Title: s-ordered phase-space correspondences, fermions, and negativities

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Series: Quantum Foundations

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Abstract: For continuous-variable systems, the negativities in the s-parametrized family of quasi-probability representations on a classical phase space establish a sort of hierarchy of non-classility measures. The coherent states, by design, display no negativity for any value of -1<=s<=1, meaning that sampling from the quantum probability distribution resulting from any measurement of a coherent state can be classically simulated, placing the coherent states as the most classical states according to this particular choice of phase space.

In this talk, I will describe how to construct s-ordered quasi-probability representations for finite-dimensional quantum systems when the phase space is equipped with more general group symmetries, focusing on the fermionic SO(2n) symmetry. Along the way, I will comment on an obstruction to an analogue of Hudson's theorem, namely that the only pure states that have positive s=0 Wigner functions are Gaussian states, and a possible remedy by giving up linearity in the phase-space correspondence.

Zoom link



s-ordered phase-space correspondences, fermions, and negativities

Quantum Foundations Seminar, Perimeter Institute, 9 May 2024

The three C's

Ninnat (Tom) Dangniam

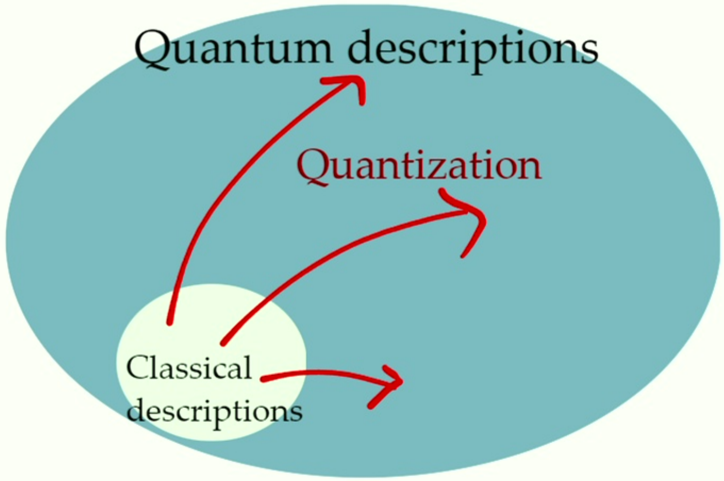
Quantization

- The standard formulations of classical and quantum mechanics look very different, with functions on a phase-space manifold in the former, and Hilbert spaces and linear operators in the latter
- Can we put the two on the same footing?

Dirac quantization prescription

$$\{f, g\} \mapsto \frac{1}{i\hbar}[\hat{f}, \hat{g}]$$

Ambiguous because of operator ordering (Groenewold's theorem)



- Quantization is in general ill-defined because many quantum descriptions have the same classical limits; dequantization throws away information
- Constraints from symmetry help pick out preferred quantization maps/ *phase-space correspondences*

The Early History

1930s-1940s

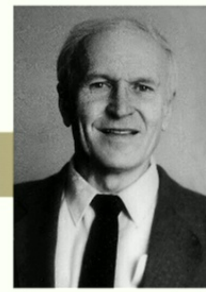
Weyl
(1927)

Groenewold
(1946)

Wigner
(1932)

Moyal
(1949)

Flat (Weyl-Heisenberg) phase space



Stratonovich
(1957)

1960s

Klauder
(1960)

Glauber
(1963)

Sudarshan
(1963)

Coherent-state revolution

A Crash Course in Wigner Functions

Phase-space density $$W(q,p) = \int \mathrm{d}\eta\, \mathrm{d}\zeta\; e^{-i(\zeta q - \eta p)} \chi(\eta, \zeta)$$ Characteristic function

$$\chi(\eta,\zeta) = \left\langle e^{i(\zeta Q - \eta P)} \right\rangle$$

Classically, these two are just inverse transformations of each other

Photo: sapfirr

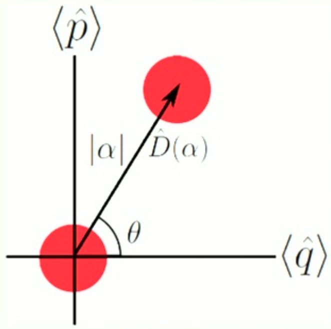
Quantum sleight of hand

Q and P are now operators and $\langle\ \rangle$ becomes the quantum expectation value

$$W_\rho(q,p) = \mathrm{Tr}\left[\rho \int \mathrm{d}\eta\, \mathrm{d}\zeta\; e^{i\zeta(Q-q) - i\eta(P-p)}\right]$$

This is the Wigner function!

Operator Ordering



Plick, 2011 (thesis)

Displacement operator (representation of the Weyl-Heisenberg group)

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = D(q,p) = e^{i(\zeta Q - \eta P)}$$

Dictionary

$$a = \frac{Q + iP}{\sqrt{2}}$$

$$Q = \frac{a + a^\dagger}{\sqrt{2}} \quad P = \frac{a - a^\dagger}{\sqrt{2}i}$$

Wigner (Symmetric)

$$\chi = \mathrm{Tr}(\rho\, e^{\alpha a^\dagger - \alpha^* a}) = \mathrm{Tr}(\rho\, D(\alpha))$$

Husimi Q (Anti-normal)

$$\chi_{\mathrm{a}} = \mathrm{Tr}(\rho\, e^{-\alpha^* a} e^{\alpha a^\dagger}) = \mathrm{Tr}(\rho\, e^{-|\alpha|^2/2} D(\alpha))$$

$$Q_\rho(\alpha) = \langle\alpha|\,\rho\,|\alpha\rangle$$

Glauber-Sudarshan P (Normal)

$$\chi_{\mathrm{n}} = \mathrm{Tr}(\rho\, e^{\alpha a^\dagger} e^{-\alpha^* a}) = \mathrm{Tr}(\rho\, e^{|\alpha|^2/2} D(\alpha))$$

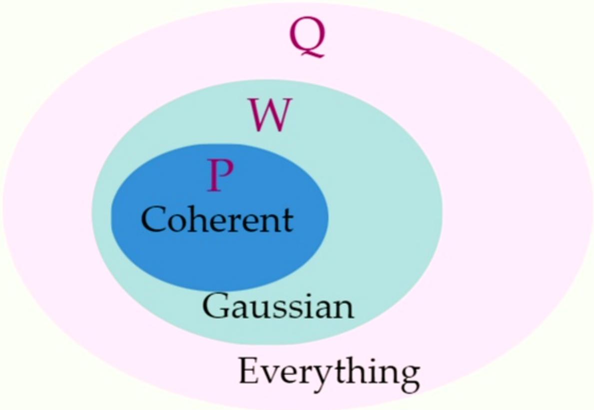
$$\rho = \int d^2\alpha\, P_\rho(\alpha)\, |\alpha\rangle\langle\alpha|$$

$$\chi_s = \mathrm{Tr}(\rho\, e^{-s|\alpha|^2/2} D(\alpha))$$

s-ordering doesn't correspond to an actual ordering anymore

Measures of Non-Classicality

- The Weyl-Heisenberg symmetry of the phase space picks out preferred correspondences
- The choice of operator ordering is still there, but all orderings are useful: hierarchy of non-classicality measures by the quasi-probability negativities



Pure states that have non-negative representations
("Hudson's theorem")

- If the states are represented by $W^{(s)}$, measurements must be represented by $W^{(-s)}$

$$\Pr(k|\rho) = \int \mathrm{d}^2\alpha\, W^{(-s)}_{E_k}(\alpha) W^{(s)}_{\rho}(\alpha)$$

- In general, there are no representations of states and measurements that are simultaneously free of negativity

The P function of the coherent states are delta functions which are non-negative

$$\rho = \int d^2\alpha\, P_\rho(\alpha)\, |\alpha\rangle\langle\alpha|$$

Is there a Hudson's theorem for fermions?

$$[a, a^\dagger] = \mathbb{I} \xrightarrow{\ ?\ } \{a, a^\dagger\} = \mathbb{I}$$

Stratonovich-Weyl Correspondences

A quasi-rep is a linear, invertible mapping from the real vector space of Hermitian operators $\mathrm{H}(\mathbb{C}^d)$ to the function space $L^2(\Lambda)$ over the label manifold (phase space) $\Omega \in \Lambda$

- (Linearity) The map $A \mapsto W_A^{(s)}(\Omega)$ is linear and one-to-one.
- (Reality)

$$W_{A^\dagger}^{(s)}(\Omega) = \left(W_A^{(s)}\right)^*$$

- (Normalization)

$$\int_\Lambda d\Omega\, W_A^{(s)} = \mathrm{Tr} A$$

- (Group covariance)

$$W_{U(g^{-1})AU(g)}^{(s)}(\Omega) = W_A^{(s)}(g \cdot \Omega)$$

- (Traciality)

$$\int_\Lambda d\Omega\, W_A^{(s)}(\Omega) W_B^{(-s)}(\Omega) = \mathrm{Tr}(AB)$$

Brif & Mann, PRA 1999

Quasi-Probability Representations

A quasi-rep is nothing but an expansion of quantum states and measurement operators in a *frame* $\{F_j\}$ (Ferrie *et al.*, J Phys A 2008, NJP 2009, PRA 2010)

$$a||A||^2 \leq \sum_j |\mathrm{Tr}(F_j^\dagger A)|^2 \leq b||A||^2, \qquad 0 < a \leq b$$

For example, the Wigner frame

$$W_\rho(q,p) = \mathrm{Tr}\left[\rho \underbrace{\int \mathrm{d}\eta\, \mathrm{d}\zeta\, e^{i\zeta(Q-q)-i\eta(P-p)}}_{F(q,p)}\right]$$

Much more convenient to switch to "*Dirac* action" of superoperators

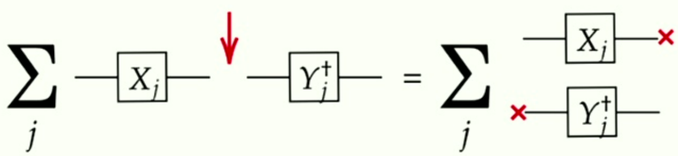
$$a\mathbf{I} \leq \underbrace{\sum_j |F_j)(F_j|}_{\text{Frame operator}} \leq b\mathbf{I},$$

Graphical Notations

Bra | Ket | Inner product | Operator | Trace | Vectorization

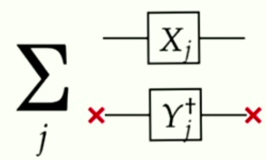
Ordinary (*Kraus*) action

$$\mathcal{A} = \sum_j X_j \odot Y_j^\dagger$$



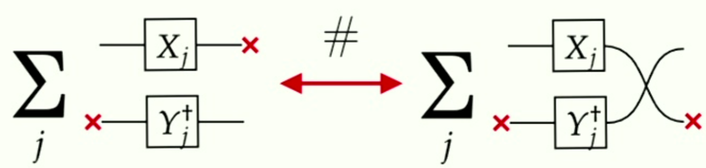
Left-right (*Dirac*) action

$$\mathcal{A} = \sum_j |X_j)(Y_j|$$



$$\mathcal{A}|B) = \sum_j \mathrm{Tr}(Y_j^\dagger B)|X_j)$$

Exchanging the two:
(the # *map*)



Graphical Notations

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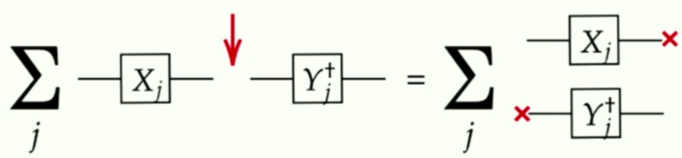

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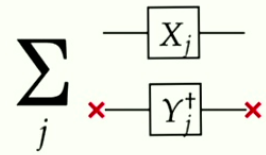

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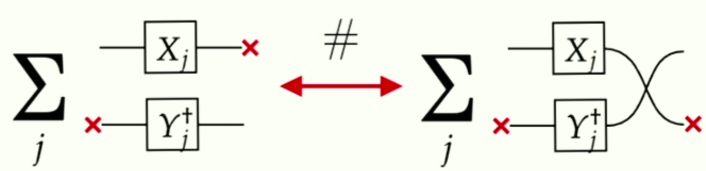
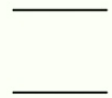
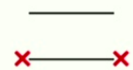


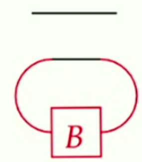
Examples

$$\mathcal{I} = \mathbb{I} \otimes \mathbb{I} = \sum_{j,k} |j\rangle\langle j| \otimes |k\rangle\langle k| =$$



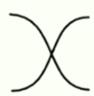
Twirl is $\mathcal{I}$ w.r.t. Dirac action
(Schur's lemma)

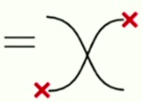


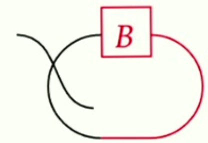


$= \mathrm{Tr(B)}\mathbb{I}$

$$\mathbf{I} = \mathrm{SWAP} = \sum_{j,k} |j\rangle\langle k| \otimes |k\rangle\langle j| =$$



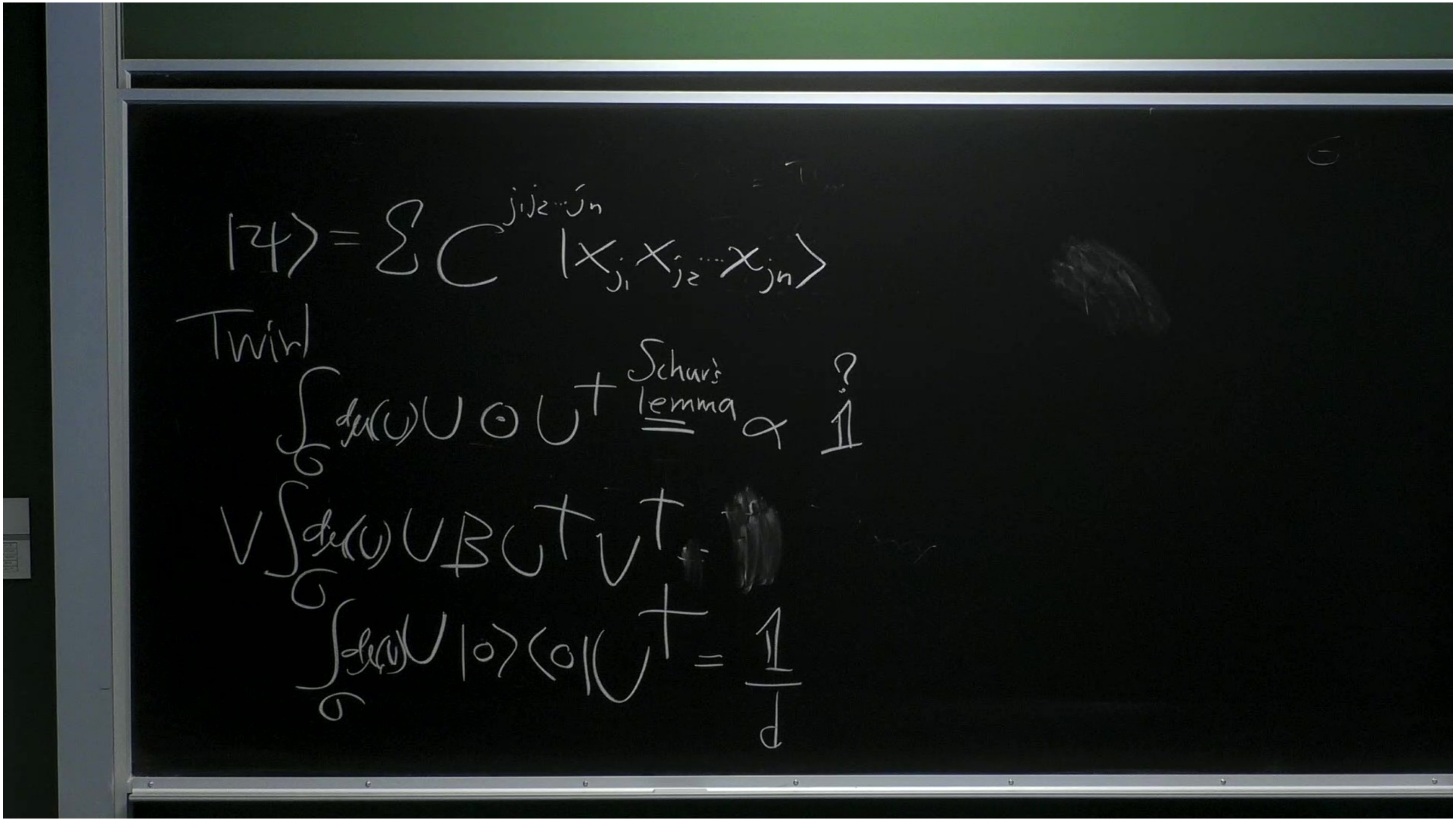


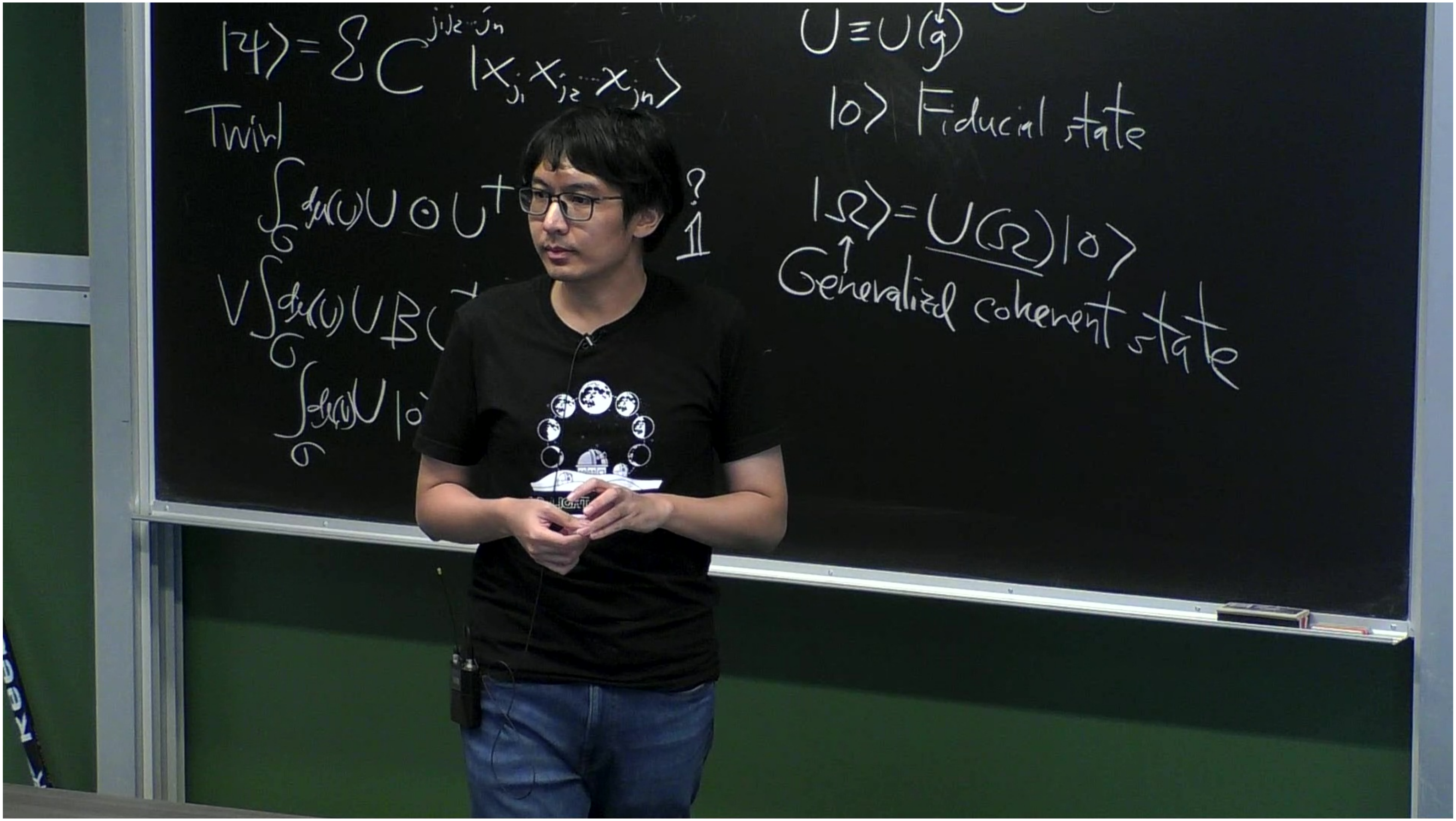


$\mathbf{I}^{\#} = \mathcal{I}$

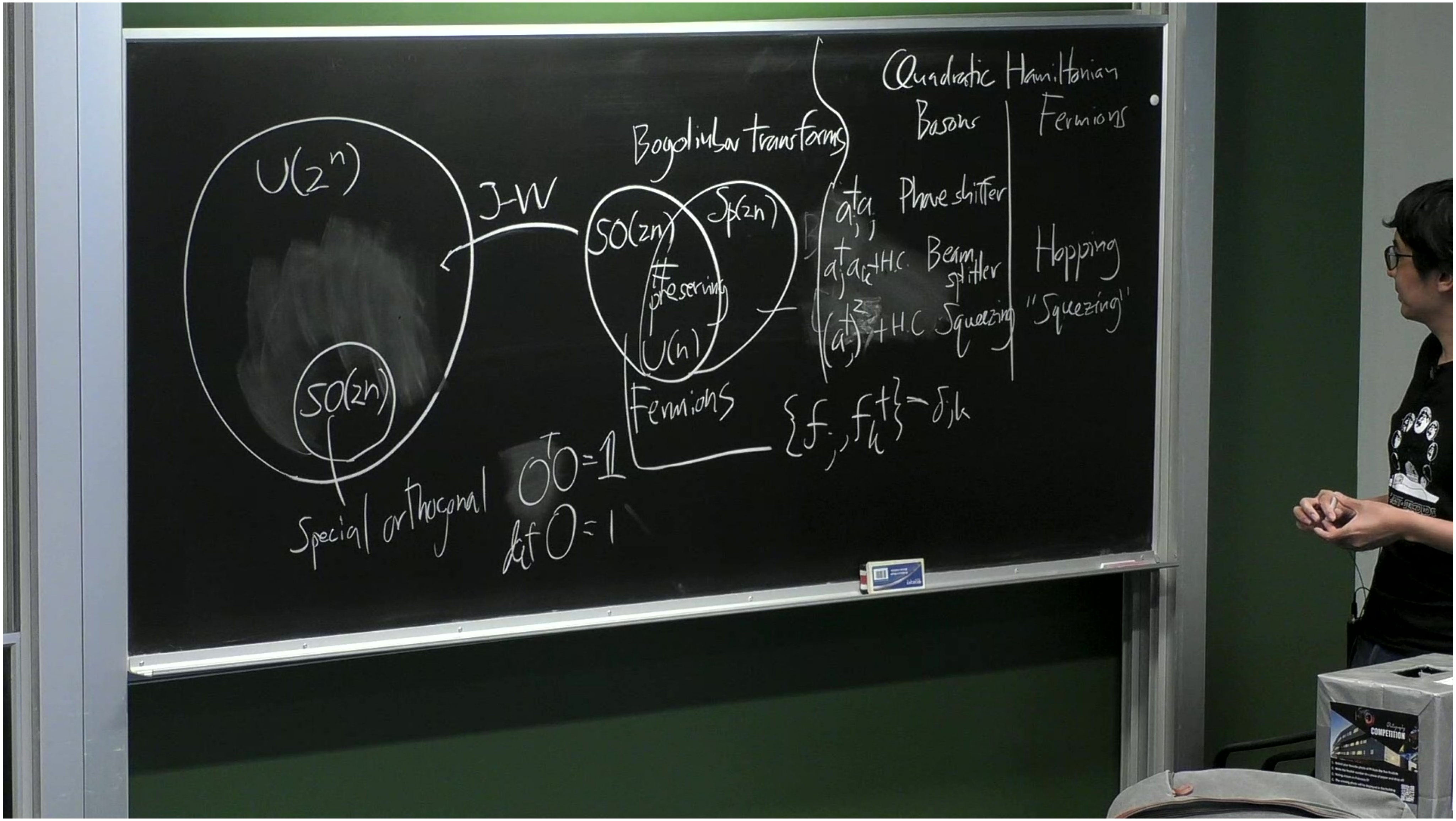
The frame operator for the Q function is the *quartic twirl*

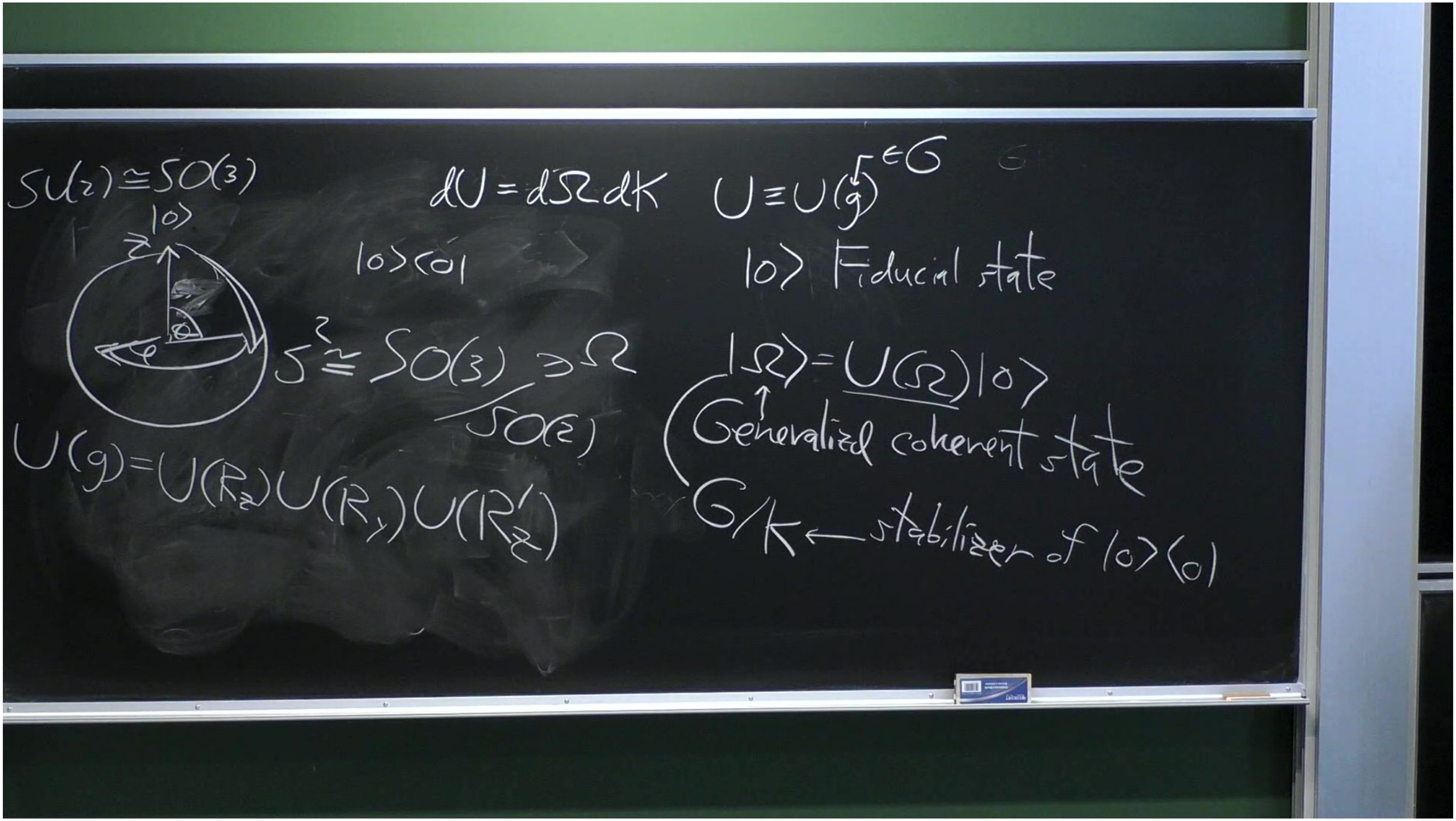
$$\int_\Lambda d\Omega\, |\Omega)(\Omega| = \int_\Lambda d\Omega\; |\Omega\rangle\langle\Omega| \odot |\Omega\rangle\langle\Omega|$$

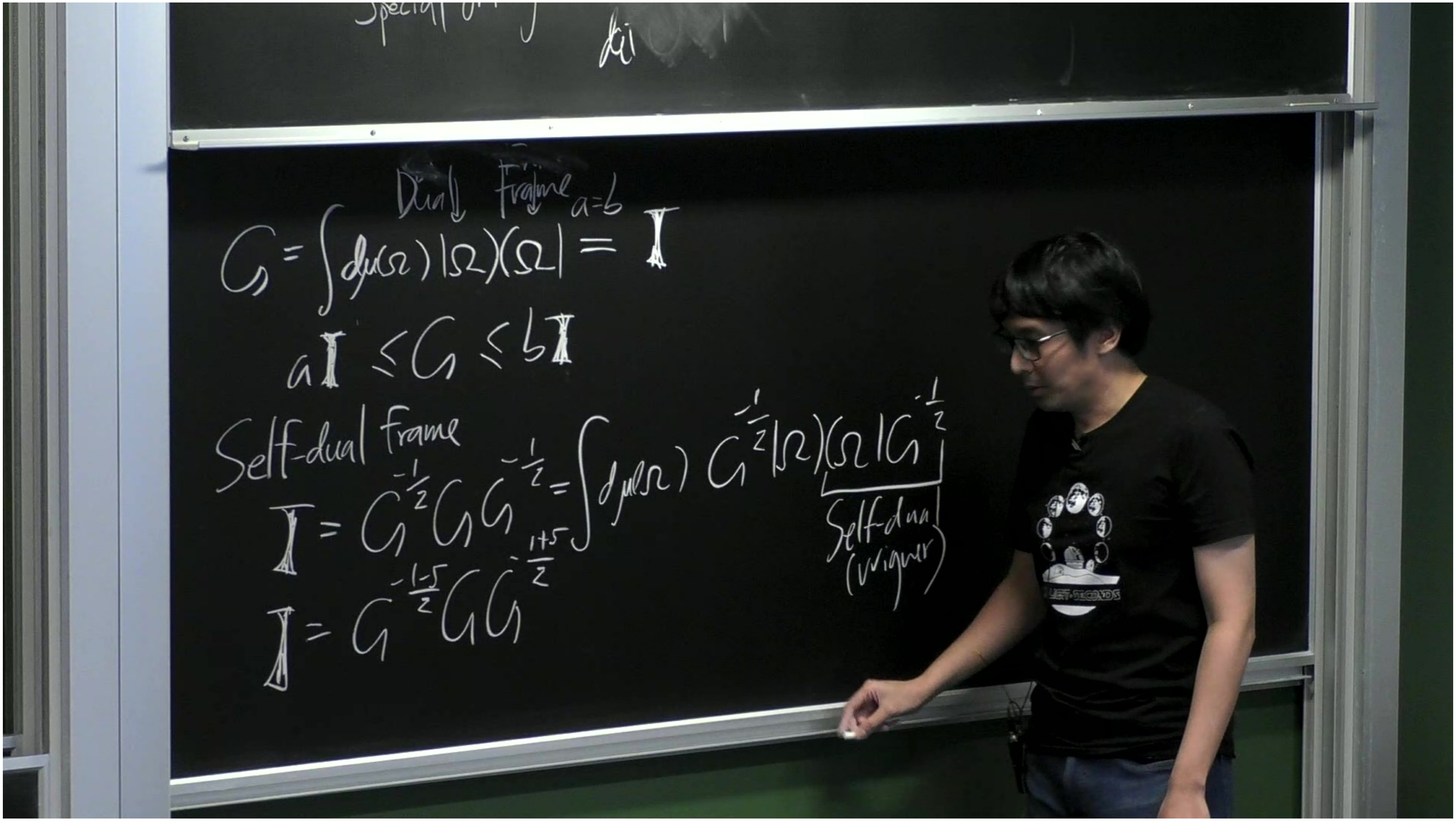

|ψ⟩ = Σ C^{j1 j2 ... jn} |x_{j1} x_{j2} ... x_{jn}⟩
Twirl
Schur's lemma
∫_G dμ(U) U O U† ∝ 1 ?
V ∫_G dμ(U) U B U† V†
∫_G dμ(U) U |0⟩⟨0| U† = 1/d



|ψ⟩ = Σ C^{j1 j2 ... jn} |x_{j1} x_{j2} ... x_{jn}⟩
Twirl
U ≡ U(g)
|0⟩ Fiducial state
|Ω⟩ = U(Ω)|0⟩
Generalized coherent state


Quadratic Hamiltonian
Bosons
Fermions
Bogoliubov transforms
U(2^n)
J-W
SO(2n)
Sp(2n)
preserving
U(n)
Fermions
{f_j, f_k^†} = δ_jk
SO(2n)
Special orthogonal
O^T O = 1
det O = 1
a_j^† a_j
Phase shifter
a_j^† a_k + H.C.
Beam splitter
Hopping
(a_j^†)^2 + H.C.
Squeezing
"Squeezing"


SU(2) ≅ SO(3)
dU = dΩ dK
U ≡ U(g) ∈ G
|0⟩
|0⟩⟨0|
|0⟩ Fiducial state
S² ≅ SO(3)/SO(2) ∋ Ω
|Ω⟩ = U(Ω)|0⟩
Generalized coherent state
U(g) = U(R_z)U(R_y)U(R'_z)
G/K ← stabilizer of |0⟩⟨0|


Dual Frame a=b
$G = \int d\mu(\Omega) |\Omega)(\Omega| = \mathbb{I}$
$a\mathbb{I} \le G \le b\mathbb{I}$
Self-dual Frame
$\mathbb{I} = G^{-\frac{1}{2}} G G^{-\frac{1}{2}} = \int d\mu(\Omega)\, G^{-\frac{1}{2}} |\Omega)(\Omega| G^{-\frac{1}{2}}$
Self-dual (Wigner)
$\mathbb{I} = G^{-\frac{1-s}{2}} G G^{-\frac{1+s}{2}}$