Title: Analytical Methods for Inflation Correlators.

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Series: Particle Physics

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Abstract: The correlation functions of large-scale fluctuations are crucial observables in modern cosmology. These boundary correlators contain rich information about the bulk dynamics and can be used to probe physics at the inflation scale. There have been considerable efforts in the analytical study of inflation correlators in recent years. In this talk, I will introduce the basic structure of inflation correlators and several analytical methods we have developed for their computation. In particular, I will introduce the method of partial Mellin-Barnes (PMB) representation. With this method, we can largely solve the massive tree correlators analytically. At the loop level, we use PMB to prove a factorization theorem for the nonanalytical part of inflation correlators, which produces a characteristic nonlocal signal. Finally, we can recover the full correlator starting from this nonanalytical part using a dispersion integral.

Zoom link

Analytical Methods for Inflation Correlators

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Perimeter Insitute | May 10th, 2024

2208.13790, 2304.13295, 2308.14802 w/ Zhong-Zhi Xianyu

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Outline

- Cosmological Collider Physics
  - Inflation correlators
  - Signal versus background
- Analytical Methods
  - Improved cosmological bootstrap
  - Partial Mellin-Barnes representation
- Factorization Theorem and Cutting Rule
  - 1-loop nonlocal signals
- Future: Dispersion Integral

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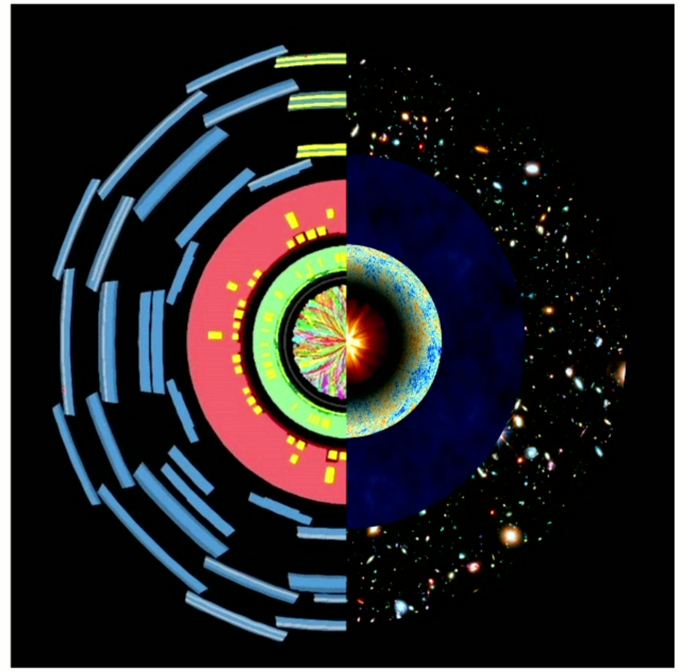
Cosmological Collider Physics

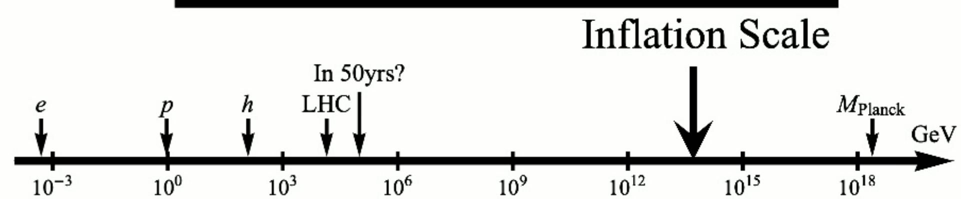
Inflation seems to be the highest energy observable process, with the energy scale $H \sim 10^{13}\text{GeV}$.

$$\mathrm{d}s^2 = \frac{1}{H^2\tau^2}(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2)$$



We want to make use of the high energy of the early universe to study fundamental physics at such a high scale.

How? One answer is the “Cosmological Collider” program.

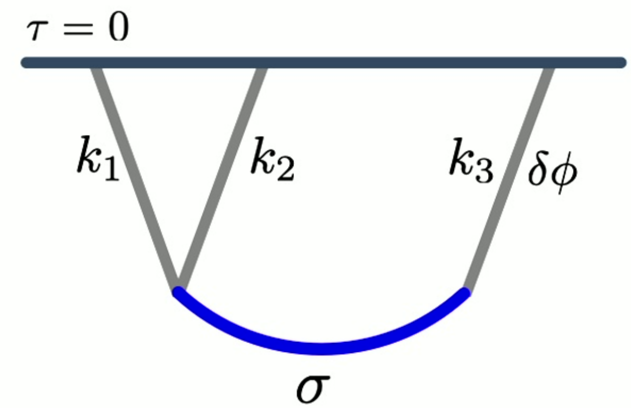


Arkani-Hamed, Maldacena, 1503.08043
Chen, Namjoo, Wang, 1509.03930

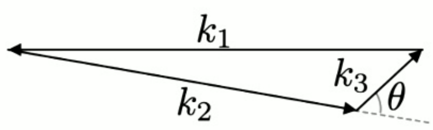
Cosmological Collider Signal

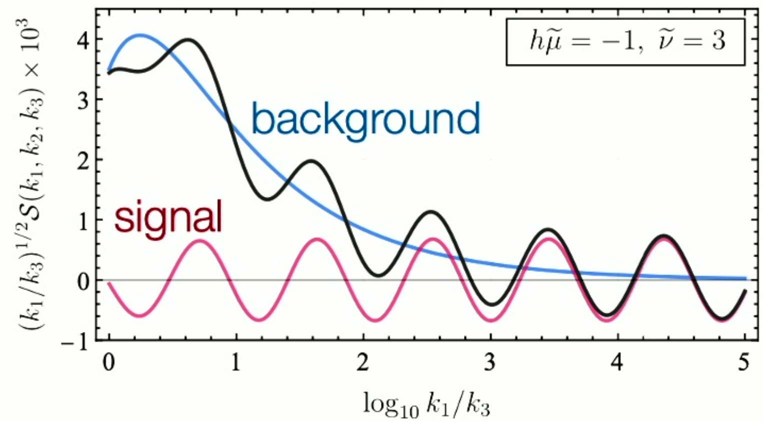
Observable: Inflation correlators.

Equal-time correlation functions of inflaton fluctuation (massless scalar).



Signal: Oscillatory pattern in the logarithmic scale in the squeezed limit.





Intuition: Pair production (Schwinger effect) & superhorizon interference.

Tong, Wang, Zhu, 2112.03448

Qin, Xianyu, 2208.13790

Nonlocal Signal

signal size

scaling exponent

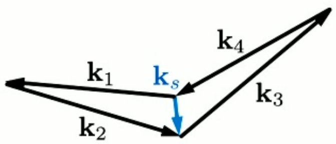
phase

$$\lim_{k_s \to 0} \mathcal{S}(k_1, k_2, k_3, k_4, k_s) = \mathcal{A} \times K(\theta) \times \left(\frac{k_s^2}{k_{12}k_{34}}\right)^{\alpha} \times \cos\left(\omega \log \frac{k_s^2}{k_{12}k_{34}} + \vartheta\right)$$

$$+\text{terms analytic in } k_s$$

kinematic factor

frequency



What can we learn from the nonlocal signal?

angular momentum, tree/loop, mass, spin, interaction, …

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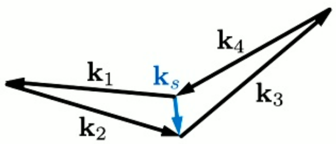
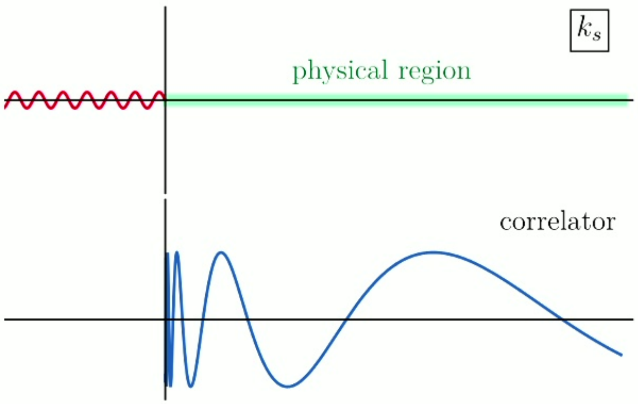
Nonlocal Signal as Nonanalyticity

$$\lim_{k_s \to 0} \mathcal{S}(k_1, k_2, k_3, k_4, k_s) = \mathcal{A} \times K(\theta) \times \left(\frac{k_s^2}{k_{12}k_{34}}\right)^{\alpha} \times \cos\left(\omega \log \frac{k_s^2}{k_{12}k_{34}} + \vartheta\right)$$

$$+\text{terms analytic in } k_s$$





The oscillation also corresponds to nonanalyticity in unphysical region.

$$\mathcal{C}\left(\frac{k_s^2}{k_{12}k_{34}}\right)^{i\omega} + \text{c.c.} = 2|\mathcal{C}| \cos\left(\omega \log \frac{k_s^2}{k_{12}k_{34}} + \arg \mathcal{C}\right)$$

Progress Nowadays

Observations

21cm: Meerburg, Münchmeyer, Muñoz, Chen: 1610.06559
LSS: Dizgah, Lee, Muñoz, Dvorkin: 1801.07265
Galaxy imaging: Kogai, Akitsu, Schmidt, Urakawa, 2009.05517
P-violating trispectrum: Philcox: 2206.04227; Cabass, Ivanov, Philcox, 2210.16320

And real data are coming!
BOSS: Cabass, Philcox, Ivanov, Akitsu, Chen, Simonović, Zaldarriaga, 2404.01894
Planck: Sohn, Wang, Fergusson, Shellard, 2404.07203

Particle Physics & Phenomenology

SM: Chen, Wang, Xianyu: 1604.07841, 1610.06597, 1612.08122; Hook, Huang, Racco, 1907.10624, 1908.00019;
New scalars: An et al: 1706.09971; Wang, 1911.04459; Bodas, Kumar, Sundrum, 2010.04727; Lu, Reece, Xianyu, 2108.11385;
Higgs: Wu, 1812.10654; Kumar, Sundrum, 1711.03988, Lu, Wang, Xianyu, 1907.07390;
CP violation: Liu et al, 1909.01819; Cui, Xianyu, 2112.10793;
Spin: Lee, Baumann, Pimentel, 1607.03735; Chen, Wang, Xianyu, 1805.02656; Alexander et al, 1907.05829; Wang, Xianyu, 1910.12876, 2004.02887; Tong, Xianyu, 2203.06349, Niu et al., 2211.14324, 2211.14331;
Beyond slow-roll: Tong, Wang, Zhou, 1801.05688; Kumar and Sundrum, 1908.11378; Chen, Ebadi, Kumar, 2205.01107;
DM: Li et al: 1903.08842, 2002.01131; Lu, 2103.05958, and many more

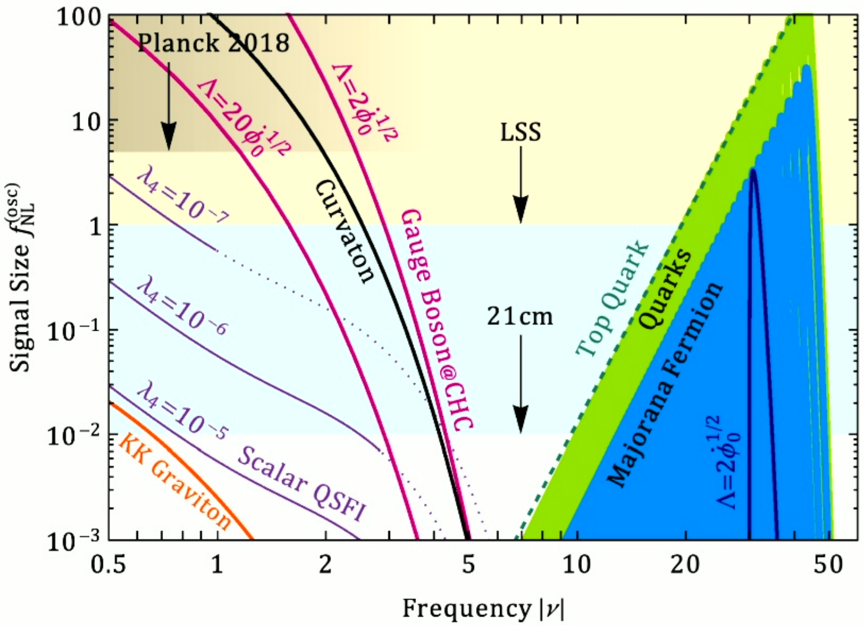
Structure and Calculation

Bootstrap: Arkani-Hamed, Baumann, Lee, Pimentel, et al: 1811.00024, 1910.14051, 2005.04234; Pajer et al: 2007.00027, 2010.12818, 10189; Pimentel, Wang, 2205.00013; Jazayeri, Renaux-Petel, 2205.10340, Wang, Pimentel, Achúcarro 2212.14035; Qin, Xianyu, 2301.07047
Mellin-Barnes: Sleight, Taronna, 1906.12302, 1907.01143, 2007.09993, 2109.02725; Qin, Xianyu, 2205.01692, 2208.13790;
Spectral: Xianyu, Zhang, 2211.03810; Numerical: Wang, Xianyu, Zhong, 2109.14635
Analytical properties: Pajer et al: 2009.02898, 2103.09832, 2104.06587; Di Pietro, Gorbenko, S. Komatsu, 2108.01695; Tong, Wang, Zhu, 2112.03448, and many more

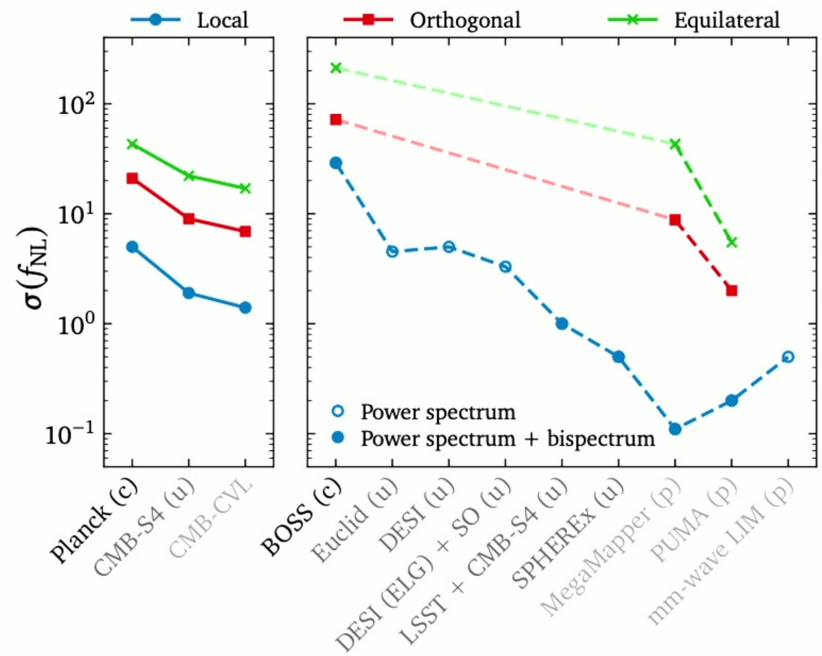
Progress Nowadays

To have better forecasts and templates for future observations,

we are led to calculate the inflation correlators **analytically**!



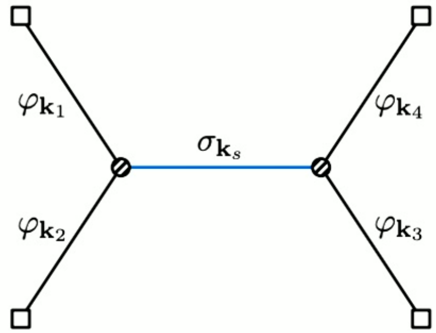
Wang, Xianyu, 1910.12876



Achúcarro, et al: 2203.08128

Difficulty in Calculations

In-In Formalism:



$$\Delta\mathcal{L} = \frac{1}{2}\lambda a^2 \varphi'^2 \sigma$$

$$\langle \varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}\rangle'_s$$

$$= \lambda^2 \sum_{\mathsf{a},\mathsf{b}=\pm} (\mathrm{i}\mathsf{a})(\mathrm{i}\mathsf{b}) \int_{-\infty}^{0} \frac{\mathrm{d}\tau_1}{\tau_1^2}\frac{\mathrm{d}\tau_2}{\tau_2^2} G'_{\mathsf{a}}(k_1,\tau_1)G'_{\mathsf{a}}(k_2,\tau_1)G'_{\mathsf{b}}(k_3,\tau_2)G'_{\mathsf{b}}(k_4,\tau_2)D_{\mathsf{ab}}(k_s;\tau_1,\tau_2)$$

$$D_{-+}(k;\tau_1,\tau_2) = \sigma(k,\tau_1)\sigma^*(k,\tau_2),$$

$$D_{+-}(k;\tau_1,\tau_2) = \left[D_{+-}(k;\tau_1,\tau_2)\right]^*,$$

$$D_{\pm\pm}(k;\tau_1,\tau_2) = D_{\mp\pm}(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2) + D_{\pm\mp}(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1),$$

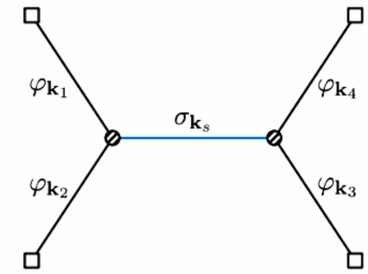
$$\sigma(k,\tau) = \frac{\sqrt{\pi}}{2}e^{-\pi\widetilde{\nu}/2}(-\tau)^{3/2}\mathrm{H}^{(1)}_{\mathrm{i}\widetilde{\nu}}(-k\tau), \qquad \widetilde{\nu} \equiv \sqrt{m^2 - \frac{9}{4}}$$

$$G_{\mathsf{a}}(k,\tau) = \frac{1}{2k^3}(1-\mathsf{a}\mathrm{i}k\tau)e^{\mathsf{a}\mathrm{i}k\tau}$$

$$H \equiv 1$$

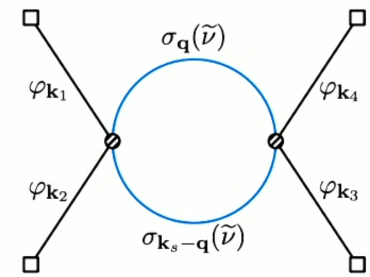
Difficulty in Calculations

- Tree is hard
  - Propagators in terms of special functions
  - In-in formalism in hybrid coordinates $(\tau, \mathbf{k})$
  - Time-ordered integrals
  - Reduced symmetry with dS-boost breaking



Arkani-Hamed, Baumann, Lee, Pimentel, 1811.00024

- Loop is even harder, but also important
  - Signal produced by charged particles



Xianyu, Zhang, 2211.03810

Cosmological Bootstrap

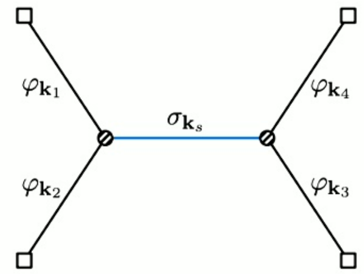
Cosmological Bootstrap

Baumann, Lee, Pimentel, et al: 1811.00024, 1910.14051, 2005.04234

Original idea:

Constraints from boundary conformal symmetry.

Solving the Ward identity of SCTs with a clever ansatz.



Cons:

Full dS isometry is needed.

Hard to dispose of the spurious folded poles.

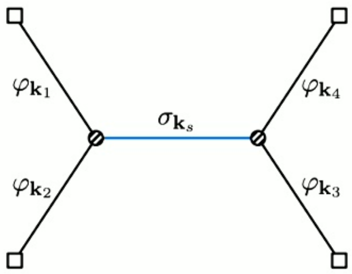
Hard to set ansatz for more complicated graphs.

Improved Bootstrap

Our observation:

EoM of the bulk propagator implies ODE for the correlator.

Qin, Xianyu: 2208.13790, 2301.07047



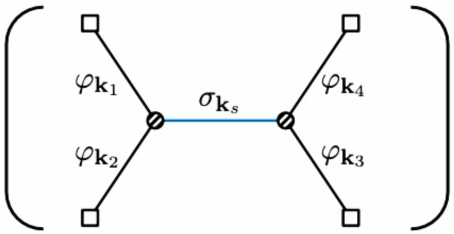
Pros:

Applicable to dS-boost-breaking cases.

Closed-form results for 3pt and 2pt correlators.

Can be generalized to multiple-exchange diagrams.

Improved Bootstrap

$\mathcal{D}$  $= 0$ or a local term

$$\mathcal{I}_{\mathsf{ab}}^{p_1p_2}(r_1,r_2) \equiv -\mathsf{ab}\, k_s^{5+p_1+p_2} \int_{-\infty}^{0} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, (-\tau_1)^{p_1} (-\tau_2)^{p_2} e^{\mathrm{i}\mathsf{a}k_{12}\tau_1 + \mathrm{i}\mathsf{b}k_{34}\tau_2} D_{\mathsf{ab}}(k_s;\tau_1,\tau_2)$$

The opposite-sign propagators satisfy the homogeneous equation — Factorized part: Signal

$$(\Box - m^2) D_{\pm\mp} = 0$$

The same-sign propagators satisfy the imhomogeneous equation — Time-ordered part: EFT piece

$$(\Box - m^2) D_{\pm\pm} = \delta(x-y)$$

Bootstrap Equation

$$(\tau_1^2\partial_{\tau_1}^2 - 2\tau_1\partial_{\tau_1} + k_s^2\tau_1^2 + m^2)D_{\pm\mp}(k_s;\tau_1,\tau_2) = 0$$

$$(\tau_1^2\partial_{\tau_1}^2 - 2\tau_1\partial_{\tau_1} + k_s^2\tau_1^2 + m^2)D_{\pm\pm}(k_s;\tau_1,\tau_2) = \mp\mathrm{i}\tau_1^2\tau_2^2\delta(\tau_1-\tau_2)$$

$$\mathcal{I}_{\mathsf{ab}}^{-2,-2}(r_1,r_2) \equiv -\mathsf{ab}\,k_s\int_{-\infty}^{0}\frac{\mathrm{d}\tau_1}{\tau_1^2}\frac{\mathrm{d}\tau_2}{\tau_2^2}\,e^{\mathrm{i}\mathsf{a}k_{12}\tau_1+\mathrm{i}\mathsf{b}k_{34}\tau_2}D_{\mathsf{ab}}(k_s;\tau_1,\tau_2).$$

Insert the differential operator in the integrand before the propagator, then commute it with the time integral using IBP.

$$\left[(r_1^2-r_1^4)\partial_{r_1}^2 - 2r_1^3\partial_{r_1} + \left(\widetilde{\nu}^2+\frac{1}{4}\right)\right]\mathcal{I}_{\pm\mp}^{-2,-2}(r_1,r_2) = 0$$

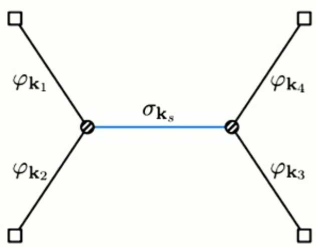
$$\left[(r_1^2-r_1^4)\partial_{r_1}^2 - 2r_1^3\partial_{r_1} + \left(\widetilde{\nu}^2+\frac{1}{4}\right)\right]\mathcal{I}_{\pm\pm}^{-2,-2}(r_1,r_2) = \frac{r_1r_2}{r_1+r_2}$$

$$r_1 \equiv \frac{k_s}{k_{12}},\qquad r_2 \equiv \frac{k_s}{k_{34}},\qquad k_{12}\equiv k_1+k_2,\qquad k_{34}\equiv k_3+k_4,\qquad \widetilde{\nu}\equiv\sqrt{m^2-\frac{9}{4}}$$

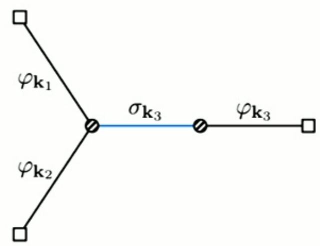
Qin, Xianyu, 2208.13790, 2301.07047

Closed-Form Formulae

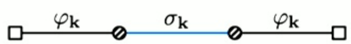
With a clever change of variable: $u_i = \frac{2r_i}{1+r_i}$, we are able to pick out the spurious divergence part at the folded limit $u_i(r_i) \to 1$ easily, and derive **closed-form** formulae for 3pt and 2pt functions.



$k_4 \to 0$
$r_2 \to 1$
$u_2 \to 1$



$k_2 \to 0$
$r_1 \to 1$
$u_1 \to 1$



series → series → numeric

series | closed-form | closed-form

Partial Mellin-Barnes

Mellin Transform

- For QFT in flat spacetime, Fourier transform is useful, since its the kernel is the eigenmode of translation.

- In dS, there is no time translation symmetry. Instead, we have dilation symmetry. The corresponding integral transform is the so-called (inverse) Mellin transform:

$$f(x) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, x^{-s}\mathcal{F}(s), \qquad \mathcal{F}(s) = \int_0^\infty \mathrm{d}x\, x^{s-1} f(x)$$

Partial Mellin-Barnes Representation

PMB rep: Apply inverse Mellin transform to all internal massive modes.

Qin, Xianyu, 2208.13790

Scalar propagators in PMB representation:

$$D_{+-}(k;\tau_1,\tau_2) = \frac{\pi}{4} e^{-\pi\widetilde{\nu}} (\tau_1\tau_2)^{3/2} \mathrm{H}^{(1)}_{\mathrm{i}\widetilde{\nu}}(-k\tau_1) \mathrm{H}^{(2)}_{-\mathrm{i}\widetilde{\nu}}(-k\tau_2)$$

$$= \frac{1}{4\pi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}s_1}{2\pi \mathrm{i}} \frac{\mathrm{d}s_2}{2\pi \mathrm{i}} e^{-\mathrm{i}\pi(s_1-s_2)} \boxed{\left(\frac{k}{2}\right)^{-2(s_1+s_2)}} \boxed{(-\tau_1)^{-2s_1+3/2}(-\tau_2)^{-2s_2+3/2}}$$

$$\times \Gamma\left(s_1 - \mathrm{i}\frac{\widetilde{\nu}}{2}\right) \Gamma\left(s_1 + \mathrm{i}\frac{\widetilde{\nu}}{2}\right) \Gamma\left(s_2 - \mathrm{i}\frac{\widetilde{\nu}}{2}\right) \Gamma\left(s_2 + \mathrm{i}\frac{\widetilde{\nu}}{2}\right)$$

Time integral and momentum integral are factorized.

Integrands are greatly simplified.

Inflation Correlator in PMB

For a graph with B external lines, V internal vertices, I internal lines, L loops:

E is the energy inflow at each vertex (magnitude sum of external momenta)

$$\mathcal{G}(\{\mathbf{k}\}) = \sum_{\mathsf{a}_1,\cdots \mathsf{a}_V=\pm} \int \prod_{i=1}^{V} \Big[\mathrm{i}\mathsf{a}_i \mathrm{d}\tau_i \, (-\tau_i)^{p_i}\Big] \prod_{j=1}^{B} \Big[C_{\mathsf{a}_j}(k_j;\tau_j)\Big] \times \int \prod_{k=1}^{L} \Big[\frac{\mathrm{d}^3\mathbf{q}_k}{(2\pi)^3}\Big] \prod_{\ell=1}^{I} \Big[D^{(\widetilde{\nu}_\ell)}_{\mathsf{a}_{\ell_1}\mathsf{a}_{\ell_2}}(p_\ell;\tau_{\ell_1},\tau_{\ell_2})\Big]$$

$$= \mathcal{A} \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \prod_{\ell=1}^{I} \Big[\frac{\mathrm{d}s_{\ell_1}}{2\pi\mathrm{i}} \frac{\mathrm{d}s_{\ell_2}}{2\pi\mathrm{i}}\Big] \Big[\sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V} \mathcal{T}_{\mathsf{a}_1\cdots\mathsf{a}_V}\big(\{s\},\{p\},\{E\}\big)\Big] \times \mathcal{L}\big(\{s\},\{\mathbf{K}\}\big)$$

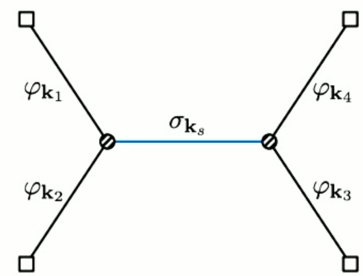
$$\times \prod_{\ell=1}^{I} \Gamma\Big[s_{\ell_1} - \frac{\mathrm{i}\widetilde{\nu}_\ell}{2}, s_{\ell_1} + \frac{\mathrm{i}\widetilde{\nu}_\ell}{2}, s_{\ell_2} - \frac{\mathrm{i}\widetilde{\nu}_\ell}{2}, s_{\ell_2} + \frac{\mathrm{i}\widetilde{\nu}_\ell}{2}\Big].$$

Dependence on momentum transfer is fully encoded in the momentum integral:

$$\mathcal{L}\big(\{s\},\{\mathbf{K}\}\big) \equiv \int \prod_{i=1}^{L} \Big[\frac{\mathrm{d}^3\mathbf{p}_i}{2\pi\mathrm{i}}\Big] \prod_{\ell=1}^{I} \Big|\frac{\mathbf{p}_\ell}{2}\Big|^{-2s_{\ell_1\ell_2}}$$

PMB at Tree Level

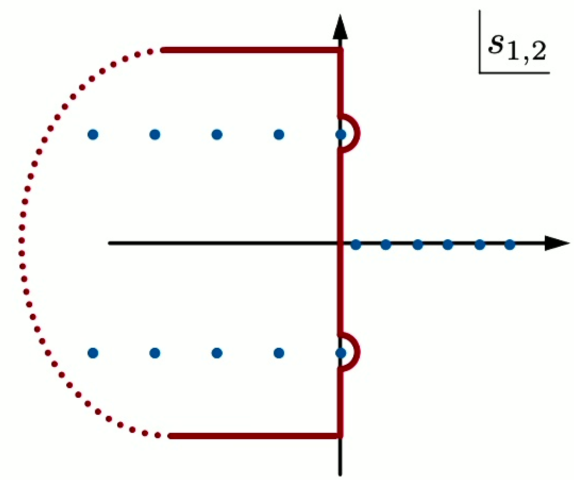
Sum the residue at IR poles!



Factorized integral: Signal

$$s_1 = -n_1 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}, \quad s_2 = -n_2 \pm \frac{\mathrm{i}\widetilde{\nu}}{2}$$ => local signal

$$s_1 = -n_1 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}, \quad s_2 = -n_2 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}$$ => nonlocal signal



Time-ordered integral: Background

$$s_1 = -n_1 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}, \quad s_2 = -n_2 \pm \frac{\mathrm{i}\widetilde{\nu}}{2}$$ => background

$$s_1 = -n_1 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}, \quad s_2 = -n_2 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}$$ => 0

PMB at Tree Level

Tree-level lessons：

- (Nonlocal) Signal is always factorized.

- Nonlocal signal equals to residue sum at nonlocal IR poles:

$$s_1 = -n_1 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}, \quad s_2 = -n_2 \mp \frac{\mathrm{i}\widetilde{\nu}}{2}$$

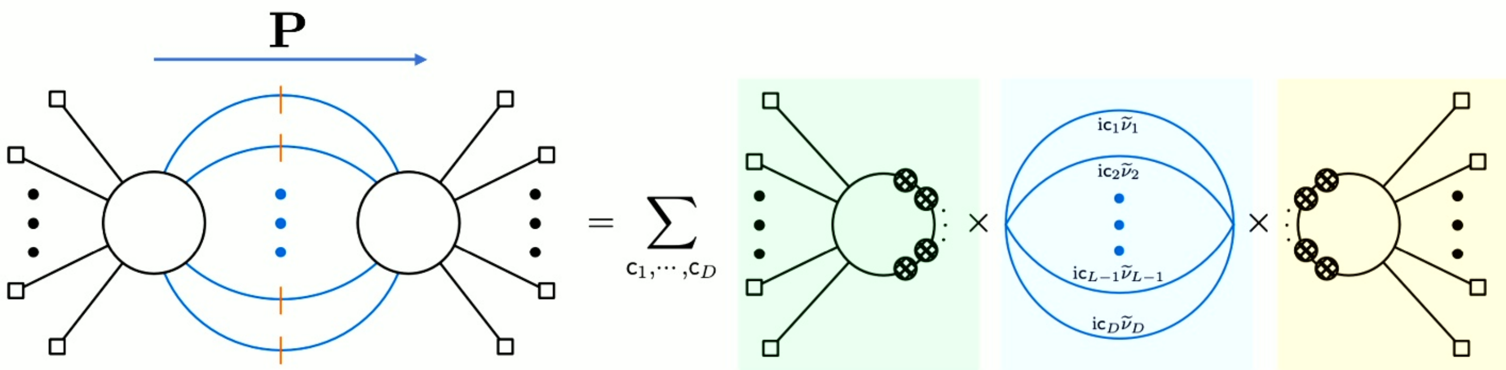
- At these poles, the four propagators $D_{\mathsf{ab}}(k_s;\tau_1,\tau_2)$ become the same and equal to their real part, which is automatically factorized:

$$\mathrm{Re}\, D_{\mathsf{ab}}(k_s;\tau_1,\tau_2) = \frac{1}{2}\Big[D_{+-}(k_s;\tau_1,\tau_2) + D_{-+}(k_s;\tau_1,\tau_2)\Big]$$

Factorization Theorem

Factorization Theorem for Nonlocal Signal

- Factorization theorem:



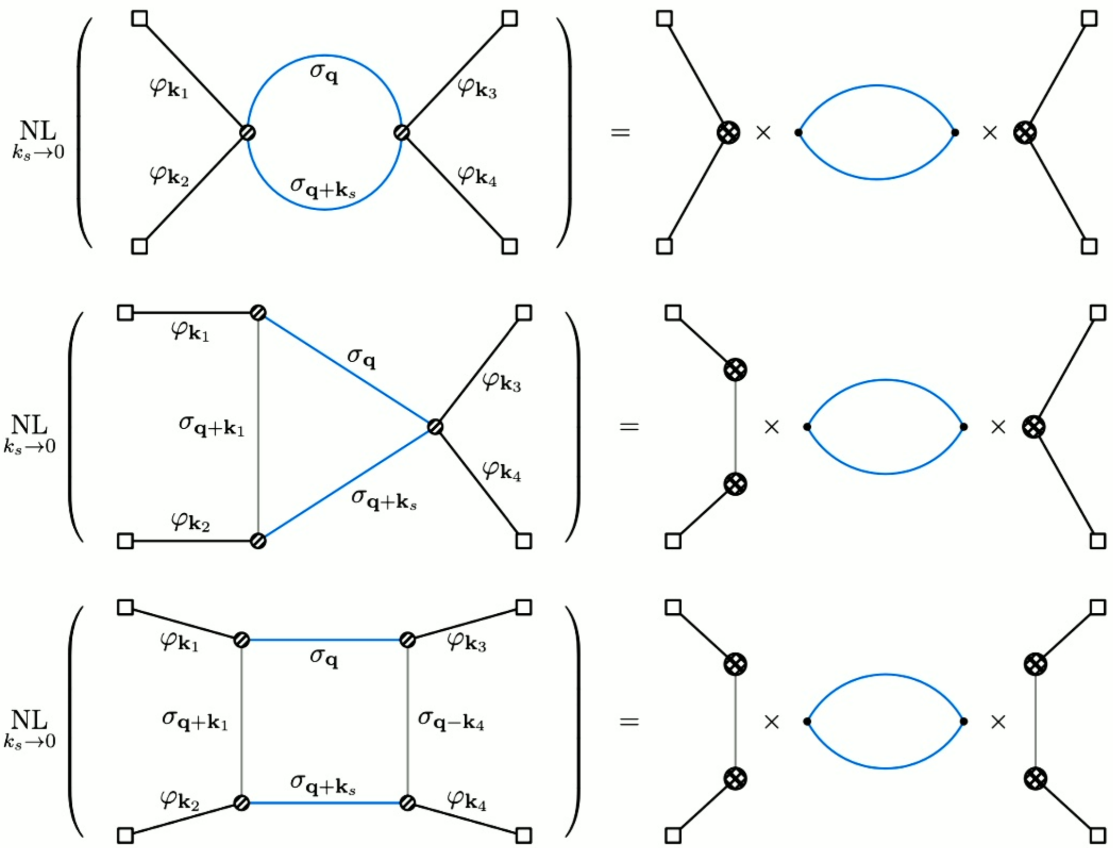
$$\lim_{P\to 0}\mathcal{T}(\{\mathbf{k}\})\supset\sum_{\substack{c_1,\cdots,c_D=\pm\\ \text{nonlocal}}}\mathcal{G}^{(\mathrm{L})}_{c_1\cdots c_D}\left(\{\mathbf{k}^{(\mathrm{L})}\}\right)\mathcal{G}^{(\mathrm{R})}_{c_1\cdots c_D}\left(\{\mathbf{k}^{(\mathrm{R})}\}\right)\mathfrak{M}_{c_1\cdots c_D}(P)$$

$$+\,(\text{subleading nonlocal signals})+(\text{terms analytic in } P)$$

- Cutting rule: replace all cut propagators by their real parts.

Qin, Xianyu, 2304.13295, 2308.14802

Applications



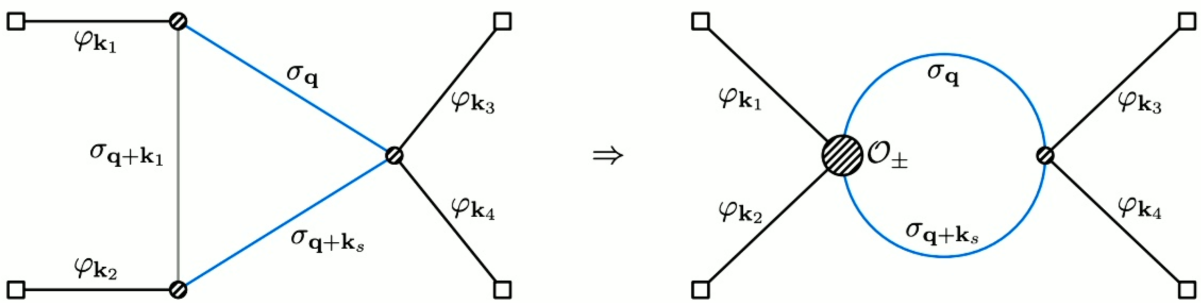
Analytical results for nonlocal signal from all typical 1-loop processes, with the help of closed-form expression of 2pt functions.

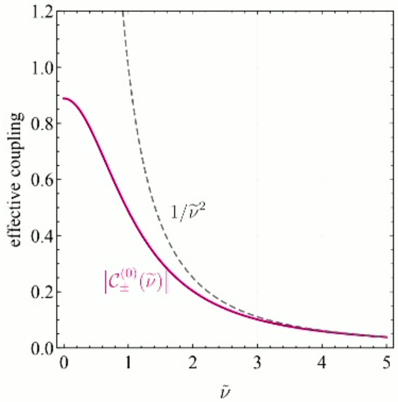
Qin, Xianyu, 2304.13295

OPE and Consistency

Factorization theorem in an OPE perspective:

- Push the soft bulk lines to the boundary.
- Perform OPE for boundary operators.





$$\mathcal{O}_\pm = \frac{1}{4} a^2 \mathcal{C}_\pm^{(0)}(m) \varphi'^2 \sigma^2 + \cdots , \qquad \lim_{m\to\infty} \mathcal{C}_\pm^0(m) \sim \frac{1}{m^2}$$

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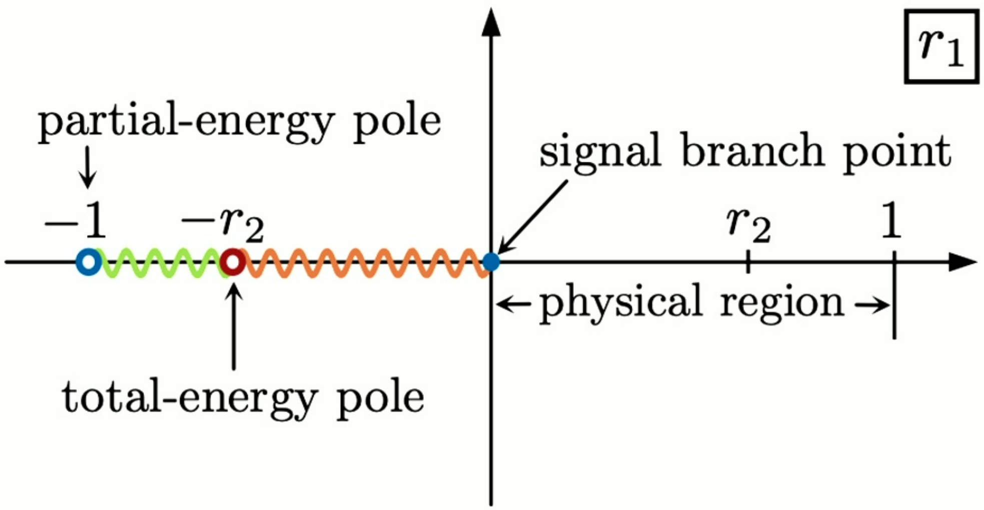
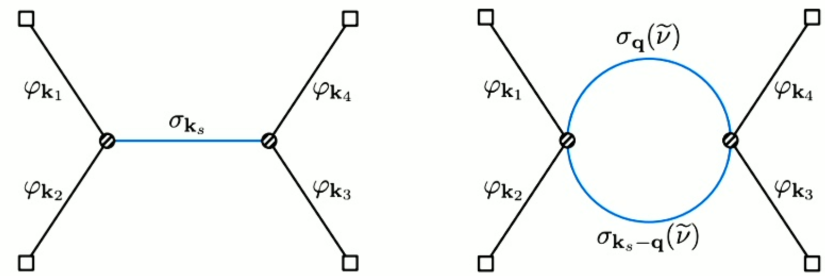
How can we do better?

Analytical Structure





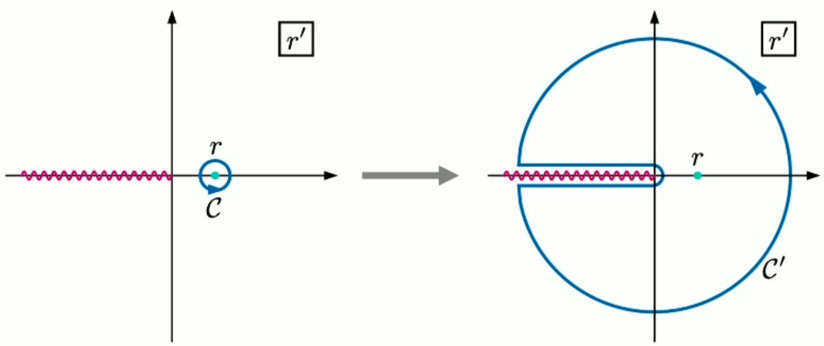
$$r_1 \equiv \frac{k_s}{k_1 + k_2}, \qquad r_2 \equiv \frac{k_s}{k_3 + k_4}$$

Total & partial energy poles: time integral over conserved energy.

Signal: logarithmic oscillation.

Dispersion Integral: from Signal to Correlator

We can recover the full correlator from the signal via a dispersion integral, for both tree and loop diagrams!



$$f(r) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \mathrm{d}r' \, \frac{f(r')}{r' - r} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{0} \mathrm{d}r' \, \frac{\mathrm{Disc} f(r')}{r' - r}$$

Analytical Structure + Signal $\xrightarrow{\text{dispersion integral}}$ Correlator

Liu, Qin, Xianyu, ongoing work

Summary and Outlooks

- Tree-level correlators:

Basically solved with bootstrap/PMB.

See Xianyu, Zang, 2309.10849;
Aoki, Pinol, Sano, Yamaguchi, Zhu, 2404.09547
for applications in multiple exchanges.

- Loop-level correlators:

Nonlocal signal found for arbitrary graph.
Other types of signals? Analytical structures?

See Xianyu, Zhang, 2211.03810
for full bubble diagram results
using spectral decomposition.
(PMB also works)

- Other methods: Dispersion? CFT?

- Pheno: Large signal? SUSY signal? String amplitude?