

Title: Principal 2-group bundles and the Freed--Quinn line bundle

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Series: Mathematical Physics

Date: May 16, 2024 - 11:00 AM

URL: <https://pirsa.org/24050078>

Abstract: A 2-group is a categorical generalization of a group: it's a category with a multiplication operation which satisfies the usual group axioms only up to coherent isomorphisms. The isomorphism classes of its objects form an ordinary group, G . Given a 2-group G with underlying group G , we can similarly define a categorical generalization of the notion of principal bundles over a manifold (or stack) X , and obtain a bicategory $\text{Bun}_G(X)$, living over the category $\text{Bun}_G(X)$ of ordinary G -bundles on X . For G finite and X a Riemann surface, we prove that this gives a categorification of the Freed--Quinn line bundle, a mapping-class group equivariant line bundle on $\text{Bun}_G(X)$ which plays an important role in Dijkgraaf--Witten theory (i.e. Chern--Simons theory for the finite group G). This talk is based on joint work with Daniel Berwick-Evans, Laura Murray, Apurva Nakade, and Emma Phillips.

I will not assume previous knowledge of 2-groups: I will provide a quick overview in the main talk, as well as a more detailed discussion during a pre-talk on Tuesday.

Zoom link

jt with Dan Berwick-Evans

Laura Murray

Apurva Nakode

Emma Phillips

1) Background / motivation

2) Classification of \mathcal{G} -bundles

3) Recover $\mathbb{F}\mathbb{Q}$ line bundle

§1.

Def A 2-group \mathcal{G} is a monoidal groupoid in which all objects have tensor inverses.

• Classified by Sinh

Ex Let G be a (finite) group,

$\alpha: G \times G \times G \rightarrow U(1)$ a cocycle

$$(*) \quad \frac{\alpha(h, k, l) \alpha(g, hk, l) \alpha(g, h, k)}{\alpha(gh, k, l) \alpha(g, h, kl)} = 1$$

Define a monoidal category $\mathcal{C}_g = \mathcal{C}(G, \text{UV}, \alpha)$

with objects $g \in G$.

morphisms $\text{Hom}_{\mathcal{C}_g}(g, h) = \begin{cases} \emptyset & \text{if } g \neq h \\ \text{UCI} & \text{if } g = h \end{cases}$ ← skeletal

monoidal structure $g \otimes h = gh$

associator: $(g \otimes h) \otimes k \xrightarrow{\alpha(g, h, k)} g \otimes (h \otimes k)$
 $\uparrow \quad \quad \quad \parallel$
 $ghk \quad \quad \quad ghk$

cocycle condition \leftrightarrow pentagon axiom

Dijkgraaf-Witten theory associated

to G (finite)
 α cocycle

\times 3D TQFT \mathcal{T}

In particular, for Σ a Riemann surface, $\mathcal{T}(\Sigma) \in \text{Vect}$

Freed-Quinn construct a line bundle \mathcal{L}_{FQ}

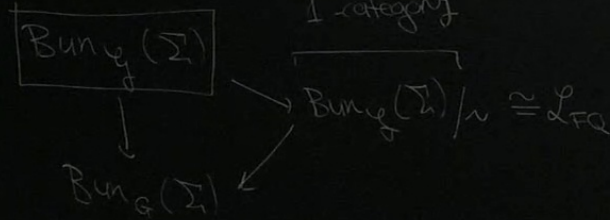
$\mathcal{L}_{\text{FQ}, \Sigma} \rightarrow \mathcal{L}_{\text{FQ}}$
 $\downarrow \quad \quad \downarrow$
 $\text{Bun}_G(\Sigma) \hookrightarrow \text{Bun}_G(\text{surfaces})$
 $\Gamma(\mathcal{L}_{\text{FQ}, \Sigma}) = \mathcal{T}(\Sigma)$

$$\alpha(g, h, k, l) \quad \alpha(g, h, k, l)$$

Goal relate \mathcal{G} & \mathcal{T}

More precisely we will categorify \mathcal{L}_{FA}

Define a bicategory

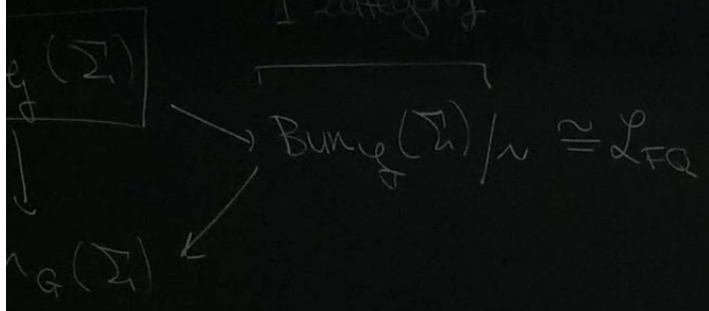


§ 2.

Def

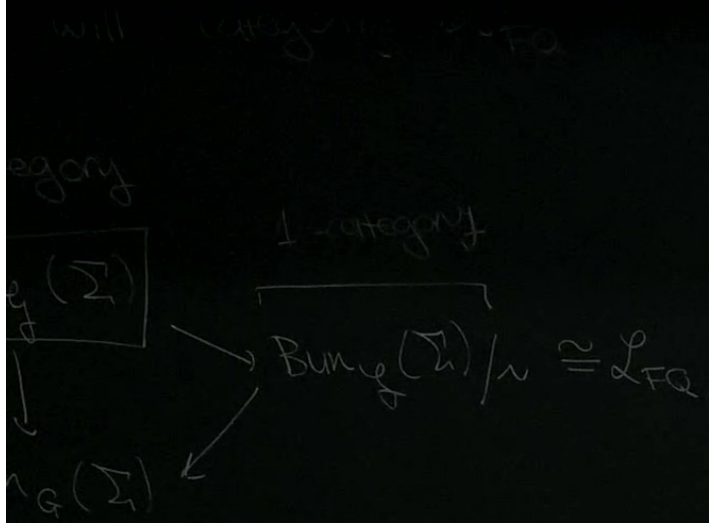
will categorify $\mathcal{L}FQ$

category



over a differentiable stack \mathcal{Y}
(Lie groupoid)

is a stack \mathcal{P} equipped with
 $\downarrow \pi$
 \mathcal{Y}
 an action of \mathcal{Y} / \mathcal{Y}
 $act: \mathcal{P} \times_{\mathcal{Y}} \mathcal{Y} \rightarrow \mathcal{P}$
 + natural transformations
 eg $\pi \circ Act \cong \pi$
 $Act \circ (Act \cdot id) \cong Act \circ (id \cdot id)$



over a differentiable stack \mathcal{Y} (Lie groupoid)

is a stack \mathcal{P} equipped with an action of \mathcal{Y} / \mathcal{Y}

which is locally trivial.

locally: $\mathcal{P} \times_{\mathcal{Y}} \mathcal{P} \xrightarrow{\cong} \mathcal{P}$

equivariant (data!)

$\downarrow \pi$

\mathcal{Y}

act: $\mathcal{P} \times_{\mathcal{Y}} \mathcal{P} \rightarrow \mathcal{P}$

+ natural transformations

e.g. $\pi \circ \text{Act} \cong \pi$

$\text{Act} \circ (\text{Act} \times \text{id}) \cong \text{Act} \circ (\text{id} \times \text{id})$

Classification via Čech data:

Warm-up G bundles over $Y \xrightarrow{\pi} Y$

• atlas $u_0: Y_0 \rightarrow Y$ & iso $Y_0 \times G \xrightarrow{\Phi} u^*P$

look at $Y_0 \times Y_0 \xrightarrow{\sim} Y_0$ = $\{(t, s, \varphi) \mid t, s \in Y_0, u(s) \xrightarrow{\varphi} u(t)\}$
 Lie groupoid presentation of Y

have $Y_1 \times G \xrightarrow{s^* \Phi} s^* u^* P \xrightarrow{\sim} t^* u^* P \xrightarrow{t^* \Phi^{-1}} Y_1 \times G$

of Y
 P
 ations
 $\text{Ad}^*(\text{id})$
 $\times m$

A morphism $\mathcal{P} \xrightarrow{H} \mathcal{Q}$

\Leftrightarrow (over some atlas)

$h: Y_0 \rightarrow G$ s.t.

$$h(y) s_P(y, y_0) = s_Q(y, y_0) h(y_0).$$

$y \times y \rightarrow y$
 $\swarrow \searrow$
equivariant
(data)

• objects: $(u_0: Y_0 \rightarrow Y, \quad g: Y_1 \rightarrow G, \quad \gamma: Y_2 \xrightarrow{\text{loc const}} U(1))$

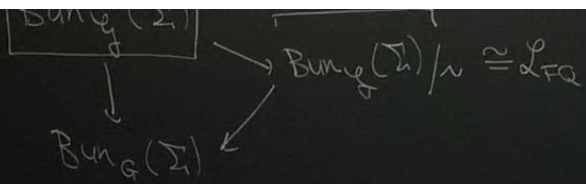
① becomes $g(Y_1, Y_2) \xleftarrow{\phi} g(Y_2, Y_3) \xleftarrow{\psi} g(Y_1, Y_3) \xleftarrow{\gamma(Y_1, Y_2, Y_3)} g(Y_1, Y_3)$ in \mathcal{G}_g

s.t. $d\gamma = g^* \alpha$

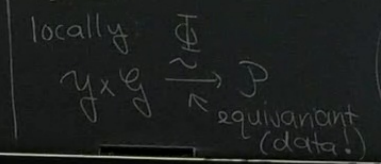
• 1-morphisms

$(g_1, \gamma_1) \xrightarrow{\quad} (g_2, \gamma_2)$ (over same atlas Y_0)
 $(h: Y_0 \rightarrow G, \quad \mu: Y_1 \rightarrow U(1))$ (loc const.)

② become $h(Y_1) \xleftarrow{\phi} g_1(Y_1, Y_2) \xleftarrow{\gamma(Y_1, Y_2)} g_2(Y_1, Y_2) \xleftarrow{\mu(Y_1, Y_2)} h(Y_2) + \text{condition on } d\gamma$



which is locally trivial.



an action of \mathcal{G} / \mathcal{Y}
 $\text{act: } \mathcal{P} \times \mathcal{G} \rightarrow \mathcal{P}$
 + natural transformations
 eg $\pi \circ \text{Act} \cong \pi$
 $\text{Act} \circ (\text{Act} \times \text{id}) \cong \text{Act} \times \text{id}$

A morphism $\mathcal{P} \xrightarrow{H} \mathcal{Q}$

\Leftrightarrow (over some atlas)

$h: Y_0 \rightarrow G$ s.t.

$$(2) \quad h(y) g_{\mathcal{P}}(y, y_0) = g_{\mathcal{Q}}(y, y_0) h(y_0)$$

2-morphisms w: $Y_0 \rightarrow U(1)$ (loc. const)

Theorem [BCMNPF] For $\mathcal{G} = \mathcal{G}(G, U(1), \omega)$, principal \mathcal{G} -bundles

form a bicategory with

• objects. $(u, Y_0 \rightarrow Y_1, g: Y_1 \rightarrow G, \gamma: Y_2 \xrightarrow{\text{loc. const}} U(1))$

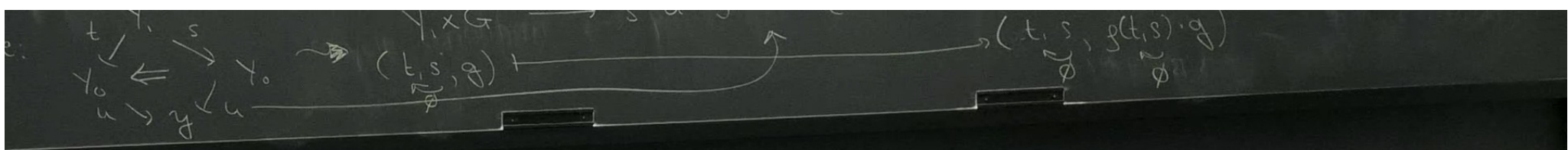
$$(1) \text{ becomes } g_1(y_1, y_2) \xleftarrow{\sim} g_2(y_1, y_2) \xleftarrow{\sim} g_3(y_1, y_2) \text{ in } \mathcal{G}$$

s.t. $d\gamma = g^* \omega$

• 1-morphisms

$$(g_1, \gamma_1) \xrightarrow{\text{loc. const}} (g_2, \gamma_2) \text{ (over same atlas } Y_0)$$

$$(2) \text{ become } h_1(y) g_1(y, y_0) \xleftarrow{\sim} h_2(y) g_2(y, y_0) h_1(y_0) + \text{condition on } d\gamma$$



example: $Y = X$ manifold with $\tilde{X} \xrightarrow{p} X$
 contractible universal cover

$$Y_1 = \tilde{X} \times \tilde{X} \xrightarrow{\beta} G \text{ cocycle} \iff \text{homomorphism } \hat{\beta} : \pi_1(X) \rightarrow G$$

$$\tilde{X} \times \pi_1(X)$$

$$Y_2 = \tilde{X} \times \tilde{X} \times \tilde{X} \xrightarrow{\gamma} U(1) \text{ loc. const} \iff \hat{\gamma} : \pi_1(X)^2 \rightarrow U(1)$$

$$\tilde{X} \times \pi_1(X)^2$$

providing $\hat{\beta}(a \cdot b) \cong \hat{\beta}(a) \hat{\beta}(b)$

Def. $(\hat{\beta}, \hat{\gamma})$ is a homomorphism $\pi_1(X) \rightarrow \mathcal{G}_G$
 "holonomy"

Theorem [BCMNP] (X a $K(\pi, 1)$)

$$\text{Bun}_G(X) \cong \text{Fun}_{\text{Bicat}}(*//\pi_1(X), *//\mathcal{G}_G)$$

Ex #2 if $G = \{1\}$ $\mathcal{G}_G = *//U(1)$
 $\cdot (\beta, \gamma) - \beta = 1, d\gamma = \beta^* \alpha = 1$
 $\hookrightarrow \text{Bun}_G(Y) = \text{Gerbe}_{U(1)}(Y) \text{ (flat)}$

② become $h(y_1) \xrightarrow{g(y_1, y_2)} h(y_2) + \text{condition on } dy$

Coating Functor

$$\begin{array}{ccc} \text{Bun}_Y(Y) & \ni & (u, g, \alpha) \\ \pi \downarrow & & \downarrow \\ \text{Bun}_G(Y) & \ni & (u, g) \end{array}$$

↳ answer for sym. modular invariance

Gerbe $\text{Gerbe}_{\text{U(1)}}(Y)$

Fibres $\{ (u, g, \alpha) \mid d\alpha = g^* \alpha \} \cong \text{Gerbe}_{\text{U(1)}}(Y)$

$$\{ \delta \mid d\delta = 1 \}$$

$$(u, g, \alpha) \cdot \delta = (u, g, \alpha \delta)$$

$$d(\alpha \delta) = d\alpha \delta = g^* \alpha \cdot 1 \quad \checkmark$$

§3 Recovering FQ

Let $Y = \Sigma$ Riemann surface

Take iso classes along fibres of π

$$\text{Gerbe}_{\text{U(1)}}(\Sigma) / \sim = H^2(\Sigma, \text{U(1)}) \cong \text{U(1)}$$

$X \times \pi_1(X)$
 $f(a \cdot x) = f(a) \cdot f(x)$
 Def. $(\tilde{p}, \tilde{\gamma})$ is a homomorphism $\pi_1(X) \rightarrow \mathcal{G}$
 "holonomy"

Obtain $\mathcal{E} = \text{Bun}_G(\Sigma) / \sim$

\downarrow
 $\text{Bun}_G(\Sigma)$ principal U(1)-bundle
 MCG equivariant

Thm [BCMNP] \mathcal{E} is isomorphic to the
 U(1)-bundle associated to the \mathbb{F}_Q line bundle
 Iso respects the action of MCG

Applications / Future directions

1) (At least for me) it is easier to
 work out examples in \mathcal{E}

Ex $\Sigma = \mathbb{T}$ $\pi_1(\Sigma) = \mathbb{Z}^2$
 $\text{MCG} = \text{SL}_2(\mathbb{Z})$

$\tilde{p} \in \text{Bun}_G(\Sigma)$ look at $\text{Act}_{\text{MCG}}(\tilde{p})$
 acting on fibre over \tilde{p}

For $[\tilde{p}, \tilde{\gamma}] \in \mathcal{E}$, $[\tilde{p}, \tilde{\gamma}] \cdot A$ in same fibre
 differs from $[\tilde{p}, \tilde{\gamma}]$ by $\phi(\lambda, \tilde{p}) \in \mathcal{U}(1)$

$$\text{Genus } g(\mathcal{L}) = \frac{1}{2} \pi(\mathcal{L}, \mathcal{L}) \cong \mathcal{L}(1)$$

↳ this element is:

$$\frac{\hat{\rho}([0], [0])}{\hat{\rho}([0], [0])} \bigg/ \frac{\hat{\rho}(A[0], A[0])}{\hat{\rho}(A[0], A[0])}$$

Rmk for certain α , the value $\phi(A, \hat{p})$ is the square of the corresponding value for the determinant line bundle (Freed)

Future move to compact (non-flat) case