Title: Circuit-to-Hamiltonian from tensor networks and fault tolerance

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Series: Perimeter Institute Quantum Discussions

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Abstract: We define a map from an arbitrary quantum circuit to a local Hamiltonian whose ground state encodes the quantum computation. All previous maps relied on the Feynman-Kitaev construction, which introduces an ancillary 'clock register' to track the computational steps. Our construction, on the other hand, relies on injective tensor networks with associated parent Hamiltonians, avoiding the introduction of a clock register. This comes at the cost of the ground state containing only a noisy version of the quantum computation, with independent stochastic noise. We can remedy this - making our construction robust - by using quantum fault tolerance. In addition to the stochastic noise, we show that any state with energy density exponentially small in the circuit depth encodes a noisy version of the quantum computation with adversarial noise. We also show that any 'combinatorial state' with energy density polynomially small in depth encodes the quantum computation with adversarial noise. This serves as evidence that any state with energy density polynomially small in depth has a similar property. As an application, we give a new proof of the QMA-completeness of the local Hamiltonian problem (with logarithmic locality) and show that contracting injective tensor networks to additive error is BQP- hard. We also discuss the implication of our construction to the quantum PCP conjecture, combining with an observation that QMA verification can be done in logarithmic depth. Based on joint work with Anurag Anshu and Nikolas P. Breuckmann. (https://arxiv.org/abs/2309.16475)

Zoom link

Circuit-to-Hamiltonian from tensor networks and fault tolerance

Quynh Nguyen (Harvard)

Joint with Anurag Anshu (Harvard) and Nikolas Breuckmann (Bristol)

arXiv:2309.16475 (ver 2 soon)

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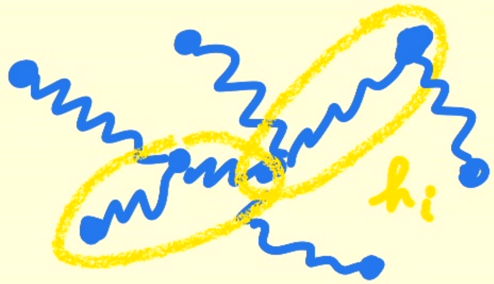
Perimeter Institute, May 15, 2024

Circuit-to-Hamiltonian

Given a (classical/quantum) circuit, derive a **local** Hamiltonian
such that we can “easily” extract computation output from its ground state

Dynamic

Static



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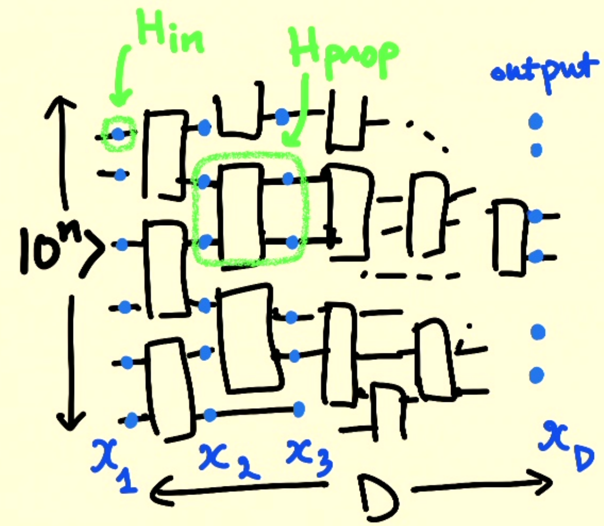
Classical: Cook-Levin mapping

[Theorem (1970s)]:
Given classical circuit, can construct classical local Hamiltonian H_{CL} s.t. unique ground state encodes history of computation.



Ground state $x_{hist} = x_1 \otimes x_2 \ldots \otimes x_D$

~ nD bits

Hamiltonian (Pauli Z) $H_{CL} = H_{in} + H_{prop}$

Foundational result in TCS! *Computational complexity: NP-completeness, PCP theorem.*
Physics: NP-hardness of Ising model, Gibbs states.

Quantum: Feynman-Kitaev mapping

Can we do Cook-Levin history state $\psi_{hist} = |\psi_0\rangle \otimes |\psi_1\rangle \ldots \otimes |\psi_D\rangle$?

No! Quantum states are **not locally distinguishable!**

Example: Say $\psi_{t-1} = |CAT_n^+\rangle$ and $U_t = Z_n$
Then expect $\psi_t = |CAT_n^-\rangle$, but they locally look the same.

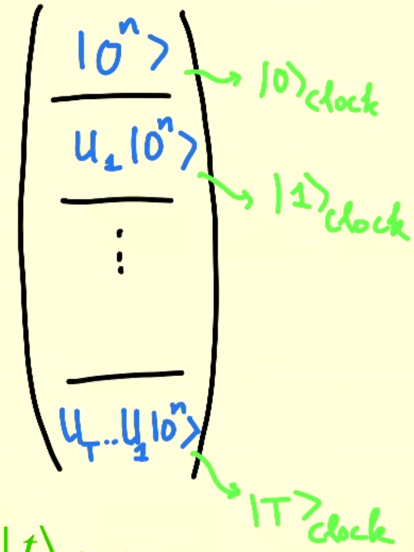
[Theorem (Kitaev 99)]:
Given quantum circuit, can construct quantum local Hamiltonian s.t. unique ground state encodes history of computation.

Idea: lay out history in superposition
(Feynman 85)

$$\Psi_{hist} = \sum_{t=0}^{T} |t\rangle_{clock} \otimes U_t \ldots U_1 |0^n\rangle_{comp}$$

$$T \sim nD$$

Quantum: Feynman-Kitaev mapping



FK history state

$$\Psi_{hist} = \sum_{t=0}^{T} |t\rangle_{clock} \otimes U_t \ldots U_1 |0^n\rangle_{comp}$$

is unique ground state of $H_{FK} = H_{in} + \sum_t H_{prop,t}$

Init. state is $|0^n\rangle$ in $|0\rangle_{\text{clock}}$

Gate U_t between $|t-1\rangle$ and $|t\rangle_{\text{clock}}$

Quantum complexity ⟷ **Quantum many-body physics**

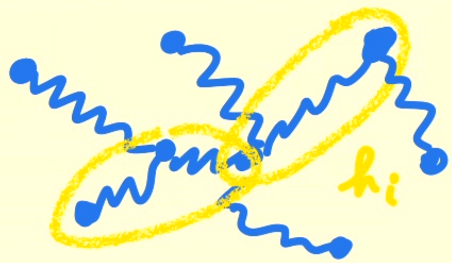
QMA complexity class, delegation of QC, equivalence of adiabatic QC

Local Hamiltonian problem, counter-examples to area law, ground state circuit complexity lowerbounds

Motivation: Quantum PCP conjecture

k-local Hamiltonian

$H = \sum_{i=1}^{m} h_i$



Is $\frac{\langle \psi_{gs} | H | \psi_{gs} \rangle}{m} \leq 0$ or $\geq a$?

[Kitaev 99]: QMA-hard if $a = 1/\text{poly}(n)$

Proved using FK circuit-to-Hamiltonian.

[Conjecture]: Still QMA-hard if $a = \Theta(1)$

TCS: analogue of classical PCP theorem, quantum proof checking, hardness of approximation
Physics: quantumness at "room" temperature. E.g. high circuit complexity in excited states (NLTS)

Raised in Quantum NP - A Survey, 2002 by Aharonov and Naveh, still open!

Our work

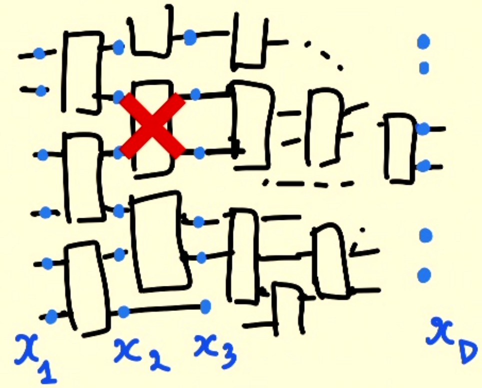
Feynman-Kitaev lacks a **robustness property** of Cook-Levin

<u>Expectation</u>: this property could be useful for **quantum PCP conjecture**, via a **connection with fault-tolerant computation**

We make progress in obtaining a new quantum circuit-to-Hamiltonian mapping with this property

New proof of QMA-completeness for log-local Hamiltonian & some results in computational complexity of tensor networks

Feynman-Kitaev is not as robust as Cook-Levin



[Cook-Levin robustness] unsatisfying bit strings (excited states) still encode **noisy** computation

E.g. if x violates only one term in H_{CL}

$$\langle x | H_{CLprop,t} | x \rangle \neq 0$$

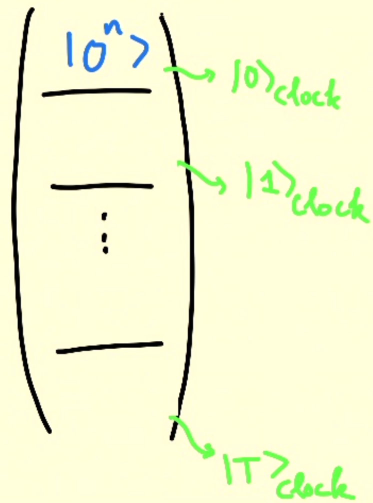
then x encodes a noisy computation with 1 faulty gate

Not true in Feynman-Kitaev!

$$\Psi = |0\rangle_{clock} |0^n\rangle_{comp}$$

Contains no information about computation

But only violates one term in $H_{FK} = H_{in} + \sum_t H_{prop,t}$



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Construction

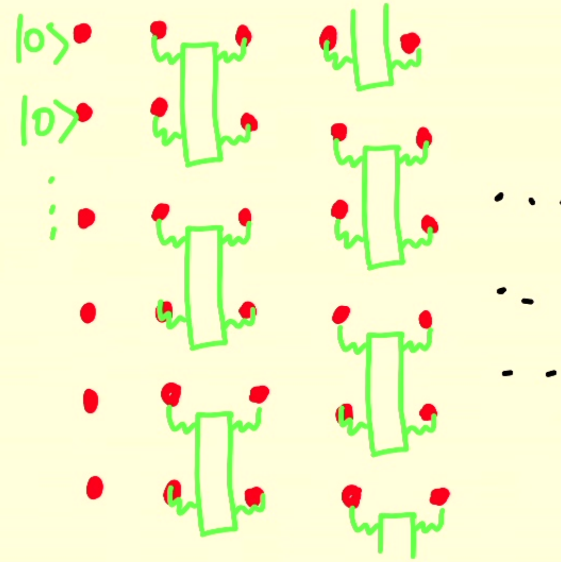
Step 1: encoding quantum circuit into a simple state

Product state works!

$$|\Psi_{prod}\rangle = |0\rangle^{\otimes n} |\phi_{U_1}\rangle \dots |\phi_{U_T}\rangle$$

$$|\phi_U\rangle = \mathrm{Id} \otimes U(|00\rangle + |11\rangle)^{\otimes 2}$$ (Choi state)

$$H_{prod} = \sum_{i=1}^{n} |1\rangle\langle 1|_i + \sum_{t=1}^{T} (1 - |\phi_{U_t}\rangle\langle\phi_{U_t}|)$$

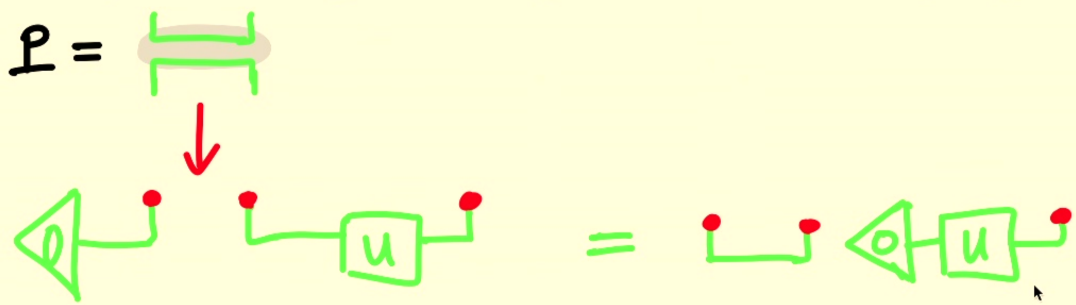


Issue: can't easily extract computation output from this state

Step 2: enforcing consistency between gates

Consider circuit with 1 qubit and 1 gate

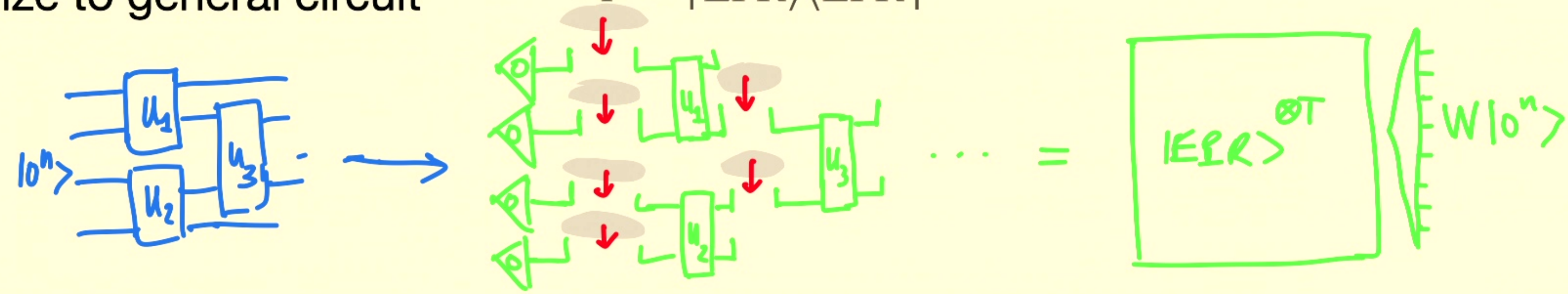
Can "implement" gate using Choi state and projector $P = |\mathrm{EPR}\rangle\langle\mathrm{EPR}|$



Step 2: enforcing consistency between gates

Generalize to general circuit

$P = |\text{EPR}\rangle\langle\text{EPR}|$



This is a tensor network known as **Projected Entangled Pair State (PEPS)**

$$|\Psi_{PEPS}\rangle = P^{\otimes nD}(|0\rangle^{\otimes n}|\phi_{U_1}\rangle \dots |\phi_{U_T}\rangle)$$

Issue: No longer unique ground state of a local Hamiltonian

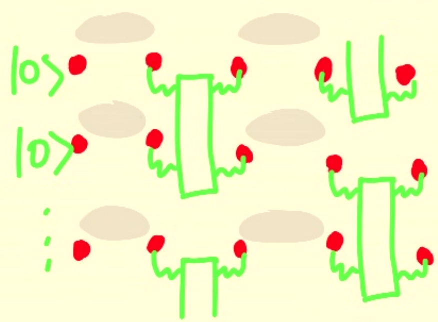
Reconciling 2 steps: local Hamiltonian from injective PEPS

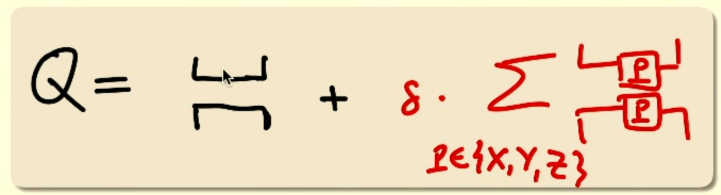
Replace EPR projector P by an **invertible** map Q → **injective** PEPS state

$$|\Psi_{iPEPS}\rangle = Q^{\otimes nD}(|0\rangle^{\otimes n}|\phi_{U_1}\rangle\ldots|\phi_{U_T}\rangle)$$

Injective PEPS is unique ground state of **parent Hamiltonian** (8-local)

$$H_{TN} = \sum_{i=1}^{n} Q^{-1}|1\rangle\langle 1|_i Q^{-1} + \sum_{t=1}^{T}(Q^{-1})^{\otimes 4}(1-|\phi_{U_t}\rangle\langle\phi_{U_t}|)(Q^{-1})^{\otimes 4}$$





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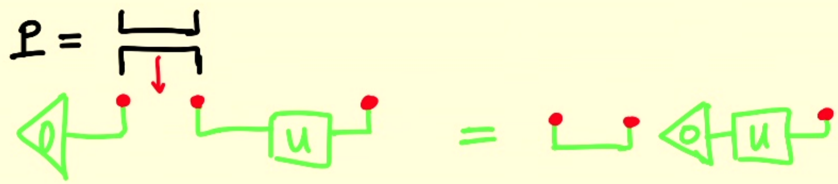
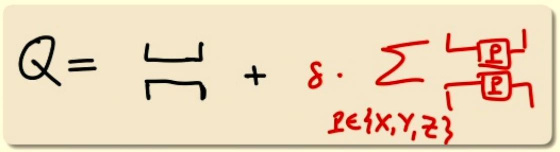
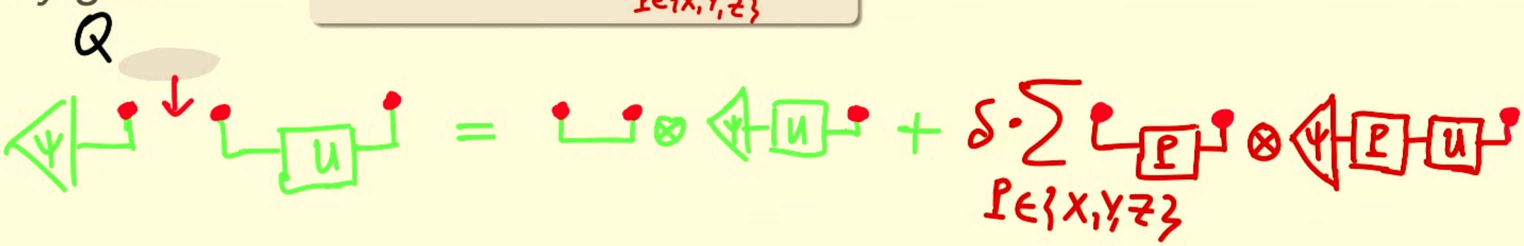
Injective PEPS encodes noisy quantum computation

Before: perfect gate



Now: noisy gate





Ground state encodes computation with **local depolarizing noise**

$$|\Psi_{iPEPS}\rangle \propto \sum_{\text{Pauli noise } \vec{P}} \delta^{|\vec{P}|} |\Phi_{\vec{P}}\rangle \otimes U_T P_T \dots U_1 P_1 |0^n\rangle$$

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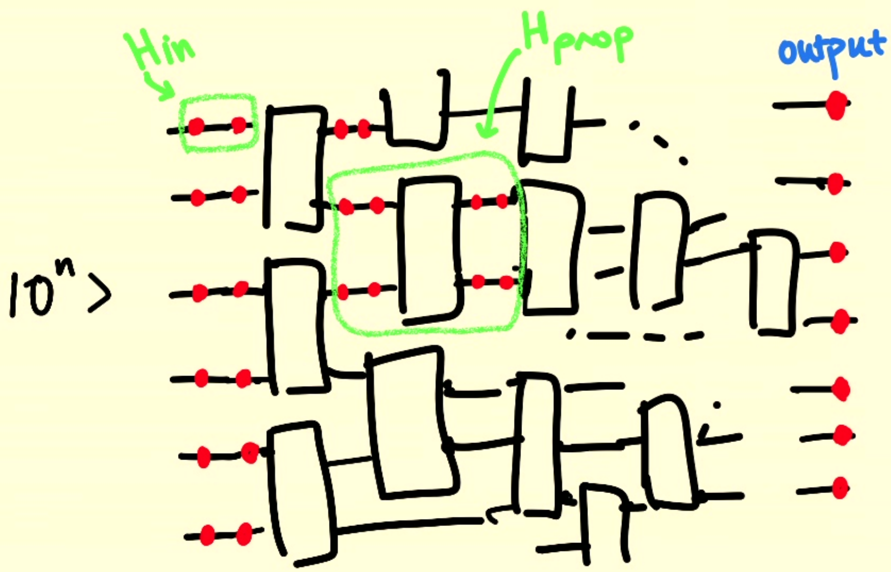
Use quantum fault tolerance!

Summary



$$H_{\text{TN}} = H_{\text{in}} + \sum_{t=0}^{T} H_{\text{prop,t}}$$

~ 2nD qubits

Ground state is an **injective PEPS**

Rightmost column contains comp. output, with **depolarizing noise** per gate

Use **fault-tolerant** version of circuit

Corollary: contracting injective PEPS is BQP-hard

Main results: robustness properties of H_{TN}

Some definitions

Def: ε-energy states have energy density of ε wrt H_{TN}.

$$H_{\mathrm{TN}} = \sum_{i=1}^{m} h_i \qquad \langle \psi | H_{\mathrm{TN}} | \psi \rangle \leq \varepsilon m$$

Def: α-combinatorial states violate α-fraction of terms in H_{TN}.

$$|\{ i : \langle \psi | h_i | \psi \rangle \neq 0 \}| \leq \alpha m$$

Def: κ-adversarial computation: In each layer, at most κ fraction of gates are faulty

Main theorems: excited states still encode noisy computation

Property 1 (semi-classical robustness): $1/\text{poly}(D)$-combinatorial states of H_{TN} encode $1/\text{polylog}(n)$-adversarial computation.

Property 2 (quantum robustness): e^{-D}-energy states of H_{TN} encode $1/\text{polylog}(n)$-adversarial computation.

Only depend on circuit depth, not circuit size

Semiclassical and quantum analogues of Cook-Levin robustness

We believe Property 2 also holds for $1/\text{poly}(D)$-energy states (ongoing work)

Soundness analysis beyond spectral gap

Conventionally, soundness analysis in Feynman-Kitaev uses its $1/\mathrm{poly}(nD)$ spectral gap. No guarantees above spectral gap.
(can't be improved, Bausch-Crosson 16)

E.g. Kitaev used spectral gap to prove QMA-completeness

Spectral gap of H_{TN} is $\Omega(e^{-D}/\mathrm{poly}(nD))$

But $1/\mathrm{poly}(D)$-combinatorial states have energy much higher
And yet Property 1 asserts they stay "close" to the computation!

Such robustness encoding of computation into excited states
could be useful for qPCP (see next slides)

QMA-completeness (using standard spectral gap analysis)

Theorem: The log-local Hamiltonian problem is QMA-complete

Observation: QMA circuit can be assumed $O(\log n)$ depth

We construct **fault-tolerant** version, keeping depth $O(\log n)$

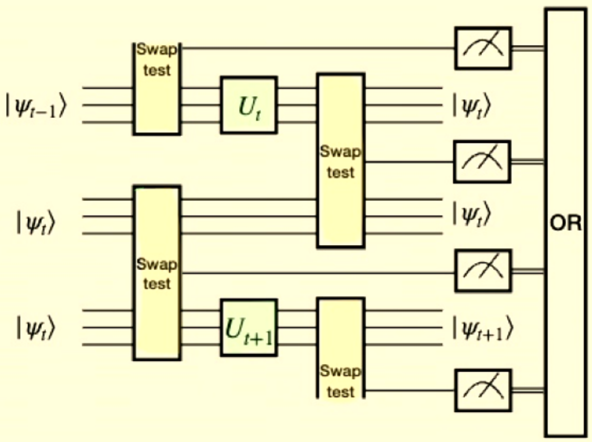
Using few layers of log-local gates

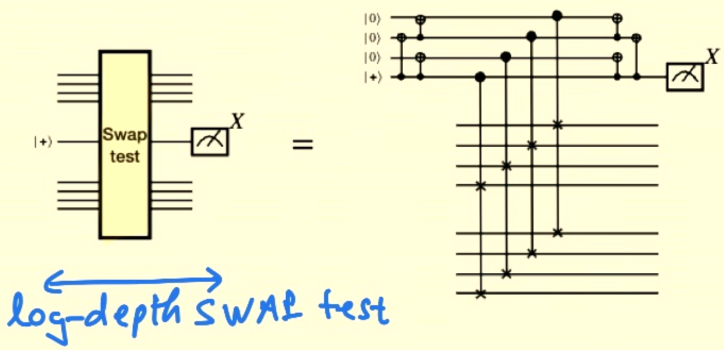
Spectral gap H_{TN} is 1/poly(n)

Use recent linear-distance qLDPC code!
(Leverrier-Zémor 23)
(with local parallel decoder)

Proof sketch: QMA in log depth

- Idea from Bill Rosgen's arXiv:0712.2595 "Distinguishing short quantum computations"
- Expect prover to send history of computation $|\psi_0\rangle\,|\psi_1\rangle^{\otimes 2}\ldots|\psi_{T-1}\rangle^{\otimes 2}\,|\psi_T\rangle$
- Use SWAP tests to check consistency





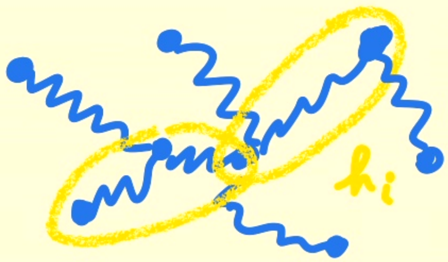
PCP and fault tolerance

Quantum PCP

k-local Hamiltonian

$H = \sum_{i=1}^{m} h_i$

$(\|h_i\| \leq 1)$



Is $\dfrac{\langle \psi_{gs} | H | \psi_{gs} \rangle}{m} \leq 0$ or $\geq a$?

$a = \dfrac{1}{\text{poly}(m)}$: QMA-hard [Kitaev'99]

$a = \dfrac{1}{\text{polylog}(m)}$: polylog-qPCP

$a = \Theta(1)$: qPCP

(open for polylog-local Hamiltonians, too)

Connection between classical polylog-PCP and adversarial fault tolerance

[Gál-Szegedy'96] observed Cook-Levin gives classical version of Property 1

Fault tolerant circuits and probabilistically checkable proofs

Anna Gál Mario Szegedy

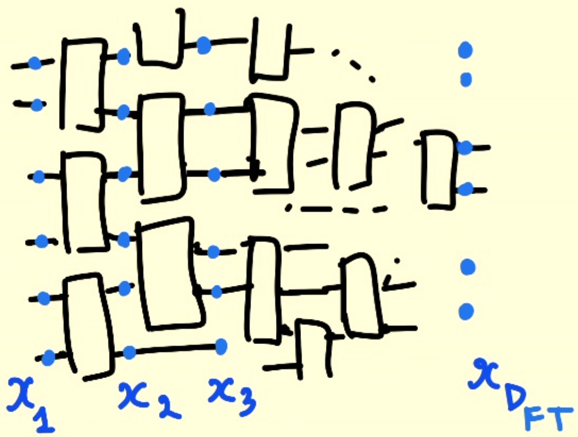
[Cook-Levin robustness]: $1/\text{poly}(D)$-combinatorial states of H_{CL} encode $1/\text{polylog}(n)$-adversarial computation.

Fact: NP circuit can be assumed $O(\log n)$ depth

Connection between classical polylog-PCP and adversarial fault tolerance

Claim: Cook-Levin robustness + adversarial fault tolerance implies polylog-PCP

such that $D_{FT} = \mathrm{poly}(D) = \mathrm{polylog}(n)$



Polylog PCP verifier:

- Expect Cook-Levin history state
- Pick random term in H_{CL}
  ACCEPT if term is satisfied
  and computation output bit is 1
- Else REJECT

→ A different proof of classical polylog-PCP

Goal: quantum version of polylogPCP-advFT connection

- Fact: QMA circuit can be assumed $O(\log n)$ depth
- Suppose Property 2 also held for $1/\text{poly}(D)$-energy states

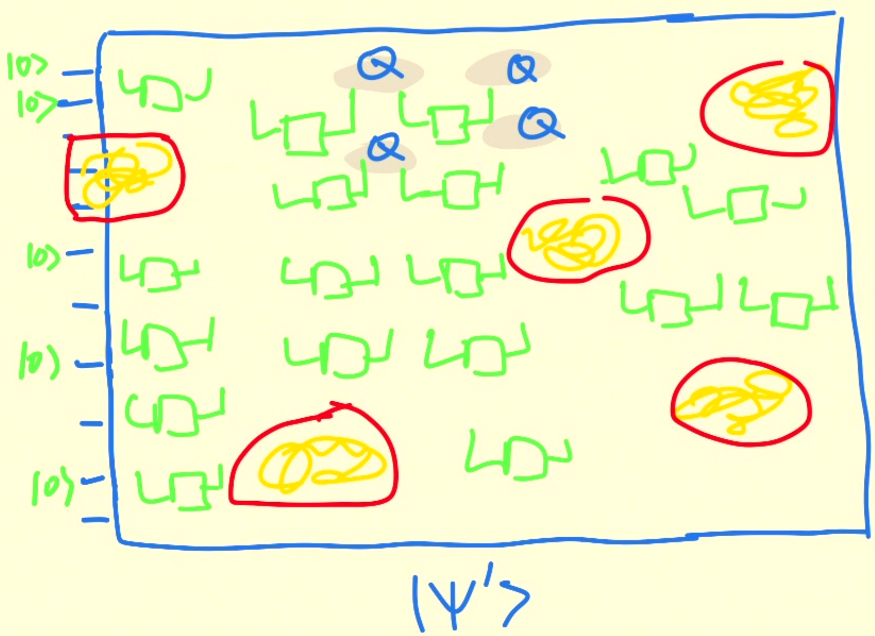
> **Main open question:** $1/\text{poly}(D)$-energy states of H_{TN} encode $1/\text{polylog}(n)$-adversarial computation

- Then a quantum FT scheme for $1/\text{polylog}(n)$-**adversarial** computation, with $\text{poly}(D)$ depth overhead, would imply polylog-qPCP.

(only need FT for QMA—should be easier than BQP)

Proof sketch: semiclassical robustness

$$H_{TN} = \sum_{i=1}^{n} Q^{-1} | 1 \rangle \langle 1 |_i Q^{-1} + \sum_{t=1}^{T} (Q^{-1})^{\otimes 4} (1 - | \phi_{U_t} \rangle \langle \phi_{U_t} |) (Q^{-1})^{\otimes 4}$$



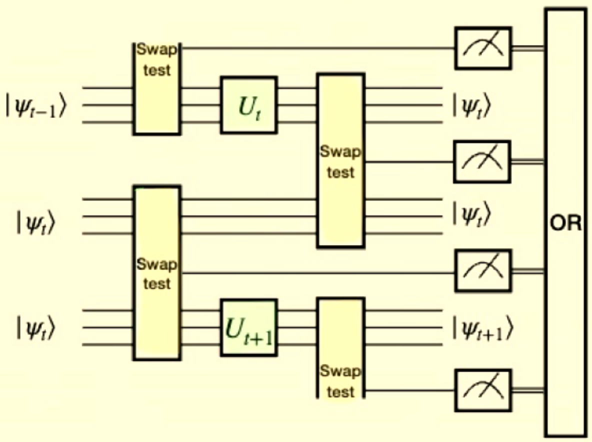
- Consider $|\psi'\rangle \propto (Q^{-1})^{\otimes T} |\psi\rangle$

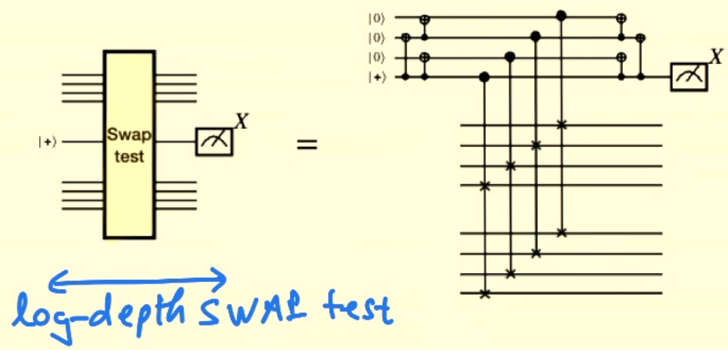
$|\psi'\rangle = |\ldots\rangle \otimes \left(\sum |\ldots\rangle \right)$ faulty Choi states

- Then apply $Q^{\otimes T}$ back to $|\psi\rangle$

Proof sketch: QMA in log depth

- Idea from Bill Rosgen's arXiv:0712.2595 "Distinguishing short quantum computations"
- Expect prover to send history of computation $|\psi_0\rangle\,|\psi_1\rangle^{\otimes 2}\ldots|\psi_{T-1}\rangle^{\otimes 2}\,|\psi_T\rangle$
- Use SWAP tests to check consistency





Summary

- New circuit-to-Hamiltonian using tensor networks. Necessitates fault tolerance
- New proof of QMA-completeness of log-local Hamiltonian
- Robustness property similar to Cook-Levin. Hope for polylog-qPCP

Open questions

- Do 1/poly(D)-energy states encode 1/polylog(n) adversarial computation?
- How to do adversarial quantum fault tolerance?
- Other applications?