Title: Principal 2-group bundles and the Freed--Quinn line bundle

Speakers: Emily Cliff

Series: Mathematical Physics

Date: May 16, 2024 - 11:00 AM

URL: https://pirsa.org/24050078

Abstract: A 2-group is a categorical generalization of a group: it's a category with a multiplication operation which satisfies the usual group axioms only up to coherent isomorphisms. The isomorphism classes of its objects form an ordinary group, G. Given a 2-group G with underlying group G, we can similarly define a categorical generalization of the notion of principal bundles over a manifold (or stack) X, and obtain a bicategory Bun_G(X), living over the category Bun_G(X) of ordinary G-bundles on X. For G finite and X a Riemann surface, we prove that this gives a categorification of the Freed--Quinn line bundle, a mapping-class group equivariant line bundle on Bun_G(X) which plays an important role in Dijkgraaf--Witten theory (i.e. Chern--Simons theory for the finite group G). This talk is based on joint work with Daniel Berwick-Evans, Laura Murray, Apurva Nakade, and Emma Phillips.

I will not assume previous knowledge of 2-groups: I will provide a quick overview in the main talk, as well as a more detailed discussion during a pre-talk on Tuesday.

Zoom link

jt with Dan Berwick-Evans
Laura Murray
Apurva Nakade
Emma Phillips

1) Background / motivation

2) Classification of $\mathcal{G}$-bundles

3) Recover FQ line bundle

§1.

<u>Def</u> A 2-group $\mathcal{G}$ is a monoidal groupoid in which all objects have tensor inverses

- Classified by Sinh

<u>Ex</u> Let G be a (finite) group,

$\alpha : G \times G \times G \to U(1)$ a cocycle

$$(*) : \frac{\alpha(h,k,l)\,\alpha(g,hk,l)\,\alpha(g,h,k)}{\alpha(gh,k,l)\,\alpha(g,h,kl)} = 1$$

Define a monoidal category $\mathcal{G} = \mathcal{G}(G, U(1), \alpha)$

with objects $g \in G$.

morphisms $\mathrm{Hom}_{\mathcal{G}}(g, h) = \begin{cases} \phi & \text{if } g \neq h \\ U(1) & \text{if } g = h \end{cases}$ ← skeletal

monoidal structure: $g \otimes h = gh$

associator: $\begin{array}{ccc} (g \otimes h) \otimes k & \xrightarrow{\sim} & g \otimes (h \otimes k) \\ \| & \uparrow & \| \\ ghk & \alpha(g,h,k) & ghk \end{array}$

cocycle condition $\leftrightarrow$ pentagon axiom

Dijkgraaf-Witten theory associated to G (finite), α cocycle

$\rightsquigarrow$ 3D TQFT $\mathcal{T}$

In particular, for Σ a Riemann surface, $\mathcal{T}(\Sigma) \in \mathrm{Vect}$

Freed-Quinn: construct a line bundle

$$\begin{array}{ccc} \mathcal{L}_{FQ,\Sigma} & \longrightarrow & \mathcal{L}_{FQ} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(\Sigma) & \hookrightarrow & \mathrm{Bun}_G(\text{surfaces}) \end{array}$$

$\Gamma(\mathcal{L}_{FQ,\Sigma}) = \mathcal{T}(\Sigma)$

$\alpha(gh, k, l)$ $\alpha(g, h, kl)$

Goal relate $\mathcal{G}$ & T

More precisely we will categorify $\mathcal{L}_{FQ}$

Define a bicategory

$$\begin{tikzcd}
\boxed{\mathrm{Bun}_{\mathcal{G}}(\Sigma)} \arrow[d] \arrow[r] & \overbrace{\mathrm{Bun}_{\mathcal{G}}(\Sigma)/\sim}^{1\text{ category}} \cong \mathcal{L}_{FQ} \arrow[dl] \\
\mathrm{Bun}_G(\Sigma) &
\end{tikzcd}$$

§2

Def

... will categorify $\mathcal{L}_{FQ}$

...egory

1 category

$$\boxed{\ldots_{\mathcal{G}}(\Sigma)} \longrightarrow \overbrace{\mathrm{Bun}_{\mathcal{G}}(\Sigma)/\sim}^{\text{1 category}} \cong \mathcal{L}_{FQ}$$

$$\ldots_{G}(\Sigma)$$

over a differentiable stack $\mathcal{Y}$
(Lie groupoid)

is a stack $\mathcal{P}$ equipped with

$\mathcal{P} \xrightarrow{\pi} \mathcal{Y}$

an action of $\mathcal{G}/\mathcal{Y}$

act: $\mathcal{P} \times \mathcal{G} \to \mathcal{P}$

+ natural transformations

e.g. $\pi \circ \mathrm{Act} \cong \pi$

$\mathrm{Act} \circ (\mathrm{Act} \times \mathrm{id}) \cong \mathrm{Act} \circ (\mathrm{id} \times m)$

will categorify ... -FQ

...egory

$\boxed{\ldots_{\mathcal{G}}(\Sigma)}$ $\longrightarrow$ $\underbrace{\mathrm{Bun}_{\mathcal{G}}(\Sigma)/\sim}_{\text{1-category}} \cong \mathcal{L}_{FQ}$

$\downarrow$

$\ldots n_G(\Sigma)$

over a differentiable stack $\mathcal{Y}$
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is a stack $\mathcal{P}$ equipped with
$\downarrow \pi$ an action of $\mathcal{G}/\mathcal{Y}$
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$\mathrm{act}: \mathcal{P} \times \mathcal{G} \to \mathcal{P}$
+ natural transformations
e.g. $\pi \circ \mathrm{Act} \cong \pi$
$\mathrm{Act} \circ (\mathrm{Act} \times \mathrm{id}) \cong \mathrm{Act} \circ (\mathrm{id} \times m)$

which is locally trivial.

locally: $\mathcal{Y} \times \mathcal{G} \xrightarrow[\sim]{\Phi} \mathcal{P}$
equivariant (data!)

Classification via Čech data:

Warm-up G bundles over $\mathcal{Y}$: $\mathcal{P} \xrightarrow{\pi} \mathcal{Y}$

· $\exists$ atlas $u: Y_0 \longrightarrow \mathcal{Y}$ & iso $Y_0 \times G \xrightarrow{\Phi} u^*\mathcal{P}$

· look at $\underset{\substack{\| \\ Y_1}}{Y_0 \times_{\mathcal{Y}} Y_0} = \{ (t, s, \varphi) \mid t, s \in Y_0, \ u(s) \xrightarrow[\sim]{\varphi} u(t) \}$

$$\left.\begin{array}{c} Y_1 \\ t \downarrow \ \downarrow s \\ Y_0 \end{array}\right\} \text{Lie groupoid presentation of } \mathcal{Y}$$

have.
$$\begin{tikzcd} & Y_1 \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, phantom, "\Leftarrow"] & \\ Y_0 \arrow[dr, "u"'] & & Y_0 \arrow[dl, "u"] \\ & \mathcal{Y} & \end{tikzcd} \rightsquigarrow \quad Y_1 \times G \xrightarrow{s^*\Phi} s^*u^*\mathcal{P} \xrightarrow{\sim} t^*u^*\mathcal{P} \xrightarrow[\sim]{t^*\Phi^{-1}} Y_1 \times G$$

$\mathcal{P} \xrightarrow{\pi} \mathcal{Y}$

& iso $Y_0 \times G \xrightarrow{\Phi} u^* \mathcal{P}$

$\{(t, s, \varphi) \mid t, s \in Y_0,\ u(s) \xrightarrow[\sim]{\varphi} u(t)\}$

...poid presentation of $\mathcal{Y}$

$\xrightarrow{s^* \Phi} s^* u^* \mathcal{P} \xrightarrow{\sim} t^* u^* \mathcal{P} \xrightarrow[\sim]{t^* \Phi^{-1}} Y_1 \times G$

$\longmapsto (t, s, \rho(t,s) \cdot g)$

$\rho : Y_1 \longrightarrow G$ is a Čech cocycle:

① $\rho(y_1, y_2)\, \rho(y_2, y_3) = \rho(y_1, y_3)$

Conversely, cocycle ρ on $Y_1 \rightrightarrows Y_0$, determines a principal bundle

$Y_1 \times G \ni (y_1, y_2, g)$

$\mathcal{P} = \quad Y_1 \times G \overset{t}{\underset{s}{\rightrightarrows}} Y_0 \times G \qquad (y_2, g) \xrightarrow{\sim} (y_1, \rho(y_1, y_2) g)$

$\mathcal{Y} \times \mathcal{G} \longrightarrow \mathcal{P}$

equivariant

(data!

A morphism $\mathcal{P} \xrightarrow{H} Q$

$\longleftrightarrow$ (over some atlas)

$h : Y_0 \longrightarrow G$ s.t.

$h(y_1)\, g_P(y_1, y_2) = g_Q(y_1, y_2)\, h(y_2)$.

s.t.

$h(y_0)$.

$$Y_0 \times_{\mathcal{Y}} Y_0 \times_{\mathcal{Y}} Y_0$$

• <u>objects.</u> $\left(u_0 : Y_0 \to \mathcal{Y},\ \rho : Y_1 \to G,\ \gamma : Y_2 \xrightarrow{\text{loc const}} U(1)\right)$

(1) becomes $\rho(y_1,y_2)\,\rho(y_2,y_3) \xleftarrow[\gamma(y_1,y_2,y_3)]{\sim} \rho(y_1,y_3)$ in $\mathcal{G}$

ϕ, ψ, $\phi \circ \psi$

s.t. $d\gamma = \rho^*\alpha$

• <u>1-morphisms.</u> $(\rho_1,\gamma_1) \longrightarrow (\rho_2,\gamma_2)$ (over same atlas Y_0)

$$\left(h : Y_0 \to G,\ \eta : Y_1 \xrightarrow{\text{loc const.}} U(1)\right)$$

(2) become $h(y_1)\,\rho_1(y_1,y_2) \xleftarrow[\eta(y_1,y_2)]{\sim} \rho_2(y_1,y_2)\,h(y_2)$ + condition on $d\eta$.

$\boxed{\mathrm{Bun}_{\mathcal{G}}(\Sigma)} \to \mathrm{Bun}_{\mathcal{G}}(\Sigma)/\sim \;\cong \mathcal{L}_{FQ}$

$\downarrow$

$\mathrm{Bun}_G(\Sigma)$

is a stack

which is locally trivial.

locally: $\mathcal{Y}\times\mathcal{G} \xrightarrow[\sim]{\Phi} \mathcal{P}$ equivariant (data!)

$\mathcal{P} \xrightarrow{\pi} \mathcal{Y}$

an action of $\mathcal{G}$ / $\mathcal{Y}$

$\mathrm{act}: \mathcal{P}\times\mathcal{G} \to \mathcal{P}$

+ natural transformations

e.g. $\pi \circ \mathrm{Act} \cong \pi$

$\mathrm{Act}\circ(\mathrm{Act}\times\mathrm{id}) \cong \mathrm{Act}\circ(\mathrm{id}\times m)$

A morphism $\mathcal{P} \xrightarrow{H} Q$

$\longleftrightarrow$ (over some atlas)

$h: Y_0 \to G$ s.t.

(2) $h(y_1)\, g_P(y_1,y_2) = g_Q(y_1,y_2)\, h(y_2)$.

2-morphisms $\omega: Y_0 \xrightarrow{\text{loc const}} U(1)$

Theorem [BCMNP] For $\mathcal{G} = \mathcal{G}(G, U(1), \alpha)$, principal $\mathcal{G}$-bundles form a bicategory with

- objects: $(u_0: Y_0 \to \mathcal{Y},\; g: Y_1 \to G,\; \gamma: Y_2 \xrightarrow{\text{loc const}} U(1))$, $Y_0 \times_{\mathcal{Y}} Y_0 \times_{\mathcal{Y}} Y_0$

(1) becomes $\overset{\Phi}{g(y_1,y_2)}\, \overset{\Psi}{g(y_2,y_3)} \xleftarrow[\gamma(y_1,y_2,y_3)]{\sim} \overset{\Phi\cdot\Psi}{g(y_1,y_3)}$ in $\mathcal{G}$

s.t. $d\gamma = g^*\alpha$

- 1-morphisms $(g_1,\gamma_1) \longrightarrow (g_2,\gamma_2)$ (over same atlas Y_0)

$(h: Y_0 \to G,\; \eta: Y_1 \xrightarrow{\text{loc const}} U(1))$

(2) become $h(y_1)\, \overset{\Phi}{g(y_1,y_2)} \xleftarrow[\eta(y_1,y_2)]{\sim} g_2(y_1,y_2)\, h(y_2)$ + condition on $d\eta$

$Y_1 \times G \longrightarrow$

$Y_0 \xleftarrow{t} Y_1 \xrightarrow{s} Y_0$ $\rightsquigarrow$ $(t, s, g) \longmapsto (t, s, g(t,s)\cdot g)$

$u \to \mathcal{Y} \leftarrow u$ $\qquad \phi \qquad \phi \qquad \phi$

Example. $\mathcal{Y} = X$ manifold with $\overset{Y_0}{\tilde{X}} \to X$ contractible universal cover

$Y_1 = \tilde{X} \times_X \tilde{X} \xrightarrow[\text{cocycle}]{g} G \quad \longleftrightarrow$ homomorphism

$\cong \tilde{X} \times \pi_1(X)$ $\qquad \hat{g} : \pi_1(X) \longrightarrow G$

$Y_2 = \tilde{X} \times_X \tilde{X} \times_X \tilde{X} \xrightarrow[\text{loc. const}]{\gamma} U(1) \quad \longleftrightarrow \quad \hat{\gamma} : \pi_1(X)^2 \longrightarrow U(1)$

$\cong \tilde{X} \times \pi_1(X)^2$ $\qquad$ providing $\hat{g}(a \cdot b) \cong \hat{g}(a)\, \hat{g}(b)$.

Def $(\hat{g}, \hat{\gamma})$ is a homomorphism $\pi_1(X) \longrightarrow \mathcal{G}$

"holonomy"

Theorem [BCMNP] (X a $K(\pi, 1)$)

$$\mathrm{Bun}_{\mathcal{G}}(X) \cong \mathrm{Fun}_{\mathrm{Bicat}}(*/\!/\pi_1(X), */\!/\mathcal{G})$$

Ex #2 if $G = \{1\}$, $\mathcal{G} = */\!/U(1)$

$\cdot\ (g, \gamma)$ $\quad - \ g = 1$. $\quad d\gamma = g^*\alpha = 1$

$\Rightarrow \mathrm{Bun}_{\mathcal{G}}(\mathcal{Y}) = \mathrm{Gerbe}_{U(1)}(\mathcal{Y})$ (flat)

② become $h(y_1)\, g(y_1, y_2) \xleftarrow[\eta(y_1,y_2)]{\sim} g_2(y_1, y_2)\, h(y_2)$ + condition on $d\eta$

Corollary Functor

$$\begin{array}{ccc} \mathrm{Bun}_{\mathcal{G}}(Y) & \ni & (u, g, \gamma) \\ \pi \downarrow & & \downarrow \\ \mathrm{Bun}_G(Y) & \ni & (u, g) \end{array}$$

$\leftarrow$ 2-torsor for sym monoidal category

$\mathrm{Gerbe}_{U(1)}(Y)$

Fibres $\{(u, g, \gamma) \mid d\gamma = g^*\alpha\} \circlearrowleft \mathrm{Gerbe}_{U(1)}(Y) = \{\delta \mid d\delta = 1\}$

$(u, g, \gamma)\cdot\delta = (u, g, \gamma\delta)$

$d(\gamma\delta) = d\gamma\, d\delta = g^*\alpha \cdot 1$ ✓

§3 Recovering FQ

Let $Y = \Sigma$ Riemann surface

Take iso classes along fibres of π

$\mathrm{Gerbe}_{U(1)}(\Sigma)/\sim \; = H^2(\Sigma, U(1)) \cong U(1)$

$X \times \pi_1(X)$ $\qquad \rho(a \cdot b) = \rho(a)\rho(b)$.

Def. $(\hat{\rho}, \hat{\gamma})$ is a homomorphism $\pi_1(X) \to \mathcal{G}$
"holonomy"

Obtain $\mathcal{C} = \mathrm{Bun}_{\mathcal{G}}(\Sigma)/\sim$

$\downarrow$ principal $U(1)$-bundle

$\mathrm{Bun}_G(\Sigma)$ $\quad$ MCG equivariant.

Thm [BCMNP] $\mathcal{C}$ is isomorphic to the $U(1)$-bundle associated to the FQ line bundle

Iso respects the action of MCG

Applications / Future directions

1) (At least for me), it is easier to work out examples in $\mathcal{C}$

Ex $\Sigma = \mathbb{T}$ $\qquad \pi_1(\Sigma) = \mathbb{Z}^2$

$\mathrm{MCG} = SL_2(\mathbb{Z})$

$\cdot\ \hat{\rho} \in \mathrm{Bun}_G(\Sigma)$ $\quad$ look at $A \in \mathrm{Stab}_{\mathrm{MCG}}(\hat{\rho})$ acting on fibre over $\hat{\rho}$

for $[\hat{\rho}, \hat{\gamma}] \in \mathcal{C}$, $\quad [\hat{\rho}, \hat{\gamma}] \cdot A$ in same fibre, differs from $[\hat{\rho}, \hat{\gamma}]$ by $\phi(A, \hat{\rho}) \in U(1)$

$\mathrm{Gerbe}_{U(1)}(\Sigma)/\sim \; = H^{?}(\Sigma, U(1)) \cong U(1)$

$\hookrightarrow$ this element is: $\dfrac{\hat{\gamma}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)}{\hat{\gamma}\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)} \Bigg/ \dfrac{\hat{\gamma}\left(A\begin{bmatrix}1\\0\end{bmatrix}, A\begin{bmatrix}0\\1\end{bmatrix}\right)}{\hat{\gamma}\left(A\begin{bmatrix}0\\1\end{bmatrix}, A\begin{bmatrix}1\\0\end{bmatrix}\right)}$

Rmk for certain α, the value $\phi(A, \vec{p})$ is the square of the corresponding value for the determinant line bundle (Freed)

• Future move to compact (non-flat) case