Title: Quantization of the NgÃ´ morphism (VIRTUAL)

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Series: Mathematical Physics

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Abstract: We will discuss work, joint with Victor Ginzburg, on the quantization (non-commutative deformation) of the Ngô morphism, a morphism of group schemes which plays a key role in Ngô's proof of the fundamental lemma in the Langlands program. We will also discuss how the tools used to construct this morphism can be used to prove conjectures of Ben-Zvi--Gunningham, which predict that this morphism gives "spectral decomposition" of DG categories with an action of a reductive group over the coarse quotient of a maximal Cartan subalgebra by the affine Weyl group.

Zoom link

Quantization of the Ngô Morphism (w/ Ginzburg)

$\mathfrak{Z}_G$ - universal centralizer

$\mathfrak{Z}_{GL_n} \cong \mathcal{M}_C(GL_n, O)$

Fix G a complex reductive group (eg GL_n)

Def) The group scheme of centralizers is

$$Z = \{(g,t) \in G \times \mathfrak{g} : \mathrm{Ad}_g(t) = t\}$$

This is a group scheme over $\mathfrak{g}$,

$$Z \xrightarrow{p} \mathfrak{g}.$$

$\boxed{\text{Ex}}$ $G = GL_2$, $\mathfrak{gl}_2 \cong \{2\times 2 \text{ matrices}\}$

Elements of $\mathfrak{gl}_2$	Fiber of p
$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\alpha \neq \beta$	$\mathbb{C}^\times \times \mathbb{C}^\times$
$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\mathbb{C}^\times \times \mathbb{C}$
$\begin{pmatrix} \alpha & 0 \\ \ldots \end{pmatrix}$	

Def/Prop An element $t \in \mathfrak{t}$ is _regular_ if $p^{-1}(t)$ is abelian.

We showed: for any regular $M \in \mathfrak{gl}_2$, the only elements centralizing M have the form $aM + S$ for any $a \in \mathbb{C}$ and $S \in Z(GL_2)$.

Def The _group scheme of universal centralizers_ for GL_2 is the scheme $\mathfrak{Z}_{GL_2} \longrightarrow \mathbb{A}^2 /\!/ S_2 = \mathrm{Spec}(\mathbb{C}[x,y]^{S_2})$

whose fiber at $\{z_1, z_2\}$ is

$$\left(\mathbb{C}[t]/(t-z_1)(t-z_2)\right)^\times.$$

<u>Thm</u> (Ngô) There exists a smooth abelian group $\mathfrak{Z}_G \to \mathfrak{g}/\!/G$ and an isomorphism

$$\mathfrak{g}_{reg} \times_{\mathfrak{g}/\!/G} \mathfrak{Z}_G \xrightarrow{\sim} p^{-1}(\mathfrak{g}_{reg}) \subseteq G \times \mathfrak{g}_{reg}.$$

Moreover, this isomorphism extends to a morphism of group schemes $\mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{Z}_G \longrightarrow G \times \mathfrak{g}$.

Note that $\mathfrak{g}^* \times_{\mathfrak{g}^*/\!/G} \mathcal{M}_C(G^\vee, \mathbb{O}) \xrightarrow{\text{Ngô}} T^*G = \mathrm{Spec}(\mathcal{O}(T^*G))$

$? \longleftarrow \mathcal{O}(T^*G)$

if we choose a Killing form. Each of these terms admits a quantization.

Conj: quantize.

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Conj (Nadler) The Ngô morphism quantizes.

$G \curvearrowright X$ +

$\operatorname{Spec}(\mathcal{O}(T^*G))$

In rep'n theory, given $H \curvearrowright Y$, its often useful to consider this action as a $H \curvearrowright Y \curvearrowleft Z(H)$.

j^*

In general, for reductive groups, this trick isn't so useful: $Z(PGL_n) = \{1\}$.

Hamiltonian G-spaces $\in QCoh(\mathfrak{g}^*)^G$

$$G \curvearrowright X \xrightarrow{\mu} \mathfrak{g}^* \rightsquigarrow \begin{array}{ccc} G \curvearrowright X \curvearrowleft & Z & \longleftarrow \mathfrak{g}^* \times_{\mathfrak{g}^*/\!/G} \mathfrak{Z}_G \\ \downarrow & \downarrow & \downarrow \\ \mathfrak{g}^* & \mathfrak{g}^* & \mathfrak{g}^* \end{array}$$

$$\begin{array}{c} QCoh(\mathfrak{g}^*)^G \ni \mathcal{O}(X) \\ j^* \downarrow \quad \uparrow j_* \\ QCoh(\mathfrak{g}^*_{reg})^G \\ \downarrow \wr \\ Rep(\mathfrak{Z}_G) = \mathcal{O}(\mathfrak{Z}_G)\text{-comod} \end{array}$$

Prop) $\quad$ $$\begin{array}{c} HC_G \\ \bar{J}^* \downarrow \quad \uparrow \bar{J}_* \\ Rep(W) = W\text{-comod} \end{array}$$

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Prop) (

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Prop] (Teleman, Ben-Zvi–Gunninghams)

$$\mathfrak{Z}_G \cong T^*(N^- {}_+\backslash G /_+ N^-).$$

$$\mathcal{W} := \mathcal{D}_G /\!\!/ n^{+,-} \times n^{+,-}$$

Thm] (G, '23) $\mathcal{W}$ is a cocommutative coalgebra. (Comes from

$$\mathcal{D}(G)^{\bar{N}\times\bar{N}, +\times+} \cong \mathrm{IndCoh}(\mathfrak{t}^*/\!\!/W^{\mathrm{aff}})$$

$\mathcal{W}$-comod // $\uparrow$ monoidal

$\mathcal{W}$-mod $\to \mathfrak{Z}_{\mathfrak{g}}$-mod

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$W \leftarrow Z_{\mathfrak{g}}$.

<u>Thm</u> There is a morphism of corings $D(G) \to U\mathfrak{g} \otimes_{Z\mathfrak{g}} W$ of $D(G)$-modules which is filtered and whose associated graded recovers the Ngô morphism.

$\mathcal{O}(T^*G) \to S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^G} \mathcal{O}(\mathfrak{Z}_G)$

<u>Cor</u> There is a braided monoidal functor

$D(G)^{G,\heartsuit} \hookrightarrow D(G)^{\heartsuit}$

$$D(G)^{\tilde{N}\times\tilde{N}, T\times T} \xrightarrow{\text{Ngô}} Z(D(G)).$$

$\cong$

$C^{H,\heartsuit} \hookrightarrow C^{\heartsuit}$

$/\!/ W^{\mathrm{aff}})$

Informally, all G-categories spectrally decompose over $\mathfrak{t}^*/\!/W^{\mathrm{aff}}$.

$$\mathcal{U}_{\mathfrak{g}} \otimes_{\mathcal{Z}_{\mathfrak{g}}} \widehat{W} \otimes_{W} \mathcal{Z}_{\mathfrak{g}}$$

$$\cong$$

$$\mathcal{U}_{\mathfrak{g}}$$

Ex) $G = GL_n$.

Known: $Z(D(GL_n)^{\heartsuit}) \cong D(T)^W \supseteq \mathrm{IndCoh}(\mathfrak{t}^*/W^{\mathrm{aff}})$

$$\mathrm{IndCoh}(\mathfrak{t}^*)^W \supseteq \mathrm{IndCoh}(\mathfrak{t}^*/\!/W)$$

$$\cong$$

$$\mathrm{IndCoh}(\mathfrak{t}^*/W)$$

$D(G)$, $W \leftarrow$ Cocommutative

$$\left(T^*(\bar{N}_+\backslash G) \times_{\mathfrak{g}^*} T^*(G/_+\bar{N})\right)/G \cong T^*(\bar{N}^+\backslash G/_+\bar{N})$$

$$\begin{tikzcd} HC_G \arrow[r, phantom, "\ni"] \arrow[d, "\mathcal{K}"'] & \mathcal{B} \\ Z_{\mathfrak{g}}\text{-mod}) & \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r] \arrow[d] & \mathfrak{g}^*/G \\ \mathfrak{g}^*/\!/G \arrow[ur, hook, "\mathcal{K}"'] & \end{tikzcd}$$

$$\mathcal{K}(\mathcal{B}) \cong \left(D^{*\lambda} \otimes_{U\mathfrak{g}} \mathcal{B}\right)^G.$$