak{g}^*)^{\otimes I} \otimes \mathrm{Lie}(I)^*$$

$$A \longmapsto \Big(\underbrace{x_1 \otimes \cdots \otimes x_n}_{\in \mathfrak{g}^{\otimes n}} \otimes \underset{\in S_{n-1}}{\sigma} \longmapsto A\big([x_{\sigma(1)}, \ldots, [x_{\sigma(n-1)}, x_n]\cdots]\big) \Big)$$

$$\rightsquigarrow \psi_I^* : p^*\Big(\mathfrak{g}_p^{*\,\boxtimes [n]/I} \otimes \Omega_{\hat{X}^{[n]/I}, X^{[n]/I}} \Big) \longleftrightarrow (\mathfrak{g}_p^*)^{\boxtimes n}\big|_{\hat{D}_I} \otimes \Omega_{\hat{D}_I, D_I}$$

$$\begin{array}{c} \hat{D}_I \\ \downarrow p \\ \hat{X}^{[n]/I} \end{array}$$

Fiberwise, $\psi_I^*\big|_{x \in \hat{D}_I} = \mathrm{Id} \boxtimes \varphi_I^*$

$I = [n]$

$$\varphi_I^* : \mathfrak{g}^* \longrightarrow (\mathfrak{g}^*)^{\otimes I} \otimes \mathrm{Lie}(I)^*$$

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$$\begin{array}{c} \hat{D}_I \\ \downarrow p \\ \hat{X}^{[n]/I} \end{array}$$

$$\left(\mathfrak{g}_p^*\right)^{\boxtimes n} \otimes \Omega_{\hat{X}^n, \mathring{X}^n} \xrightarrow{\mathrm{Res}_{\hat{D}_I}} \left(\mathfrak{g}_p^*\right)^{\boxtimes n}\Big|_{\hat{D}_I} \otimes \Omega_{\hat{D}_I, \mathring{D}_I}$$

Fiberwise, $\psi_I^*\Big|_{x \in \hat{D}_I} = \mathrm{Id} \boxtimes \varphi_I^*$

(3) $\hat{X}^n = \coprod_{T=[n]\text{-trees}} S_T$, where $\overline{S_T} = \hat{\mathbb{P}}_{\underline{T}} \times \hat{X}^J$, $J = \#$ connected comp of T

Def The "BG" sheaf

$$\hat{\mathcal{G}}_n := \left\{ \omega \in (\mathfrak{g}_p^*)^{\boxtimes n} \otimes \Omega_{\hat{X}^n_p,\, \hat{x}^n} \;\middle|\; \mathrm{Res}_{\hat{D}_I}\, \omega \in \mathrm{Im}\, \Psi_I^* \quad \forall\, |I| \geq 2 \right\}$$

Thm (G.)

$$\Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n} = \mathcal{J}_p^{n,PD}(\mathcal{M})$$

Pf Sketch

(1) Define analogues sheaf $\mathcal{G}_n$ on X^n such that

$$\Gamma(\hat{X}^n, \hat{\mathcal{G}}^n)^{-S_n} \underset{\text{Thm}}{=} \Gamma(\hat{X}^n, p^*\mathcal{G}_n)^{-S_n} \underset{\text{def}}{=} \Gamma(\hat{X}^n, \mathcal{G}_n)^{-S_n}$$

(2) $\exists$ canonical perfect pairing

$$\text{Rec}: \mathbb{M}^{out}_{\varphi,n} \times \Gamma(X^n, \mathcal{G}_n) \longrightarrow k$$

Thm

(2) $\exists$ Canonical perfect pairing

$$\mathrm{Res}: \mathbb{M}^{out}_{\varphi,n} \times \Gamma(X^n, G_n) \longrightarrow k. \quad \square$$

$$\bigoplus \Gamma(\hat{X}^n, (\hat{\mathfrak{g}}^*_p)^{\boxtimes n} \otimes \Omega_{\hat{X}^n, \hat{X}^n} \otimes \omega_{\hat{X}^n})^{S_n} =: \Omega^{PD}_{\hat{X}}(\mathfrak{g}^*_p) = \text{free PD-algebra gen'd by } \mathfrak{g}^*_p$$

<u>Rmks</u>

$$\mathrm{Res}_{\hat{D}_I} : \hat{G}_n|_{\hat{D}_I} \longrightarrow (\hat{G}_{[n]/I} \otimes \mathfrak{g}^{*(I-1)}_p) \boxtimes \Omega_{\hat{p}^I, \hat{p}^I}$$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ (G)\\ \Gamma(\hat{X}^n, \hat{G}_n)^{S_n} = J^{n,PD}_p(\mathcal{M})\end{array}}$$