Title: Replica topological order in quantum mixed states and quantum error correction

Speakers: Roger Mong

Series: Quantum Matter

Date: May 22, 2024 - 1:30 PM

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Abstract:

Topological phases of matter offer a promising platform for quantum computation and quantum error correction. Nevertheless, unlike its counterpart in pure states, descriptions of topological order in mixed states remain relatively under-explored. We will give various definitions for replica topological order in mixed states. Similar to the replica trick, our definitions also involve n copies of density matrix of the mixed state. Within this framework, we categorize topological orders in mixed states as either quantum, classical, or trivial, depending on the type of information they encode.

For the case of the toric code model in the presence of decoherence, we associate for each phase a quantum channel and describes the structure of the code space. We show that in the quantum-topological phase, there exists a postselection-based error correction protocol that recovers the quantum information, while in the classical-topological phase, the quantum information has decohere and cannot be fully recovered. We accomplish this by describing the mixed state as a projected entangled pairs state (PEPS) and identifying the symmetry-protected topological order of its boundary state to the bulk topology.

Zoom link

Replica Topological Order in Quantum Mixed States and Quantum Error Correction

Perimeter Institute – May22, 2024

Topological order in two-dimensions

Phases of matter that support fractionalized excitations.

Examples

- Toric code
- Fractianal quantum Hall
- Z_2 Spin liquids
- Chiral spin liquids

Topological order

- Anyons
  - Fusion, braiding, etc.
  - Topological quantum field theory (TQFT)

- Wavefunction characterizations
  - Ground state degeneracy
  - Topological entanglement entropy
  - Entanglement spectrum

Topological order in mixed states?

- Unstable to thermal effects
  - Gibbs ensemble is smoothly connected to trivial state

- Stable to small local decoherence

$$\epsilon_r^X(\rho) = (1 - p_x)\rho + p_x X_r \rho X_r$$
$$\epsilon_r^Z(\rho) = (1 - p_z)\rho + p_z Z_r \rho Z_r$$

  - Threshold in quantum error correction code $p_c \approx 11\%$

- How to characterize mixed states topological order?

Vectorization (double approach)

- Vectorization

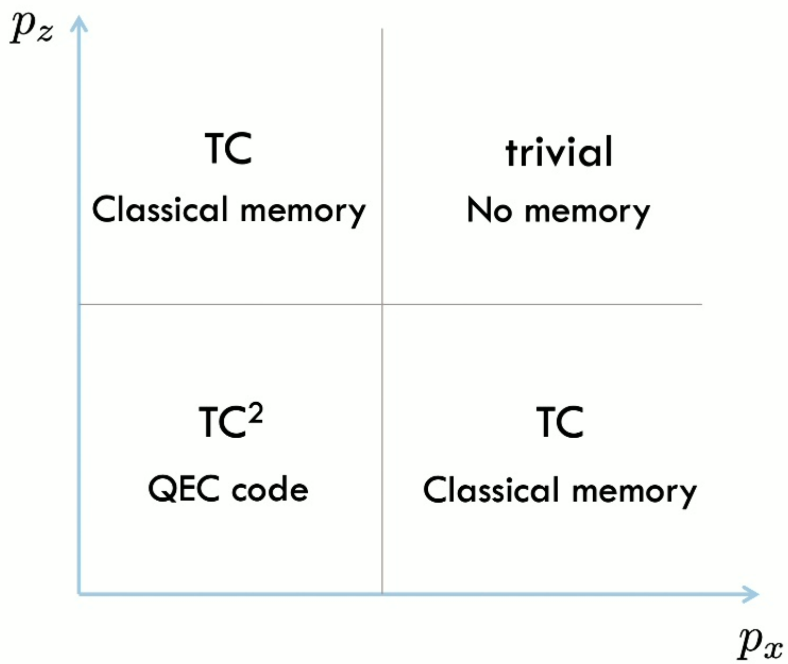
$$|a\rangle\langle b| \to |a\rangle \otimes |b^*\rangle$$

  - Pure states to double pure states
  - Density operator to wavefunctions
  - Quantum channels to operators

- Can characterize mixed state using existing tools
  - E.g., Bao, Fan, Vishwanath, Altman 2023

Vectorization (double approach)

Phase diagram



[Bao, Fan, Vishwanath, Altman 2023;
Fan, Bao, Altman, Vishwanath 2023;
Wang, Wu, Wang 2023;
Sang, Zou, Hsieh 2023]

Vectorization (double approach)

- Vectorization $|\Psi_{\rm TC}\rangle\langle\Psi_{\rm TC}| \rightarrow |\Psi_{\rm TC}\rangle \otimes |\Psi_{\rm TC}\rangle$
- X- and Z-flip errors

$$|\Psi\rangle \rightarrow \left(\prod_{\mathbf{r}} E_{\mathbf{r}}^X E_{\mathbf{r}}^Z\right)|\Psi\rangle$$

$$E_{\mathbf{r}}^X = (1-p_x) + p_x(X_{\mathbf{r}} \otimes X_{\mathbf{r}}), \quad E_{\mathbf{r}}^Z = \ldots$$

- Diagnose topological order of wavefunction
  - Observables are quadratic in ρ!

$$\langle\Psi|O_1 \otimes O_2|\Psi\rangle \propto \mathrm{Tr}(O_1\rho O_2\rho)$$

Mixed state topological order

Questions

- What does it means to for a state to have topological order?
- What is the interpretation of using n copies of the mixed state?
- How can tensor networks be used to characterize topological order?
- What are the possible phases that results from the toric code (TC) with errors?

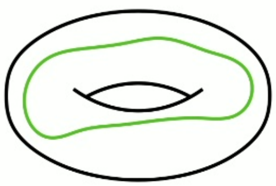
[Z. Li, RM, arXiv:2402.09516]

Pure state topological order

- Ground state degeneracy
  - Two states $|\psi\rangle$, $|\phi\rangle$ are degenerate if they are *locally indistinguishable* but *globally different.*

$$\langle\psi|A|\psi\rangle = \langle\phi|A|\phi\rangle$$
$$|\langle\psi|\phi\rangle| \neq 1$$

- Geometry
  - Set of locally indistinguishable states form $\mathbb{CP}^{d-1}$
- Wilson loops
  - Forms a matrix algebra



$|\phi_1\rangle$

Mixed state topological order

- Local distinguishability

$$\frac{\mathrm{Tr}(A_1\rho A_2\rho\cdots A_n\rho)}{\mathrm{Tr}(\rho^n)} = \frac{\mathrm{Tr}(A_1\sigma A_2\sigma\cdots A_n\sigma)}{\mathrm{Tr}(\sigma^n)}$$

for operators supported on a contractible region

- Global distinguishability
  - distance

$$\mathrm{dist}_n(\rho,\sigma) \stackrel{\mathrm{def}}{=} \frac{1}{2^{1/n}}\|\rho-\sigma\|_n$$

  - n-Schatten norm

$$\|\alpha\|_n \stackrel{\mathrm{def}}{=} \frac{1}{\mathcal{N}}\left[\mathrm{Tr}\left(|\alpha|^n\right)\right]^{1/n}$$

Mixed state topological order

Geometric definition

Let $\mathcal{S}^{(n)}(\rho)$ denote the set of n-replica states that are n-replica locally indistinguishable from ρ.

Classification

(i) ρ is called n-replica **trivial** if $\mathcal{S}^{(n)}(\rho)$ is a single point.

(ii) ρ is called n-replica **classical topologically ordered** (CTO) if $\mathcal{S}^{(n)}(\rho)$ has a finite number of extreme points.

(iii) ρ is called n-replica **quantum topologically ordered** (QTO) if the extreme points of $\mathcal{S}^{(n)}(\rho)$ form a submanifold with dimension ≥ 1.

[Z. Li, RM, arXiv:2402.09516]

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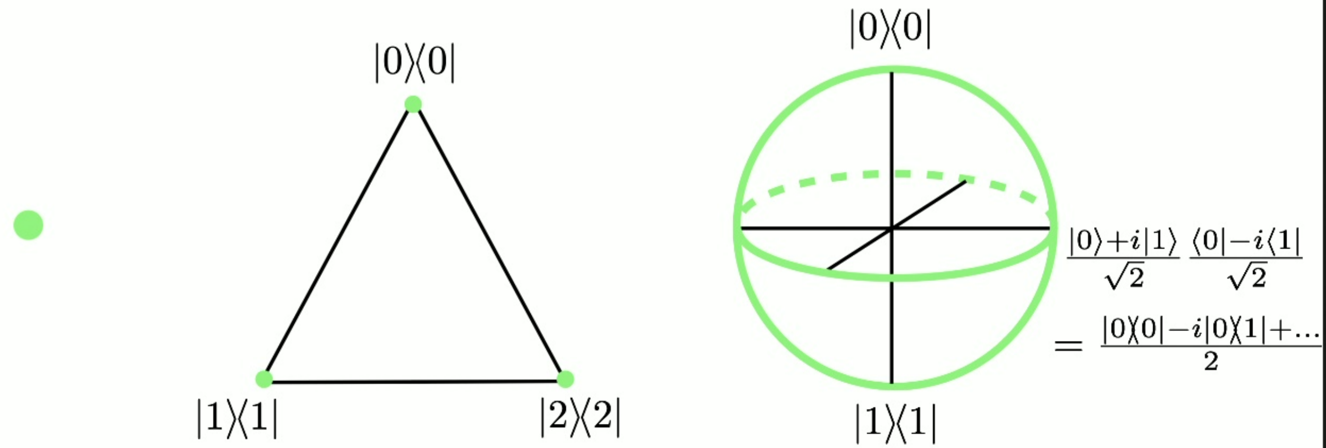
Mixed state topological order

Trivial **Classical** **Quantum**



Mixed state topological order

- **Wilson loop-based definition**

  Let $\mathcal{V}^{(n)}(\rho)$ denote the complex vector space generated by $\mathcal{S}^{(n)}(\rho)$.

- **Classification**
  - W is **classical** if it commutes with every element of $\mathcal{V}^{n}(\rho)$.
  - W is **quantum** if there exists $R \in \mathcal{V}^{(n)}(\rho)$ such that $WR \neq RW$.

(i) ρ is n-replica **trivial** if there are no such non-identity operator; i.e., all operators act trivially: $WR \propto R$ for all W and $R \in \mathcal{V}^{(n)}(\rho)$.

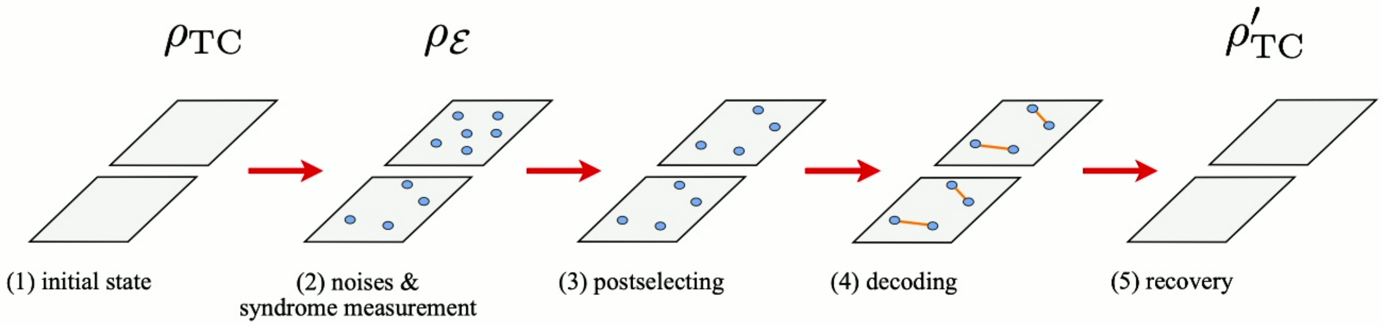
(ii) ρ is n-replica **classical topologically ordered** (CTO) if (a) there exists at least one non-identity operator, and that (b) all non-identity operators are classical: $WR = RW$ for all W and $R \in \mathcal{V}^{(n)}(\rho)$.

(iii) ρ is n-replica **quantum topologically ordered** (QTO) if there exists a quantum non-identity operator.

[Z. Li, RM, arXiv:2402.09516]

Quantum error correction (QEC)

Replica QEC protocol:



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Quantum error correction (QEC)

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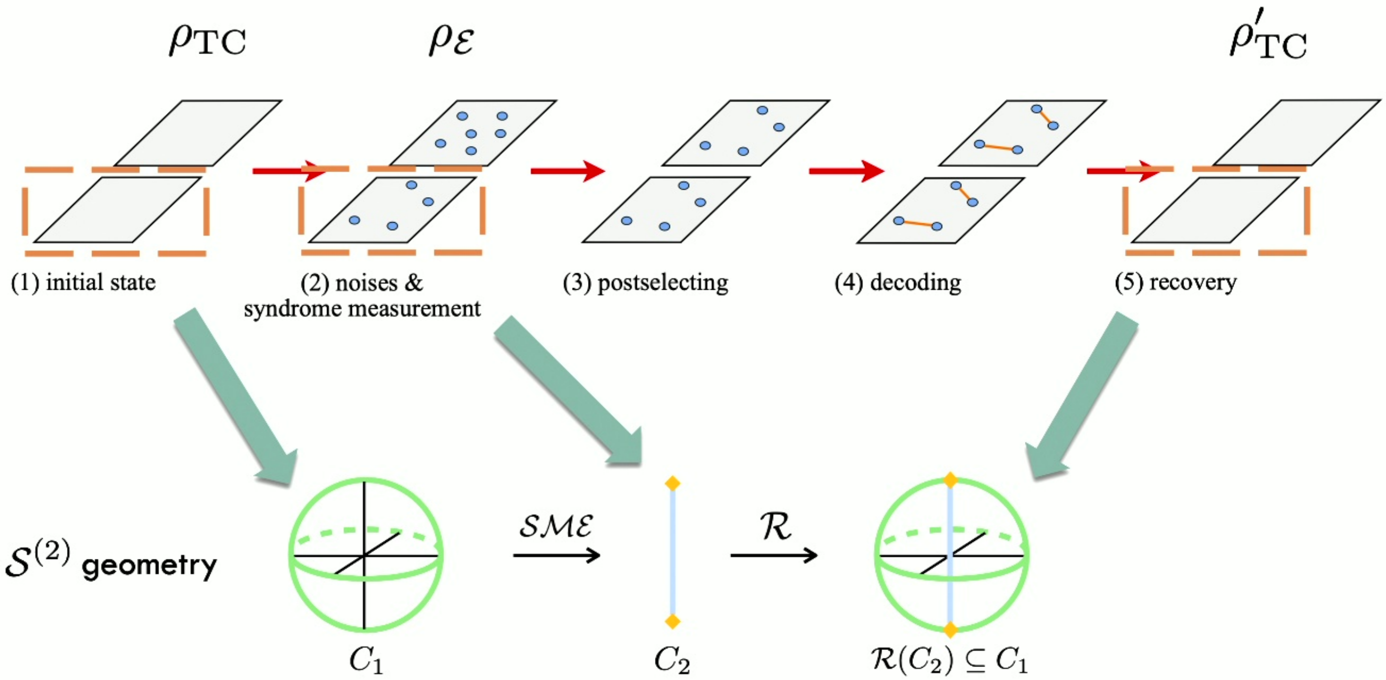
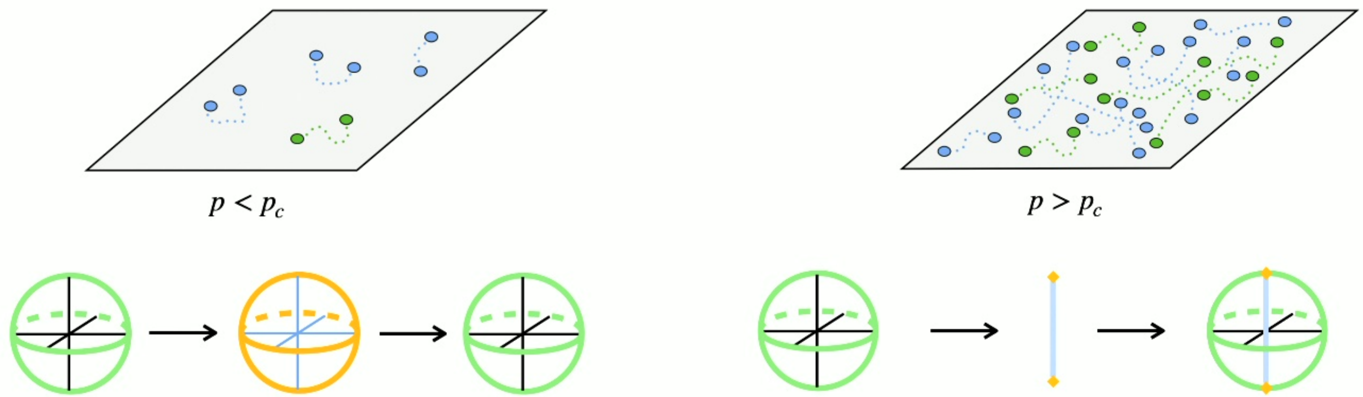


Quantum error correction (QEC)

Error threshold / Phase transition

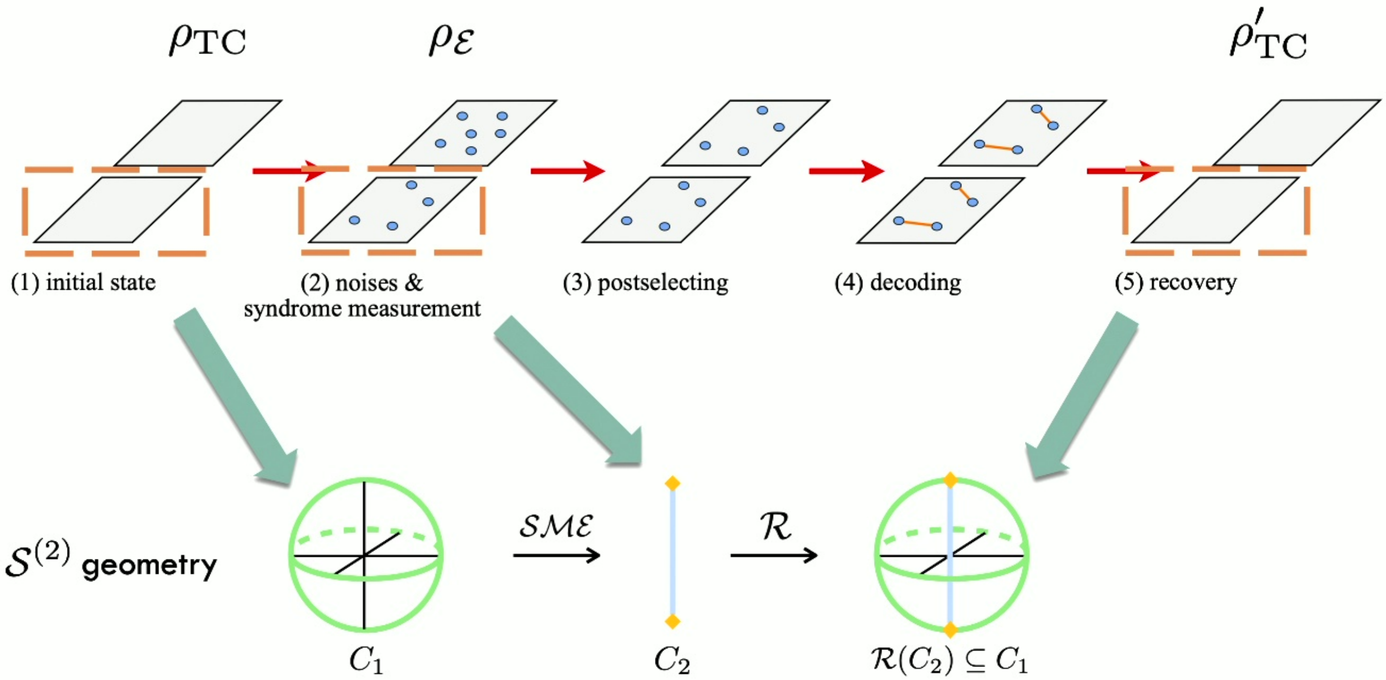


Topological classification of error mixed-state
is related to effective quantum channel after recovery

[Actual logical space is 4-dimensional]

Quantum error correction (QEC)

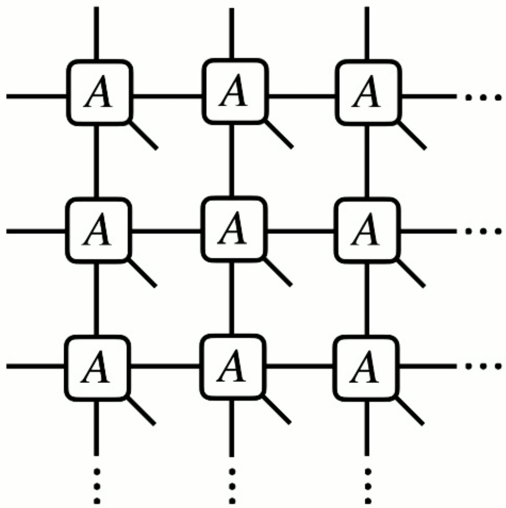
Replica QEC protocol:



Tensor networks

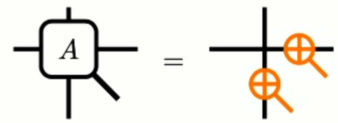
- Efficient way to capture a many-body state
- Examples
  - Matrix product states (MPS)
  - Matrix product operator (MPO)
  - Projected entangled pair states (PEPS)

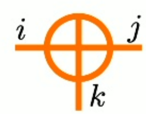
PEPS



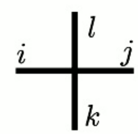
Toric code stabilizers

Toric code PEPS



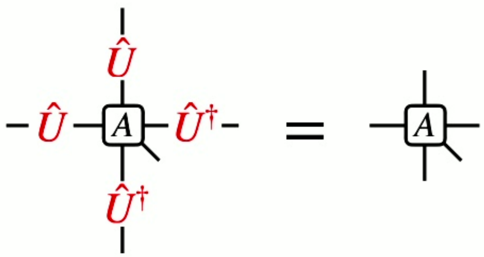


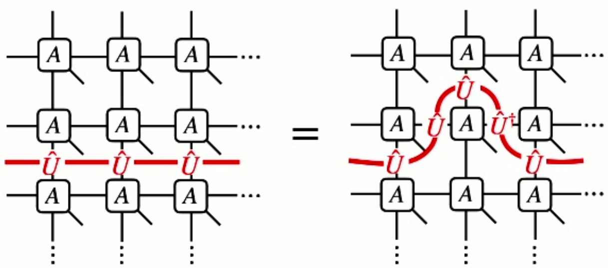
$$= \begin{cases} 1 & i+j+k \equiv 0 \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$



$$= \delta_{ij}\delta_{ik}\delta_{il}\,.$$

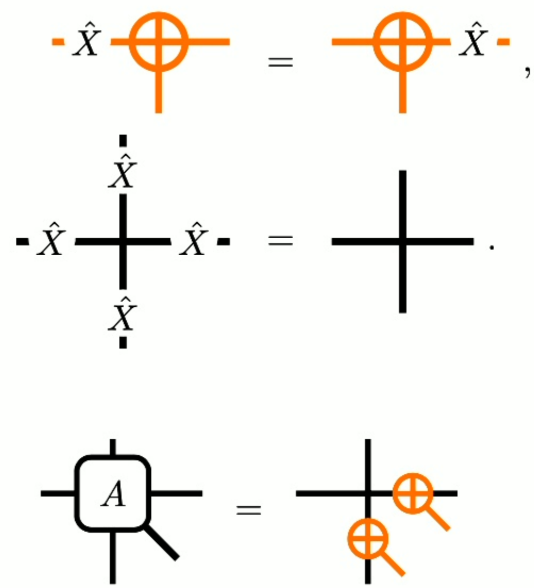
G-symmetric PEPS





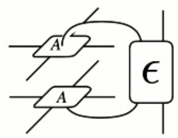
G symmetry + isometry
=> quantum double of G

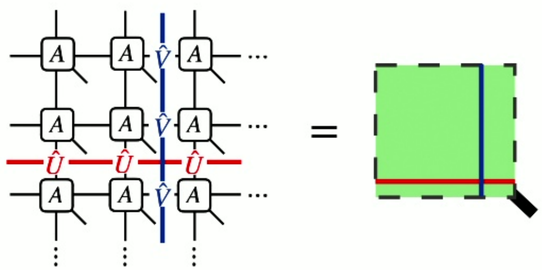
Toric code Z_2 symmetry



n-replica network

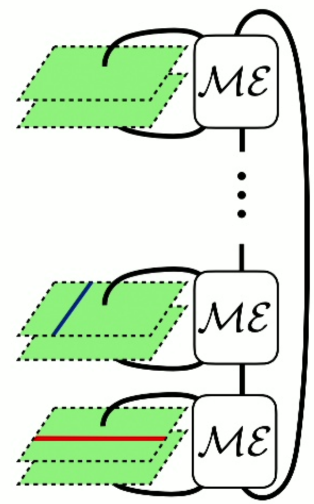
Apply error channel





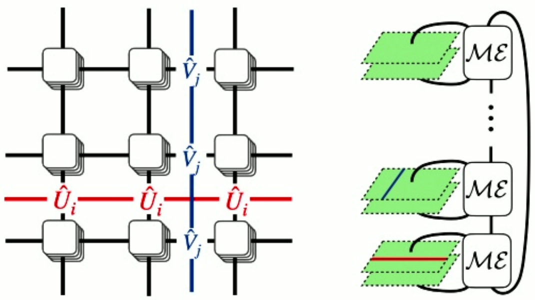
Contract 2n copies

$$\mathrm{Tr}\left(\rho_{\mathcal{E}}\right)^n \sim$$



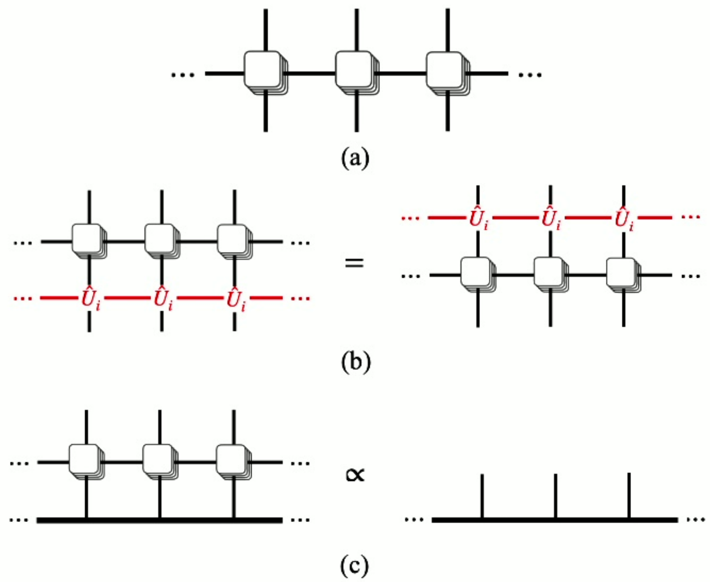
Boundary state

The n-replica network is a 2D statistical model



Its "boundary state" is the ground state of the corresponding 1+1D quantum Hamiltonian

Transfer operator



Key relations

"Master" equation

$$\left\langle \hat{X}_1^{\mathbf{t}_1} \hat{X}_2^{\mathbf{t}_2} \cdots \hat{X}_{2n}^{\mathbf{t}_{2n}} \right\rangle$$

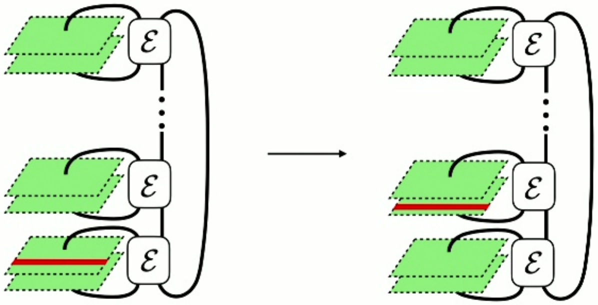
$$= \sum_\theta \mathrm{Pr}^n \, \mathrm{Tr}\Big[Q_\theta(X^{\mathbf{t}_1}\rho_{00}X^{\mathbf{t}_2})\, Q_\theta(X^{\mathbf{t}_3}\rho_{00}X^{\mathbf{t}_4}) \cdots Q_\theta(X^{\mathbf{t}_{2n-1}}\rho_{00}X^{\mathbf{t}_{2n}}) \Big] \Big/ \sum_\theta \mathrm{Pr}^n \, \mathrm{Tr}\Big[\big(Q_\theta(\rho_{00})\big)^n \Big]$$

Boundary SPT order of n-replica tensor network

Effective quantum channel within logical space

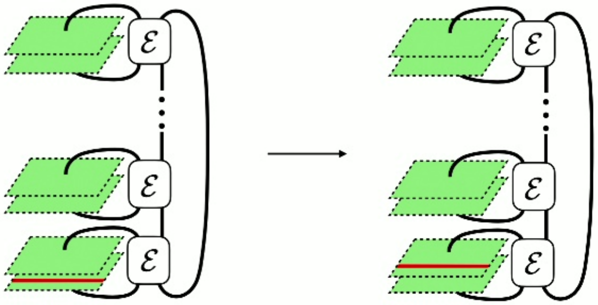
Wilson loops

Wilson loop condition



(a)

Classical condition



(b)

Toric code descendants

"Master" equation

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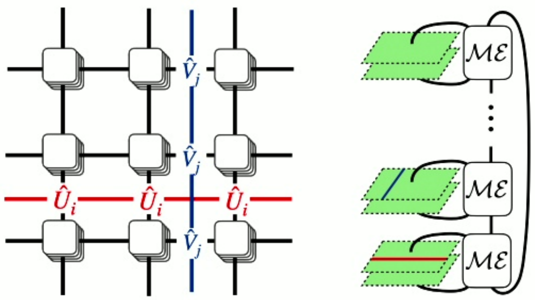
Solve for all consistent set of solutions for Z_p toric code (as parent state)

- LHS corresponds to some SPT order (group cohomology)
- RHS: Q must corresponds to quantum channels (CPTP maps)
- p + 3 solutions

Boundary state

The n-replica network is a 2D statistical model



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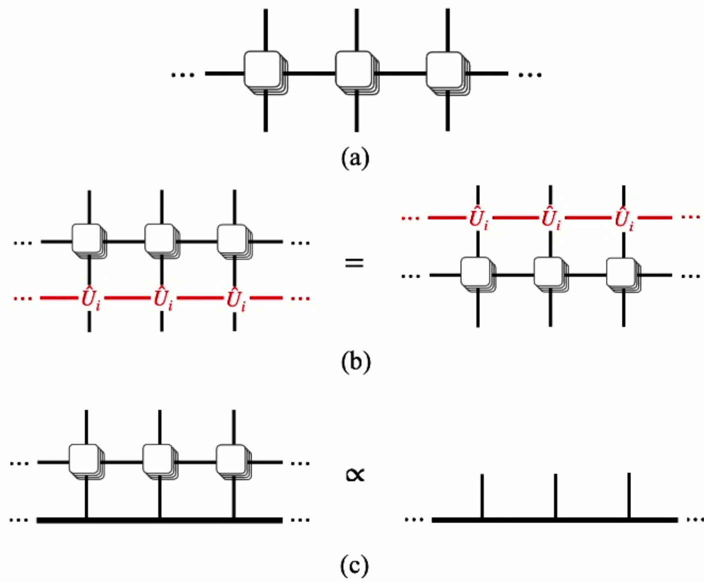
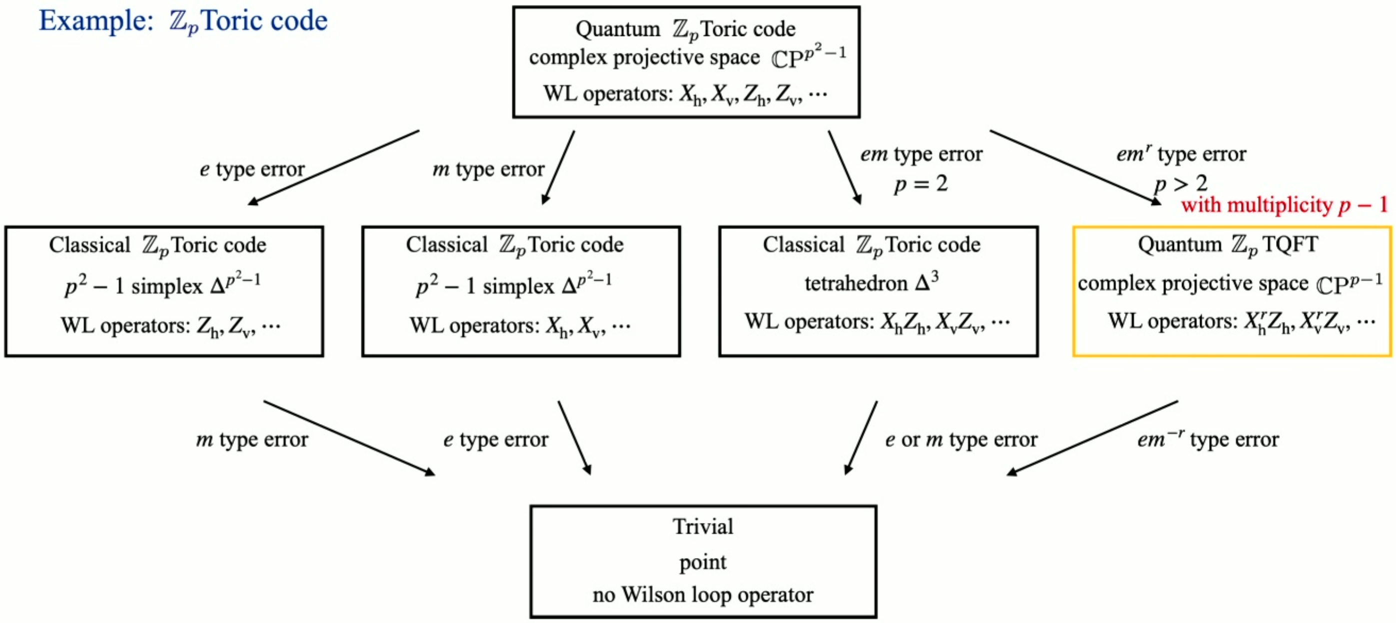
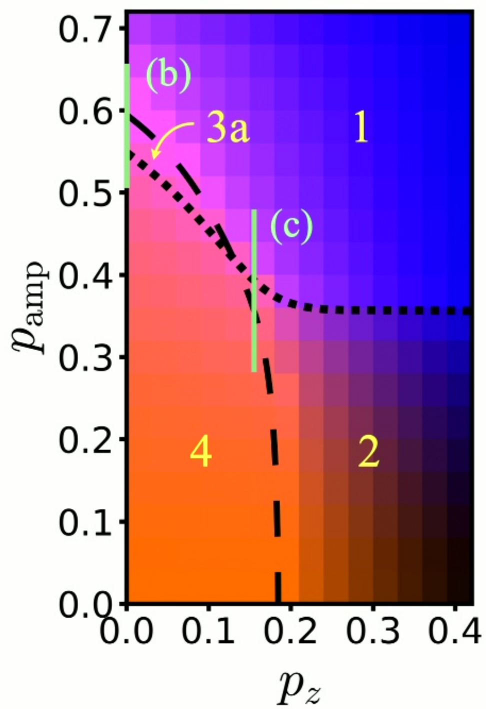


Toric code descendants

Example: $\mathbb{Z}_p$ Toric code



Numerical simulations



$$\epsilon_z(\sigma) = (1-p_z)\sigma + p_z Z\sigma Z.$$
$$\epsilon_{\mathrm{amp}}(\sigma) = M_1\sigma M_1^\dagger + M_2\sigma M_2^\dagger\,,$$

where

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p_{\mathrm{amp}}} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \sqrt{p_{\mathrm{amp}}} \\ 0 & 0 \end{pmatrix}.$$

Phases	
4	Quantum (QEC regime)
3a	Classical
2	Classical
1	Trivial

Toric code descendants

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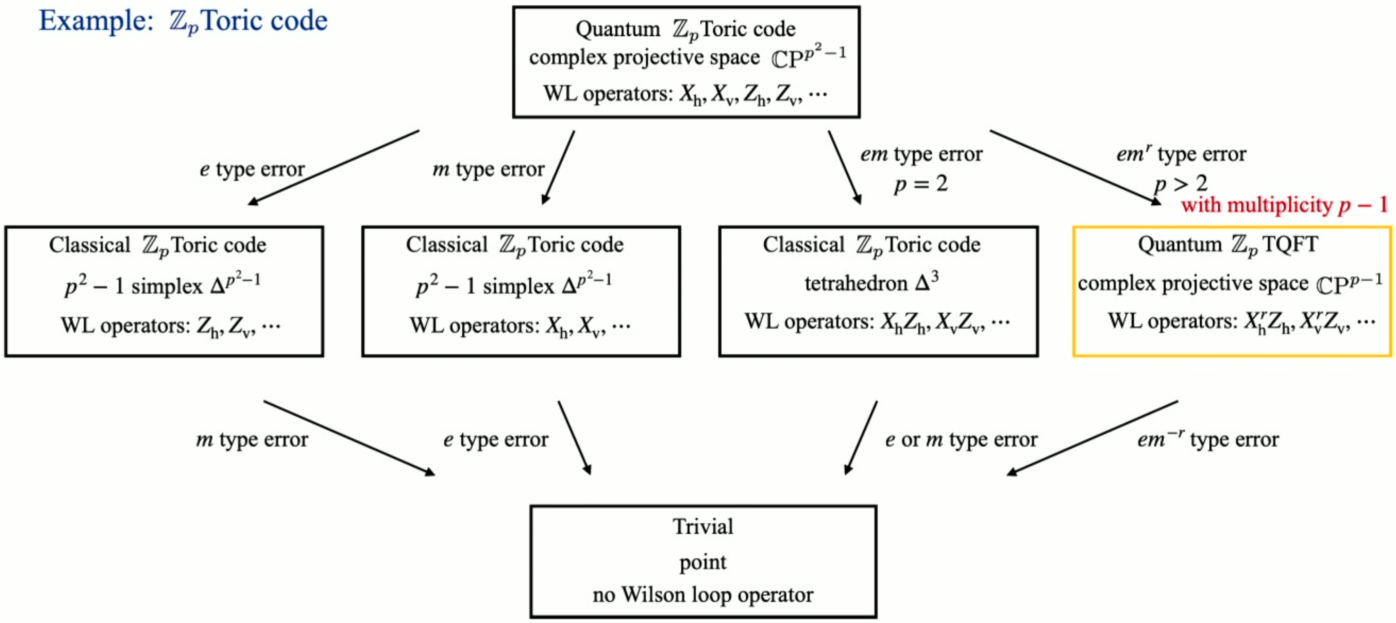

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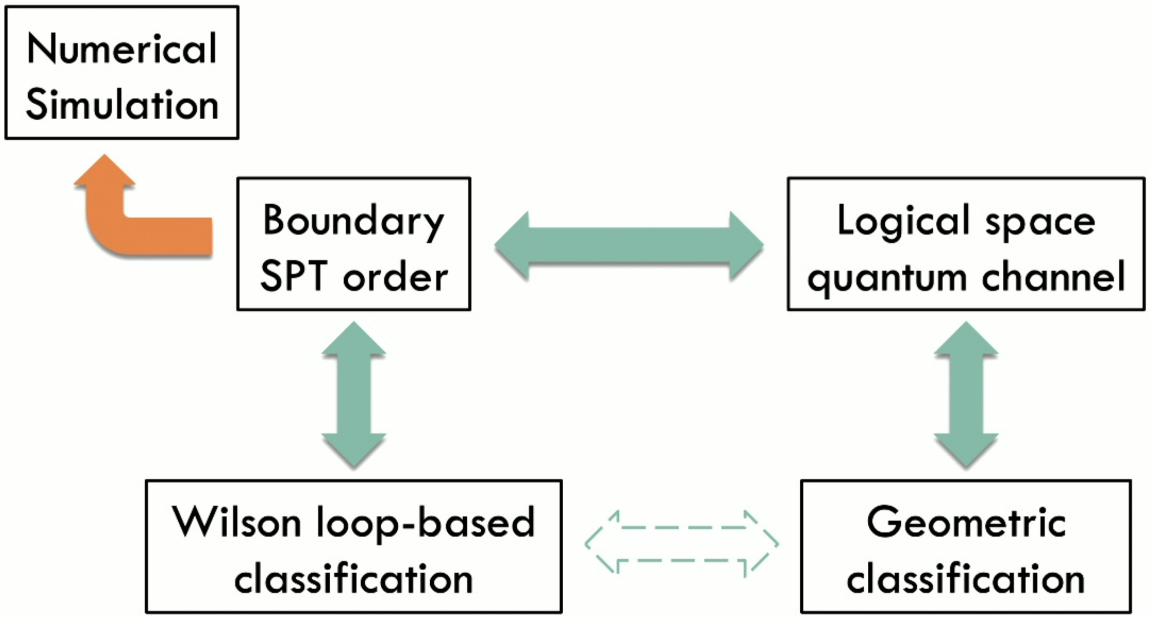

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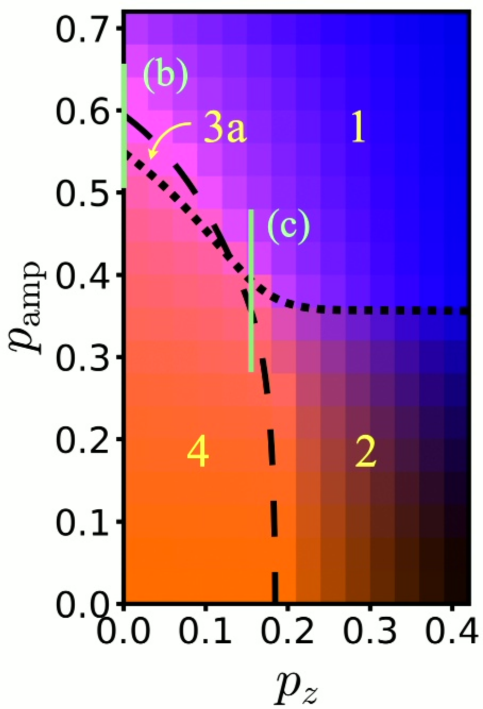

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