Title: Quantum difference equations from shuffle algebra: affine type A quiver varieties

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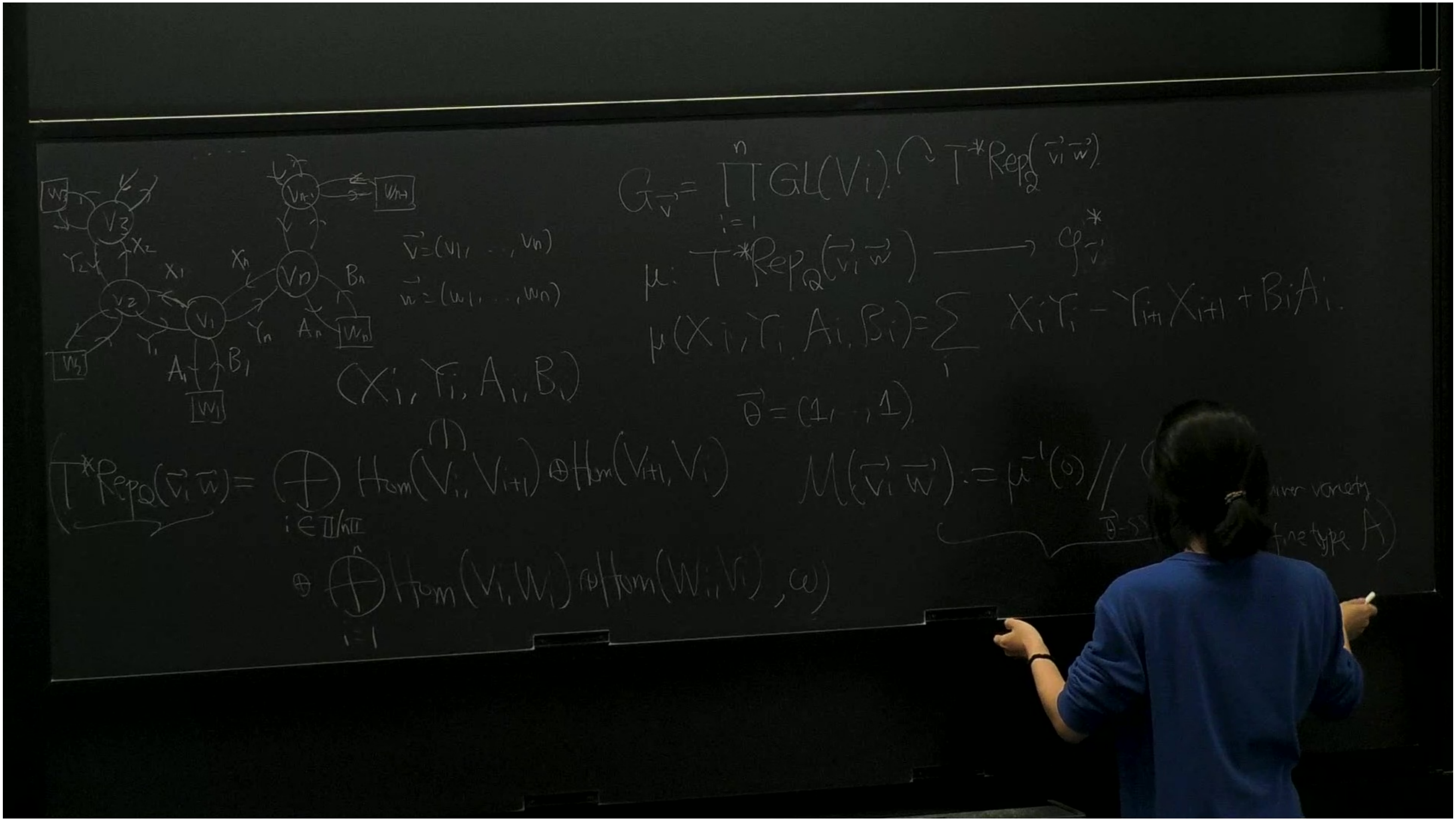
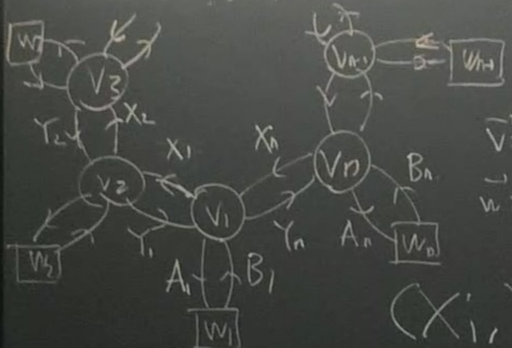
Abstract:

The quantum difference equation (qde) is the q-difference equation which is proposed by Okounkov and Smirnov to encode the K-theoretic twisted quasimap counting for the Nakajima quiver varieties. In this talk, we will give a direct quantum toroidal algebra $U_{q,t}(\hat{\hat{\mf{sl}}}_{n})$ construction for the qde of the affine type A quiver varieties. We will show that there is a really explicit and concise formula for the quantum difference operators. Moreover we will show that the degeneration limit of the quantum difference equation is equivalent to the Dubrovin connection for the quantum cohomology of the affine type A quiver varieties, which will give the description of the monodromy representation of the Dubrovin connection via the monodromy operators in the quantum difference equation.

Zoom link


$G_{\vec{v}} = \prod_{i=1}^{n} GL(V_i) \curvearrowright T^*Rep_Q(\vec{v}, \vec{w})$
$\mu : T^*Rep_Q(\vec{v}, \vec{w}) \longrightarrow \mathfrak{g}_{\vec{v}}^*$
$\mu(X_i, Y_i, A_i, B_i) = \sum_i X_iY_i - Y_{i+1}X_{i+1} + B_iA_i$
$\vec{\theta} = (1, \dots, 1)$
$\vec{v} = (v_1, \dots, v_n)$
$\vec{w} = (w_1, \dots, w_n)$
(X_i, Y_i, A_i, B_i)
$T^*Rep_Q(\vec{v}, \vec{w}) = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} Hom(V_i, V_{i+1}) \oplus Hom(V_{i+1}, V_i)$
$\oplus \bigoplus_{i=1}^{n} Hom(V_i, W_i) \oplus Hom(W_i, V_i), \omega)$
$\mathcal{M}(\vec{v}, \vec{w}) := \mu^{-1}(0) /\!/_{\vec{\theta}\text{-ss}}$
iver variety
fine type A)
W_3 V_3 X_2 Y_2 X_1 V_2 V_1 Y_1 A_1 B_1 W_1 W_2
X_n V_n B_n A_n Y_n W_n V_{n-1} W_{n-1}



$\vec{v} = (v_1, \dots, v_n)$

$\vec{w} = (w_1, \dots, w_n)$

(X_i, Y_i, A_i, B_i)

$$G_{\vec{v}} = \prod_{i=1}^{n} GL(V_i) \curvearrowright T^*\mathrm{Rep}_Q(\vec{v}, \vec{w})$$

$$\mu : T^*\mathrm{Rep}_Q(\vec{v}, \vec{w}) \longrightarrow \mathfrak{g}^*_{\vec{v}}$$

$$\mu(X_i, Y_i, A_i, B_i) = \sum_i X_i Y_i - Y_{i+1} X_{i+1} + B_i A_i$$

$$\vec{\theta} = (1, \dots, 1)$$

$$\Big(T^*\mathrm{Rep}_Q(\vec{v}, \vec{w}) = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(V_i, W_i) \oplus \mathrm{Hom}(W_i, V_i), \omega \Big)$$

$$\mathcal{M}(\vec{v}, \vec{w}) := \underbrace{\mu^{-1}(0) /\!/_{\vec{\theta}\text{-ss}} G_{\vec{v}}}$$

Quiver variety (affine type A)

$$K_{T_{\vec{w}}}(\mathcal{M}(\vec{v}, \vec{w})) \cong K_{T_{\vec{w}}}(\mathcal{M}(\vec{v}, \vec{w})^{T_{\vec{w}}}), \quad \mathcal{M}(\vec{v}, \vec{w})^{T_{\vec{w}}} = \left\{ \begin{array}{l} \text{indexed by } \vec{w}\text{-partitions} \\ (\lambda_1, \dots, \lambda_{|\vec{w}|}) \text{ s.t. } \sum_{i=1}^{|\vec{w}|} \lambda_i = |\vec{v}| \end{array} \right\}$$

$$T^*\mathrm{Rep}_Q(\vec{v},\vec{w}) = \bigoplus_{i\in \mathbb{Z}/n\mathbb{Z}} \mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(V_i, W_i) \oplus \mathrm{Hom}(W_i, V_i), \ \omega)$$

$$\underbrace{\mathcal{M}(\vec{v},\vec{w}) := \mu^{-1}(0) /\!/_{\vec{\theta}\text{-ss}}\, G_{\vec{v}}}$$

Quiver variety (affine type A)

$$K_{T_{\vec{w}}}(\mathcal{M}(\vec{v},\vec{w}))_{loc} \cong K_{T_{\vec{w}}}(\mathcal{M}(\vec{v},\vec{w})^{T_{\vec{w}}})_{loc}, \quad \mathcal{M}(\vec{v},\vec{w})^{T_{\vec{w}}} = \left\{ \begin{array}{l} \text{indexed by } \vec{w}\text{-partitions} \\ (\lambda_1, \ldots, \lambda_{|\vec{w}|}) \text{ s.t. } \sum_{i=1}^{|\vec{w}|} \lambda_i = |\vec{v}| \\ + \text{ constrained conditions} \end{array} \right\}$$

$$G_{\vec{w}} = \prod_{i=1}^{n} GL(W_i) \curvearrowright \mathcal{M}(\vec{v},\vec{w})$$

maximal torus

$$T_{\vec{w}} \triangleq A_{\vec{w}} \times \mathbb{C}^*_q \times \mathbb{C}^*_t$$

$$\sigma : \mathbb{C}^* \longrightarrow A_{\vec{w}}, \quad \text{s.t. } \vec{w} = a_1\vec{w}_1 + \cdots + a_k\vec{w}_k.$$

Fixed point

$$\mathcal{M}(\vec{v},\vec{w})^{\sigma} = \coprod_{\vec{v}_1+\cdots+\vec{v}_k=\vec{v}} \mathcal{M}(\vec{v}_1,\vec{w}_1) \times \cdots \times \mathcal{M}(\vec{v}_k,\vec{w}_k).$$

$$U_{q,t}(\hat{\hat{sl}}_n) \curvearrowright \bigoplus_{\vec{v}} K_{T_{\vec{w}}}(\mathcal{M}(\vec{v},\vec{w}))_{loc}$$

Quantum difference equations from Shuffle algebras: affine type A (2308.00550, 2405.02473)

Background: (K-theoretic Enum Geometry)

Stable quasimap, $p_1, p_2 \in \mathbb{P}^1$, $QM_{\substack{\text{nonsing } p_1 \\ \text{rel } p_2}}(\mathbb{P}^1 \longrightarrow M(\vec{v},\vec{w}))$

$$\widehat{ev}: QM^{\vec{d}}_{\substack{\text{nonsing } p_1 \\ \text{rel } p_2}} \longrightarrow M(\vec{v},\vec{w})^{\times 2} \qquad = \bigsqcup_{\vec{d}\in \mathrm{Pic}(M(\vec{v},\vec{w}))} QM^{\vec{d}}_{\substack{\text{nonsing } p_1 \\ \text{rel } p_2}}$$

$$T_{\vec{w}} \times \mathbb{C}^*_p \curvearrowright \quad \hat{\mathcal{O}}^{vir}_{\vec{d}}$$

We can define the capping operator

$$J(u,z) \triangleq \sum_{\vec{d}} z^{\vec{d}}\, ev_* \hat{\mathcal{O}}^{vir}_{\vec{d}} \in K_{T_{\vec{w}}\times\mathbb{C}^*_p}(M(\vec{v},\vec{w}))^{\otimes 2}_{loc} \otimes \mathbb{Q}[[z^{\vec{d}}]]_{\vec{d}\in \mathrm{Pic}}$$

u — equivariant variables $\in A_{\vec{w}}$

$z = (z_1, \dots, z_n)$ — Kähler variables.

$z = (z_1, \ldots, z_n)$ — Kähler variables.

$\vec{d}$

Good news: $J(u, z)$ has a p-difference equations (of variety,

$$J(u, zp^{\mathcal{L}})\,\mathcal{L} = M_{\mathcal{L}}(z)\,J(u, z)$$

$$\underset{(k_1, \ldots, k_n)}{\mathcal{L}} \in \mathrm{Pic}(\mathcal{M}(\vec{v}, \vec{w})) \cong \mathbb{Z}^n$$

$$zp^{\mathcal{L}} = (z_1 p^{k_1}, z_2 p^{k_2}, \ldots, z_n p^{k_n})$$

Q. How to compute $\underbrace{M_{\mathcal{L}}(z)}_{\text{(quantum difference operator)}}$?

(Okounkov & Smirnov)

1. $U_q^{MO}(\hat{\mathfrak{g}}_Q)$ — defined by the K-theoretic stable envelope of $\mathcal{M}(\vec{v},\vec{w})$

$$\mathrm{Stab}^s_\sigma : K_{T_{\vec{w}}}(\mathcal{M}(\vec{v},\vec{w})^{A})_{loc} \longrightarrow K_{T_{\vec{w}}}(\mathcal{M}(\vec{v},\vec{w}))_{loc}$$

$s \in \mathrm{Pic} \otimes \mathbb{R}$ I iso. II upper triangular III Degree bounding

$$R^s_\sigma \triangleq (\mathrm{Stab}^s_{-\sigma})^{-1} \circ \mathrm{Stab}^s_\sigma \quad \left(\begin{array}{c}\text{geometric R-matrix} \\ \text{YBE}\end{array}\right)$$

the MO (quantum affine alg.)

2. $U_q^{MO}(\hat{\mathfrak{g}}_Q) \supset \underbrace{U_q(\mathfrak{g}_w)}$ wall subalgebra

s, s'

$$R_w^{\pm} \stackrel{\Delta}{=} \left(\mathrm{Stab}^{s'}_{\pm\sigma}\right)^{-1} \circ \mathrm{Stab}^{s}_{\pm\sigma}$$

(YBE)

$w \in \mathrm{Pic}\otimes\mathbb{R}$.

3. On each $U_q(\mathfrak{g}_w)$ $\rightsquigarrow$ $\exists$ an ABRR eqn

$$J_w^+(z)\, z_{(1)}^{-1} T_u R_w^+ = z_{(1)}^{-1} T_u J_w^+(z) \in U_q(\mathfrak{g}_w) \otimes U_q(\mathfrak{g}_w)$$

$$\curvearrowright K(M(\vec{v}_1, \vec{w}_1) \times M(\vec{v}_2, \vec{w}_2))$$

$$z_{(1)} = z_1^{v_1^{(1)}} \cdots z_n^{v_1^{(n)}}$$

wall subalgebra

$)^{-1} \circ \mathrm{Stab}^{s}_{\pm c}$

AIBRR eqn

$T_u J^+_w(z) \in U_q(\mathfrak{g}_w) \otimes U_q(\mathfrak{g}_w) \cap K(M(\vec{v}_1, \vec{w}_1) \times M(\vec{v}_2, \vec{w}_2))$

$z^{(1)} = z_1^{v_1^{(1)}} \cdots z_n^{v_1^{(n)}}$

$T_u f(u,z) = f(up, z)$

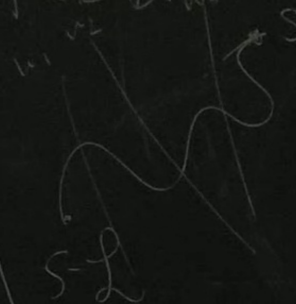
Monodromy operators

$$B_w(z) := m\left((S_w \otimes 1)(J^+_w)^{-1}_{21}\right)$$

Okounkov & Smirnov

$$M_{\mathcal{L}}(z) = \mathrm{Const.}\, \mathrm{Ad}_{\mathrm{Stab}^s_0}\, \mathbb{B}^s_{\mathcal{L}}$$

$s \in \mathrm{Pic} \otimes \mathbb{Q}_{w_2}$ $\quad \mathcal{L} \in \mathrm{Pic}$



$$\mathbb{B}^s_{\mathcal{L}} = \mathcal{L} \cdot \overrightarrow{\prod_{w \in [s, s-\mathcal{L}]}} B_w(z)$$

Quantum difference equations from Shuffle algebras: affine type A (2308.00550, 2405.02473)

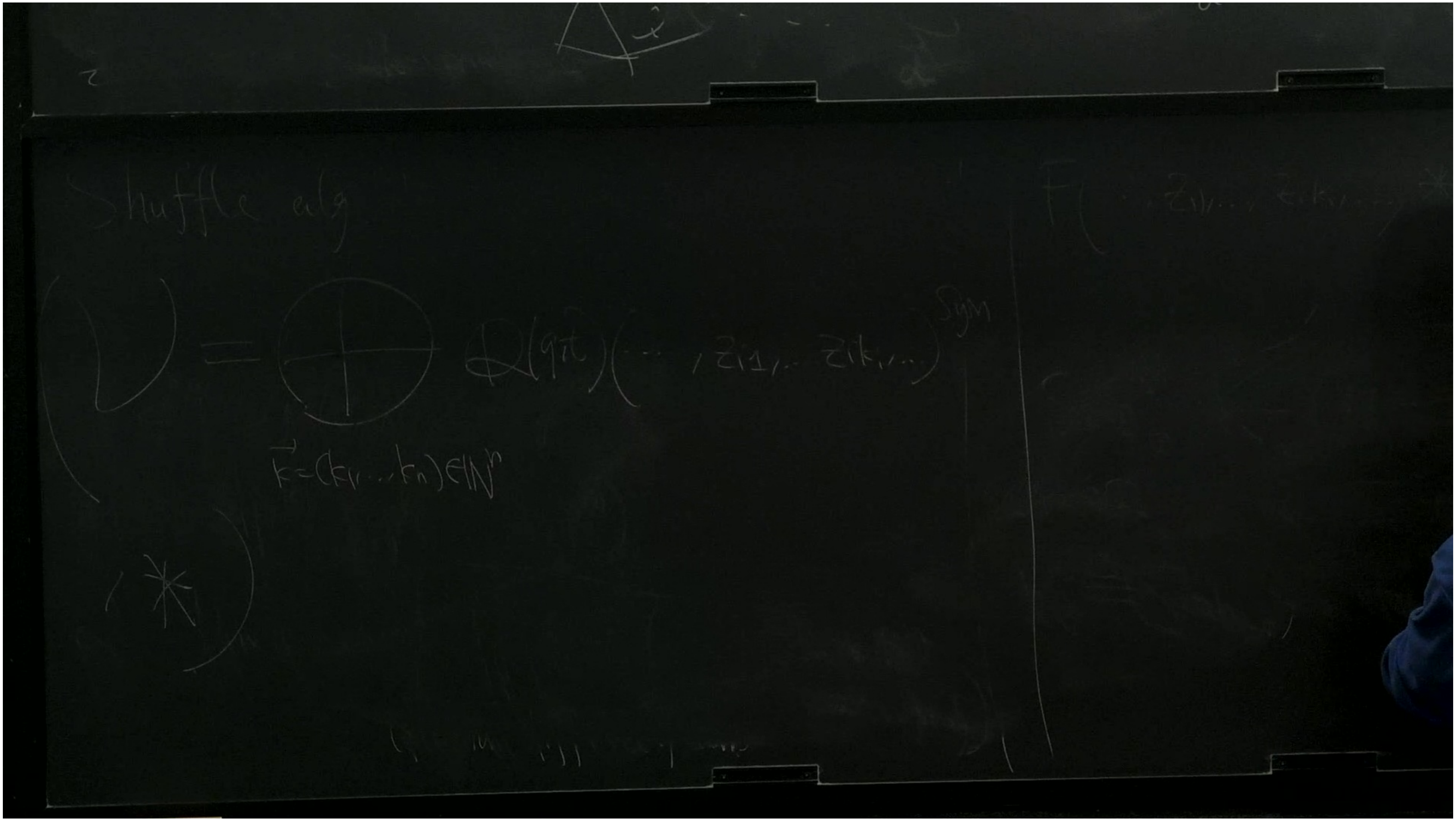
$$U_{q,t}(\hat{\mathfrak{sl}}_n) = \mathbb{Q}(q,t)\left\langle e_i^{\pm}(z), \varphi_i^{\pm}(z)\right\rangle_{1\le i\le n} \Big/ (\sim)$$

$$e_i^{\pm}(z) = \sum_{d\in\mathbb{Z}} \frac{e_{i,d}^{\pm}}{z^{d}}, \quad \varphi_i^{\pm}(z) = \sum_{d\ge 0} \frac{\varphi_{i,d}^{\pm}}{z^{\mp d}}$$

$$\zeta_{ij}(x) = \frac{\left[\frac{1}{qtx}\right]^{\delta^i_{j-1}} \left[\frac{t}{qx}\right]^{\delta^i_{j+1}}}{\left[\frac{1}{qx}\right]^{\delta^i_j} \left[\frac{1}{x}\right]^{\delta^i_j}}$$

$$[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$\begin{cases} e_i^{\pm}(z)\, e_j^{\pm}(w)\, \zeta_{ij}\left(\frac{z}{w}\right) = e_j^{\pm}(w)\, e_i^{\pm}(z)\, \zeta_{ji}\left(\frac{w}{z}\right) \\ e_i^{\pm}(z)\, \varphi_j^{\pm}(w)\, \zeta_{ij}\left(\frac{z}{w}\right) = \varphi_j^{\pm}(w)\, e_i^{\pm}(z)\, \zeta_{ji}\left(\frac{w}{z}\right) \end{cases}$$


Shuffle alg
Sym
$\vec{k}=(k_1,\dots,k_n)\in\mathbb{N}^n$

$$U = U^+ \otimes U^0 \otimes U^- \quad \Big\{ \; e_i^+(z)\,\varphi_j^+(w)\,\zeta_{ij}\!\left(\frac{z}{w}\right) = \varphi_j^+(w)\,e_i^+(z)\,\zeta_{ji}\!\left(\frac{w}{z}\right)$$

$$\| \qquad\qquad \Delta$$

$$\langle e_i^+(z) \rangle_{1 \le i \le n}$$

shuffle alg.

$$(\;) = \bigoplus_{\vec{k}=(k_1,\dots,k_n)\in\mathbb{N}^n} \mathbb{Q}(q,t)(\dots, z_{i1}, \dots, z_{ik_i}, \dots)^{\mathrm{Sym}}$$

$$(*) \supset \left(\text{generated by } \{z_{i1}^d\}_{\substack{d\in\mathbb{Z}\\ 1\le i\le n}} \right) = \mathcal{S}^+$$

$$= \mathrm{Sym}\left[\frac{F(\dots, z_{i1}, \dots z_{ik_i}, \dots)\, G(\dots)}{\vec{k}!\,\vec{l}!} \times \prod_{1\le i,j\le n}\;\prod_{\substack{1\le a\le k_i\\ k_j+1\le b\le k_j+l_j}} \zeta_{ij}\!\left(\frac{z_{ia}}{z_{jb}}\right) \right]$$

$$\underbrace{J^{+}_{w}(z)}_{\text{Fusion op}} z^{-1}_{(1)} T_u R^{+}_{w} = z^{-1}_{(1)} T_u J^{+}_{w}(z) \in U_q(\mathfrak{g}_w) \otimes U_q(\mathfrak{g}_w) \curvearrowright K(M(\vec{v}_1,\vec{w}_1) \times M(\vec{v}_2,\vec{v}_2))$$

s-L

Thm (Negut)

$$S^{+} \cong U^{+}, \quad S^{\geq} = S^{\pm} \otimes \mathbb{Q}(q,t)[\varphi^{\pm}_{i,d}] / (\sim) \quad (\in)$$

$$S = S^{\geq} \otimes S^{\leq} / (\sim)$$

$$\wr$$

$$U_{q,t}(\hat{\hat{sl}}_n)$$

grading

$$S^{+} = \bigoplus_{\substack{\vec{k} \in \mathbb{N}^n \\ d \in \mathbb{Z}}} S^{+}_{\vec{k},d}$$

Slope subalg

$$\vec{m} = (m_1, \dots, m_n) \in \mathbb{Q}^n$$

$$S^{+}_{\leq \vec{m}} \qquad \boxed{B^{+}_{\vec{m}} \triangleq \bigoplus^{\vec{m}\cdot\vec{k} = d}_{\vec{k} \in \mathbb{N}^n, d \in \mathbb{Z}} S^{+}_{\vec{k},d} \cap S^{+}_{\leq \vec{m}}}$$

slope $\leq \vec{m}$

$$\left\{ F(\dots, z_{i1}, \dots, z_{ia_i}, \xi z_{i,a_i+1}, \dots, \xi z_{i,a_i+b_i}, \dots) \text{ exists } \forall \vec{a}, \vec{b} \right\}$$

$$\lim_{\xi \to \infty} \quad \xi^{\vec{m}\cdot\vec{b} - \frac{(\vec{a},\vec{b})}{2}}$$

$$(\vec{a},\vec{b}) = \vec{a}^{T} C \vec{b}$$

$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$

$\vec{k} \in \mathbb{N}^n$, $d \in \mathbb{Z}$ $\quad k, d \quad \vec{v}$

$(\vec{a}, \vec{b}) = \vec{a}^T C \vec{b}$

2. $U_q^{MO}(\hat{\mathfrak{g}}_Q) \supset U_q(\mathfrak{g}_w)$ wall subalgebra

$R^{\pm}_{w} \stackrel{\circ}{=} (\mathrm{Stab}^{s'}_{\pm a})^{-1} \circ \mathrm{Stab}^{s}_{\pm a}$

(YBE)

$s \quad s'$

$w \subset \mathrm{Pic} \otimes \mathbb{R}$

3. On each $\mathcal{B}_{\vec{m}}$, $\rightsquigarrow \exists$ an ABRR eqn

$$\underbrace{J^{+}_{\vec{m}}(z)}_{\text{Fusion op}} z_{(1)}^{-1} T_u R^{+}_{\vec{m}} = z_{(1)}^{-1} T_u J^{+}_{\vec{m}}(z) \in \mathcal{B}_{\vec{m}} \otimes \mathcal{B}_{\vec{m}}$$

$\curvearrowright K(M(\vec{v}_1, \vec{w}_1) \times M(\vec{v}_2, \vec{w}_2))$

$z_{(1)} = z_1^{v_1^{(n)}} \cdots z_n^{v_1^{(n)}}$

$T_u f(u, z) = f(up, z)$

Monodromy operators

$$\mathbf{B}_{\vec{m}}(z) := m\left((S_{\vec{m}} \otimes 1)(J^{+}_{\vec{m}})_{21}^{-1}\right)$$

$s \in \mathrm{Pic} \otimes \mathbb{R}$, $w \quad \mathcal{L} \in \mathrm{Pic}$

Okounkov & Smirnov

$M_{\mathcal{L}}(z) = \mathrm{Const} \cdot A_{d}^{s}{}_{\mathrm{Stab}_0}$ $\quad \mathcal{B}^{s}_{\mathcal{L}}$

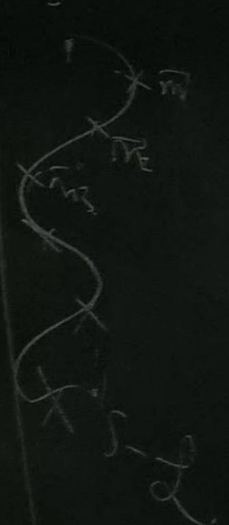
$A^{s}_{\mathcal{L}} \stackrel{\circ}{=} T_{\mathcal{L}}^{-1} \mathcal{B}^{s}_{\mathcal{L}}(z)$

$m_1 \quad m_2 \quad m_3 \quad s \quad s-\mathcal{L}$

$$\mathcal{B}^{s}_{\mathcal{L}}(z) = \mathcal{L} \cdot \overrightarrow{\prod_{\vec{m} \in [s, s-\mathcal{L}]}} \mathbf{B}_{w}(z)$$

$$\mathcal{B}_{\gamma_n} \cong \bigotimes_{h=1} U_q(\mathfrak{g}(\gamma_h)) \qquad \sum_{h=1} (l_h - 1)$$

Fix $M(\vec{v}, \vec{w})$



$\mathbb{I}_{\vec{m}}$, $\exists$ game path s.t.

$$B_{\vec{m}}(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_{q^2}!} \frac{e_\gamma^n f_\gamma^n}{\prod^n \left(1 - z^{-\vec{v}_\gamma} p^{\vec{m}\cdot\vec{v}_\gamma} q^{l(\cdots)}\right)} & U_q(sl_2)\text{-type} \\ \displaystyle\prod_{h=1}^{n} :\exp\left(-\sum_{k=0}^{\infty} \frac{n_k\, a_k^{\vec{m}_j h}\, a_{-k}^{\vec{m}_j h}}{1 - z^{k\vec{\delta}_h} p^{k\vec{m}\cdot\vec{\delta}_h}}\right): & U_q(\hat{gl}_1)\text{-type} \end{cases}$$

3. On each $\mathcal{B}_{\vec{m}}$ $\rightsquigarrow$ $\exists$ an ABRR eqn

$$\underbrace{J^{+}_{\vec{m}}(z)}_{\text{Fusion op}} z^{-1}_{(1)} T_u R^{+}_{\vec{m}} = z^{-1}_{(1)} T_u J^{+}_{\vec{m}}(z) \in \mathcal{B}_{\vec{m}} \otimes \mathcal{B}_{\vec{m}} \curvearrowright K(M(\vec{v}_1, \vec{w}_1) \times M(\vec{v}_2, \vec{v}_2))$$

$s \in P_K \otimes \mathbb{Q}_{W^2}$

$m_1, m_2, m_3, S, S-\mathcal{L}$

$$\mathcal{B}^{S}_{\mathcal{L}} \xrightarrow{\sim} \mathcal{L} \cdot \prod_{\vec{m} \in [S, S-\mathcal{L}]} B_w(z)$$

Appl: QDE $\xrightarrow{\text{degeneration}}$ Dubrovin conn.

$M(\vec{v}, \vec{w})$, $H^{*}_{T_{\vec{w}}}(M(\vec{v}, \vec{w}))$ [Maulik-Okounkov, Davison-Botta]

$$\nabla_{\mathcal{L}} = d_{\mathcal{L}} - \sum_{j>i} \frac{\mathcal{L} \cdot [i,j]}{1 - z^{[i,j]}} e_{[i,j]} e_{-[i,j]} + \underbrace{\cdots}_{\text{scaling generators}}$$

$e_{\pm[i,j]} \in \hat{sl}_n$.

Thm. [Z]

Degeneration limit $q, t \to 1$, $p \to 1$

$$\Psi(p^{\mathcal{L}} z) = M_{\mathcal{L}}(z) \Psi(z) \rightsquigarrow \text{Dubrovin conn.}$$

Difference equation — Connection matrix

$$\Psi_{\infty}^{-1} \Psi_0 = M_{\vec{m}}(z)$$

degeneration

Monodromy rep for the Dubrovin conn.

generated by

$$\mathcal{B}_{\vec{m}} = m((1 \otimes S_{\vec{m}}^{-1})(R^{-}_{\vec{m}}))$$