Title: Emergent symmetries and their application to logical gates in quantum LDPC codes

Speakers: Guanyu Zhu

Collection: Physics of Quantum Information

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Abstract: In this talk, I'll discuss the deep connection between emergent k-form symmetries and transversal logical gates in quantum low-density parity-check (LDPC) codes. I'll then present a parallel fault-tolerant quantum computing scheme for families of homological quantum LDPC codes defined on 3-manifolds with constant or almost-constant encoding

rate using the underlying higher symmetries in our recent work. We derive a generic formula for a transversal T gate on color codes defined on general 3-manifolds, which acts as collective non-Clifford logical CCZ gates on any triplet of logical qubits with their logical-X membranes having a Z2 triple intersection at a single point. The triple intersection number is a topological invariant, which also arises in the path integral of the emergent higher symmetry operator in a topological quantum field theory (TQFT): the (Z2) 3 gauge theory. Moreover, the transversal S gate of the color code

corresponds to a higher-form symmetry supported on a codimension-1 submanifold, giving rise to exponentially many addressable and parallelizable logical CZ gates. Both symmetries are related to gauged SPT defects in the (Z2) 3 gauge theory. We have then developed a generic formalism to compute the triple intersection invariants for general 3-

manifolds. We further develop three types of LDPC codes supporting such logical gates with constant or almost-constant encoding rate and logarithmic distance. Finally, I'll point out a connection between the gauged SPT defects in the 6D color code and a recently discovered non-Abelian self-correcting quantum memory in 5D.

Reference: arXiv:2310.16982, arXiv:2208.07367, arXiv:2405.11719.

Emergent symmetries and their application to logical gates in quantum LDPC codes

Guanyu Zhu

IBM Quantum, T. J. Watson Research Center

Related works

1. arXiv:2310.16982 (Logical gates on homological LDPC codes)

Sheharyar Skinder (Rutgers)

Elia Portnoy (MIT)

Andrew Cross (IBM)

Ben Brown (IBM)

2. arXiv:2208.07367 (Gauged SPT defects and higher symmetries)

Maissam Barkeshli (UMD)

Yu-An Chen (Tsinghua)

Sheng-Jie Huang (Max Planck)

Ryohei Kobayashi (UMD)

Nathanan Tantavasidakarn (Caltech)

3. arXiv:2405.11719 (Non-Abelian self-correcting memories in 5d and higher dimensions)

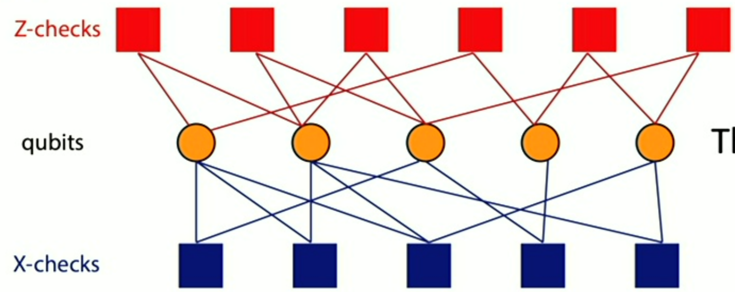
Po-Shen Hsin (King's College London)

Ryohei Kobayashi (UMD)

Introduction and motivation

- Quantum low-density parity-check (qLDPC) codes: a family of stabilizer codes such that the number of qubits participating in each check operator and the number of stabilizer checks that each qubit participates in are both bounded by a constant.

Example: A CSS LDPC codes



The Tanner graph is a sparse graph

- Classical LDPC codes (Gallager 1960's) are widely applied to communication such as 5G network.
- qLDPC codes are promising candidates to achieve low-overhead fault-tolerant quantum computing.

e.g., constant encoding rate: $k / n = \text{const}$ (k: logical qubit #, n: physical qubit #)

overcome the square-root distance: $d = O(n^{\alpha})$ $(\alpha>1/2)$

In contrast, for k copies of surface (toric) codes: $n \sim kd^2 \longrightarrow k / n \sim 1/d^2$

- Typically need long-range connection for implementation.

- Two major types:

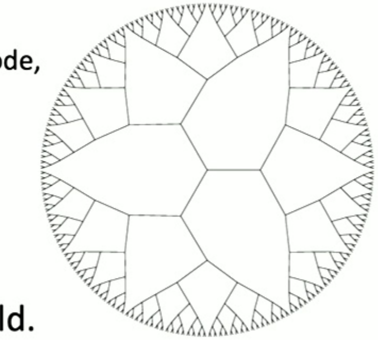
Can be mapped to each other sometimes (Freedman-Hastings' 11d manifold from codes)

1. Defined on a general chain complex, typically based on expander graphs.

   Example: Hyper-graph product code, Pantaleev-Kalachev code (good qLDPC), quantum Tanner code, balanced product code, fibre-bundle code, bivariate bicycle code (IBM) etc.

$$C_2 \xrightarrow{\partial = H_Z^T} C_1 \xrightarrow{\partial = H_X} C_0$$

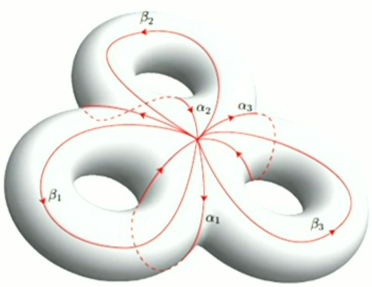
Z-check qubit X-check



2. Homological qLDPC code (this talk): defined on the tessellation of a manifold.

   Example: 2d hyperbolic code, 4d hyperbolic code (Guth and Lubotsky), Freedman-Meyer-Luo code

- Main challenge in fault-tolerant logical gates:



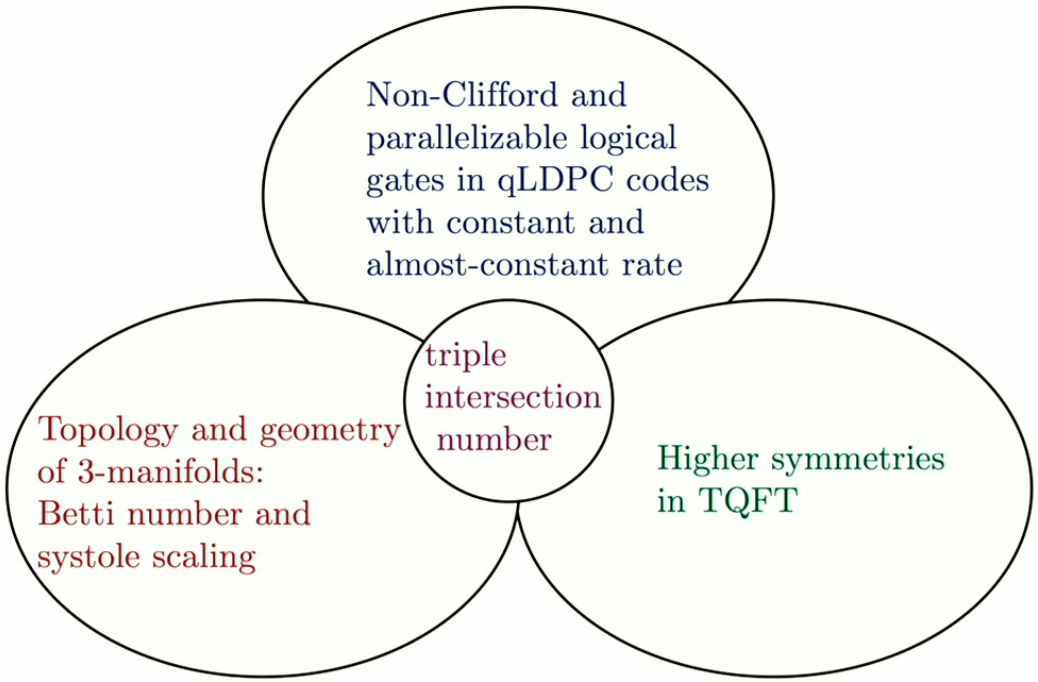
1. Individually addressable and parallelizable logical gates.

   Constant/high rate qLDPC codes encode all the logical qubits into a single code block.
   Usual transversal gates act on the entire system and hence cannot address individual logical qubits

2. Logical non-Clifford gates.

   Most of the existing qLDPC codes are extension of 2D surface codes and are hence "2D-like" (2D chain complex).
   They are only capable to perform logical Clifford gates (in analogy to the Bravyi-Konig bound).

Some interconnected concepts in this work



Outline

- Introduction to emergent symmetries, symmetry defects and logical gates
- General construction of color codes defined on 3-manifolds (LDPC color codes) and their non-Clifford and parallelizable logical gates.
- Connection to higher-form symmetries in topological quantum-field theory (TQFT).
- Construction of 3-manifold geometries and the corresponding qLDPC codes with constant or almost-constant encoding rate.
- Connection between the emergent symmetry defects and a 5d non-Abelian self-correcting quantum memory

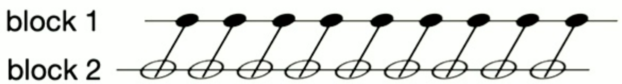
Transversal logical gates and emergent symmetries

- Consider a transversal gate $U = \otimes_j V_j$ (or more generally a constant-depth local circuit), it is a logical gate iff

$$U : \mathcal{H}_C \to \mathcal{H}_C$$

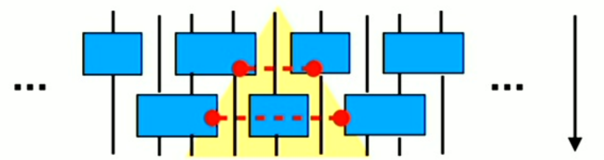
$\mathcal{H}_C$: code space

example: for any CSS code (such as surface code)



$\overline{\mathrm{CNOT}} = \prod_j \mathrm{CNOT}_j$ transversal CNOT is a logical CNOT

In general, U does not have to be the same type as V



Error propagation is bounded by a light cone

$\mathrm{supp}(E) \ll d$

code distance

- For homological LDPC codes, *U* can be considered as an emergent symmetry of the ground state subspace (code space) of a topological order described by a topological quantum field theory (TQFT).

- Furthermore, *U* is a 0-form global symmetry if it acts on the entire system of *d* spatial dimension.

  U is a higher-form (*k*-form) symmetry if it acts on a codimension-*k* submanifold $\mathcal{M}_{d-k}$

D. Gaiotto, A. Kapustin, N. Seiberg, B. Willett, JHEP 2015 (2), 1-62

B. Yoshida, Phys. Rev. B 91, 245131 (2015), Phys. Rev. B 93, 155131 (2016) Annals of Physics 377, 387 (2017)

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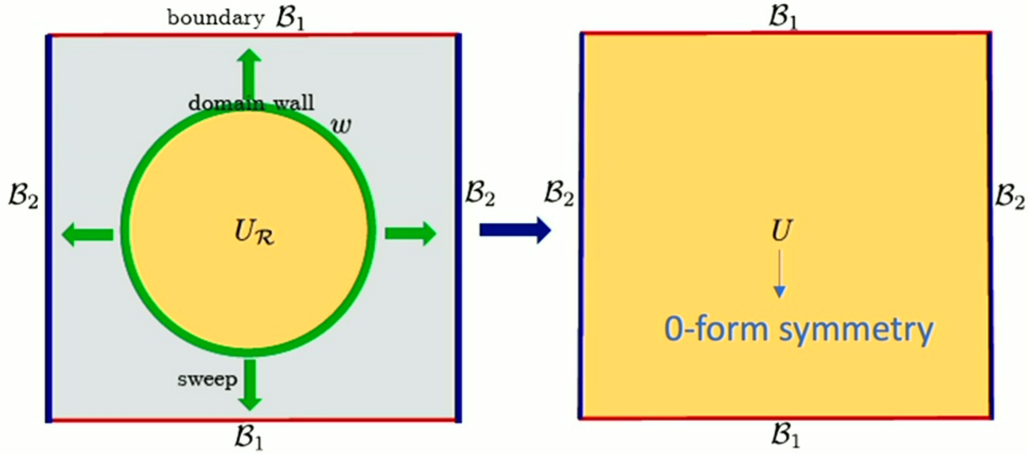
M. Barkeshli, Y.A. Chen, S.J. Huang, R. Kobayashi, N. Tantavasidakarn, GZ, arXiv:2208.07367 (2022)

M. Barkeshli, Y.A. Chen, P.S. Hsin, R. Kobayashi, arXiv:2211.11764(2022)

R. Kobayashi, GZ, arXiv:2310.06917 (2023)

Connection to defect sweeping

- The action of transversal logical gate (emergent symmetry U) is equivalent to sweeping the corresponding invertible defect (domain wall) ω :



B. Yoshida, Phys. Rev. B 91, 245131 (2015), Phys. Rev. B 93, 155131 (2016)
P. Webster and S. D. Bartlett, Phys. Rev. A 97, 012330 (2018)
GZ, M. Hafezi, and M. Barkeshli, Phys. Rev. Research 2, 013285 (2020)
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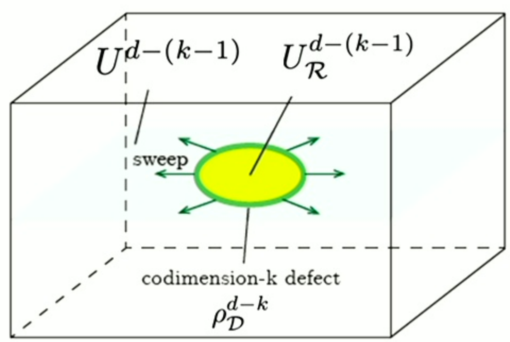
- Generalization to codimension-k defect and (k-1)-form symmetry:



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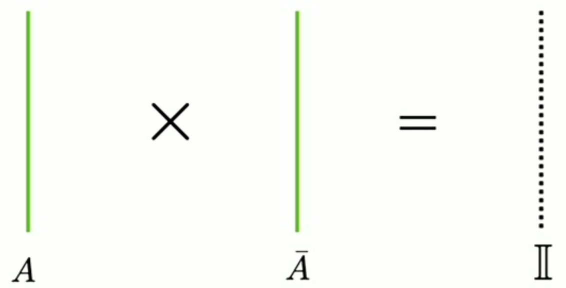
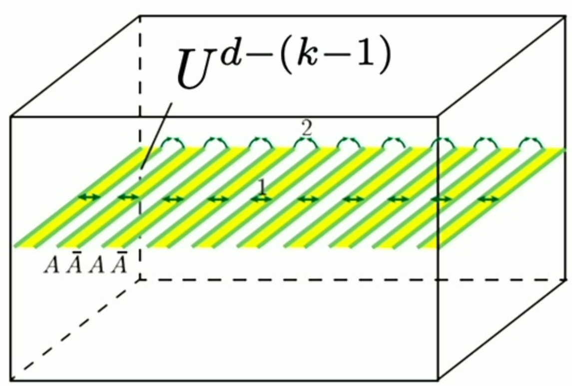
Invertible defects

- A codimension-k defect in the topological equivalence class A is invertible if there exists another codimension-k defect in an equivalence class $\bar{A}$, such that if the two codimension-k defects are near each other, they are topologically equivalent to the trivial codimension-k defect.



- The sweeping of the codimension-k invertible defect can always be implemented as a constant-depth local circuit.



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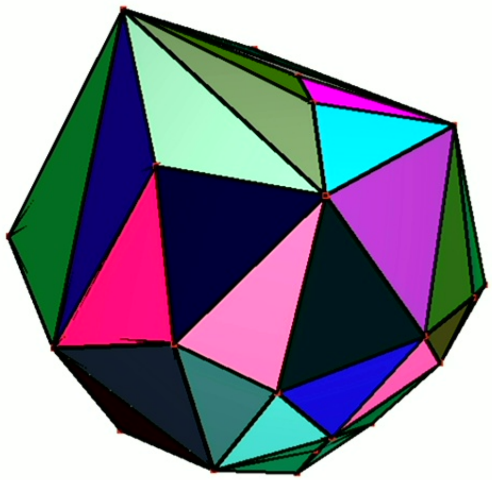
Part II. General construction of color codes defined on 3-manifolds (LDPC color codes) and their non-Clifford and parallelizable logical gates

Color codes on 3-manifolds

Start with a triangulated 3-manifold $\mathcal{M}^3$



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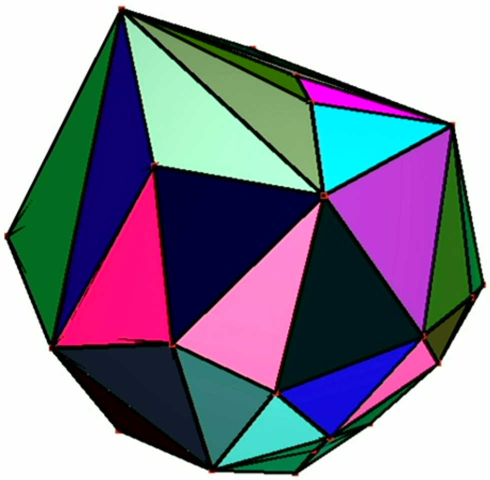
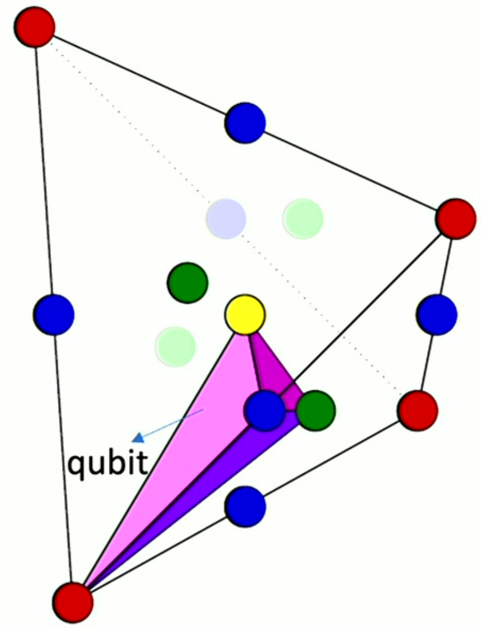


fattening

(Bombin and Delgardo)

Dual color-code lattice $\mathcal{L}_c^*$ (4-colorable)



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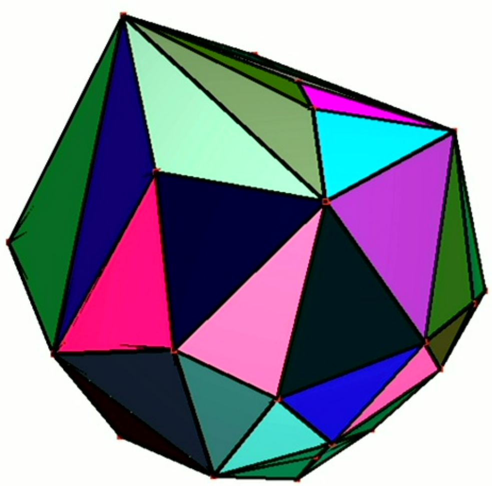

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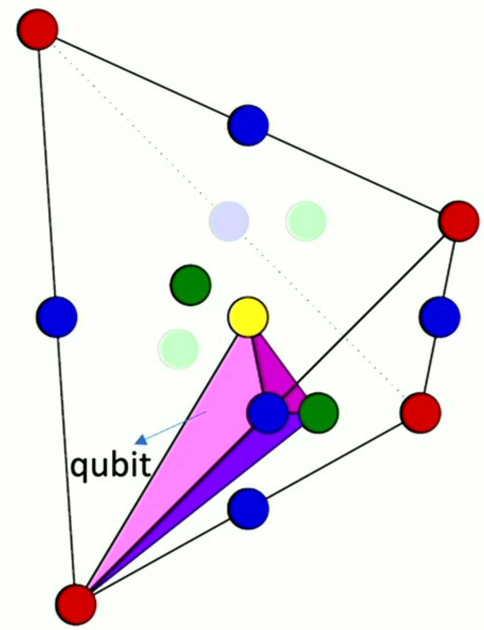


vertex v^*, edge e^*, face f^*, tetrahedron Δ^*

↓ 0-cell, 1-cell, 2-cell, 3-cell

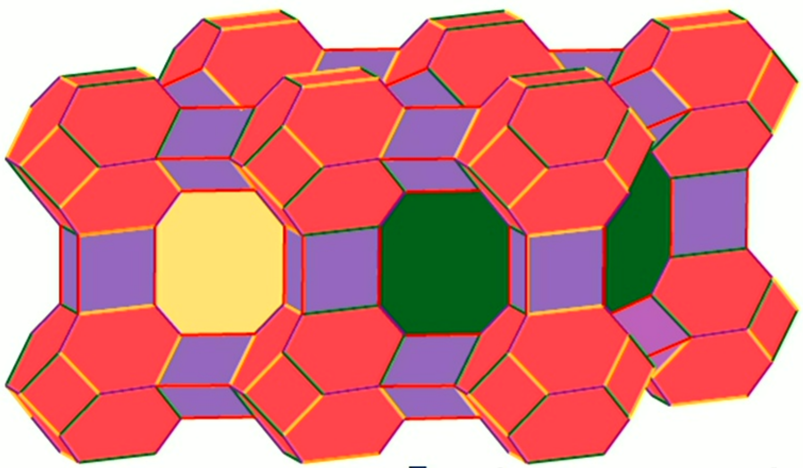
Color-code stabilizers on the dual lattice $\mathcal{L}_c^*$:

$$S_{v^*}^X = \prod_{\Delta^* \supset v^*} X_{\Delta^*} \qquad S_{e^*}^Z = \prod_{\Delta^* \supset e^*} Z_{\Delta^*}$$

vertex (0-cell) edge (1-cell)

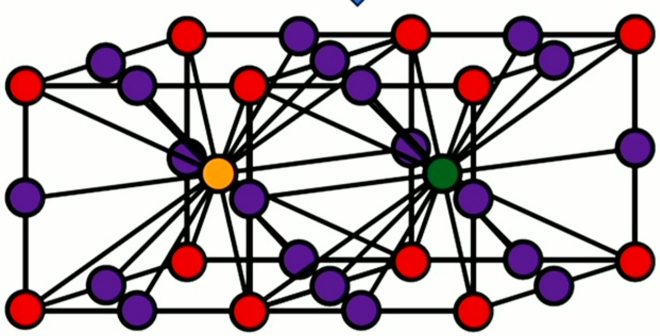
Color codes and unfolding

- Original color-code lattice $\mathcal{L}_c$: 4-colorable and 4-valent



dual lattice: i-cell → (3-i)-cell

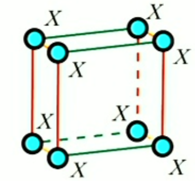
$\mathcal{L}_c^*$



- Color-code stabilizers on $\mathcal{L}_c$:

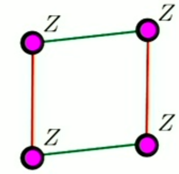
$$S_c^X = \prod_{j \in c} X_j$$

volume (3-cell)



$$S_f^Z = \prod_{j \in f} Z_j$$

face (2-cell)



- The 3D color code is constant-depth equivalent to three copies of 3D toric (surface) codes:

$$CC(\mathcal{L}_c) \cong \otimes_{i=1}^{3} TC(\mathcal{L}_i)$$

Kubica, Yoshida, Pastawski (2015)

- Constant-depth disentangling circuit V:

$$V[CC(\mathcal{L}_c) \otimes \mathcal{S}]V^\dagger = \bigotimes_{i=1}^{3} TC(\mathcal{L}_i)$$

Code space

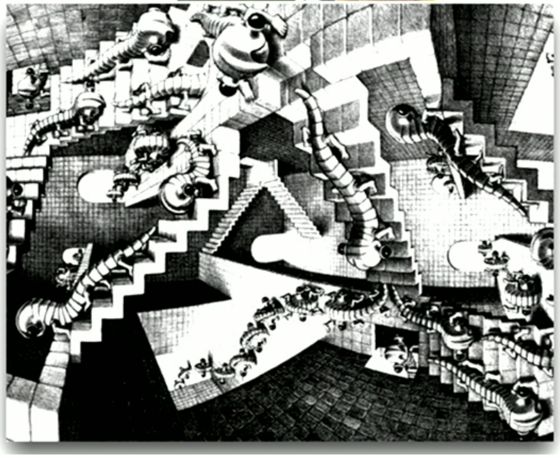
- Code space of the 3D toric code: $\mathcal{H}_{TC(\mathcal{M}^3)} = \mathbb{C}^{|H_1(\mathcal{M}^3;\mathbb{Z}_2)|}$

  $H_1(\mathcal{M}^3;\mathbb{Z}_2)$ represents the 1st Z_2-homology group of M^3, corresponding to the non-contractible 1-cycles where the logical-Z strings (worldline of e-particles) are supported

  $H_2(\mathcal{M}^3;\mathbb{Z}_2)$ represents the 2nd Z_2-homology group, corresponding to the non-contractible 2-cycles where the logical-X membranes (world-sheet of m-strings) are supported

- Poincare duality: a manifestation of the e-m (charge-flux) duality

$$H_1(\mathcal{M}^3;\mathbb{Z}_2) \cong H^2(\mathcal{M}^3;\mathbb{Z}_2) \cong H_2(\mathcal{M}^3;\mathbb{Z}_2)$$



- ith Betti number: number of "i-dimensional holes"

$$b_i(\mathcal{M}^3;\mathbb{Z}_2) = Rank(H_i(\mathcal{M}^3;\mathbb{Z}_2))$$

Code space

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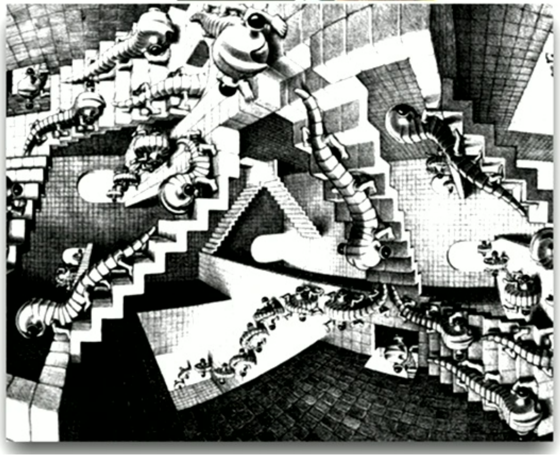
- ith Betti number: number of “i-dimensional holes”

$$b_i(\mathcal{M}^3;\mathbb{Z}_2) = Rank(H_i(\mathcal{M}^3;\mathbb{Z}_2))$$

number of logical qubit: $k = b_1(\mathcal{M}^3;\mathbb{Z}_2) = b_2(\mathcal{M}^3;\mathbb{Z}_2)$

(topological/LDPC code and topological order Eq. (1)! Kitaev and Wen)



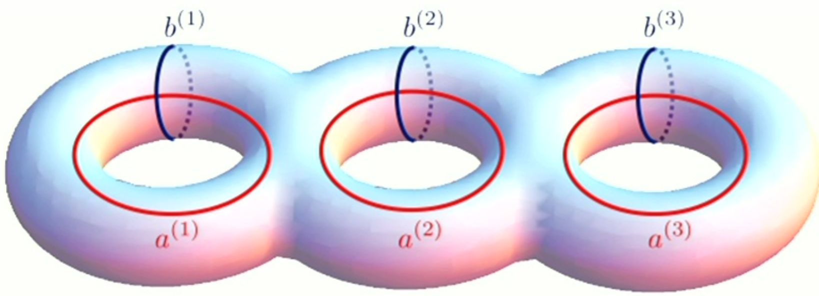
- Code space of the 3D color code:

$$\mathcal{H}_{CC(\mathcal{M}^3)} = \mathcal{H}^{\otimes 3}_{TC(\mathcal{M}^3)} \qquad k' = 3b_1(\mathcal{M}^3;\mathbb{Z}_2)$$

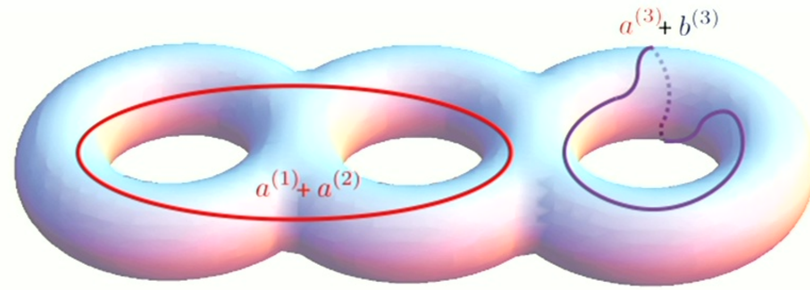
Homology basis

- **Warm-up on 2-manifolds:**

Choose a 1st homology basis $B_1 = \{\alpha_1\}$



Arbitrary 1-cycle can be decomposed to the sum of basis cycles



- **Homological basis on 3-manifolds:**

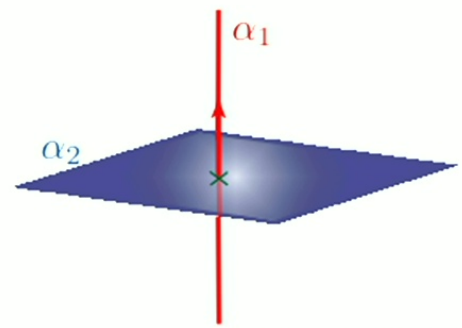
Choose a 2nd homology basis $B_2 = \{\alpha_2\}$ with $[\alpha_2] \in H_2(\mathcal{M}^3; \mathbb{Z}_2)$

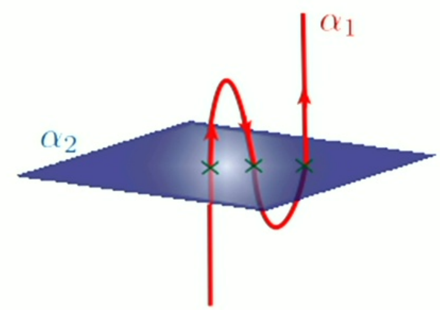
with its dual 1st homology basis $B_1 = \{\alpha_1\}$ with $[\alpha_1] \in H_1(\mathcal{M}^3; \mathbb{Z}_2)$

such that $\begin{cases} |\alpha_1 \cap \alpha_2| = 1 \\ |\alpha_1 \cap \alpha_2'| = 0 \end{cases}$ for any $\alpha_2' \in B_2$ satisfying $\alpha_2' \neq \alpha_2$

$|\cdot \cap \cdot| \in \mathbb{Z}_2 \equiv \{0, 1\}$ represents the $\mathbb{Z}_2$ intersection number

↓ generalize





topological invariant

Homology basis

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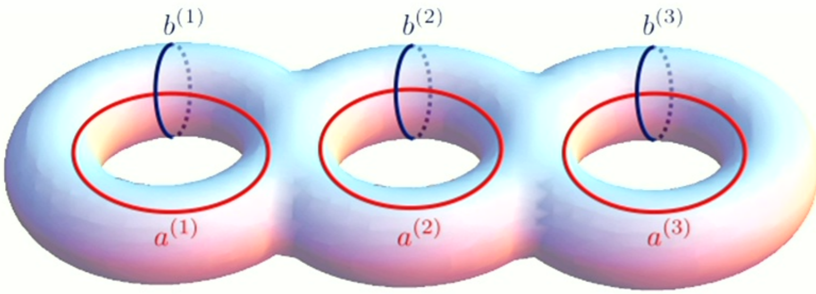

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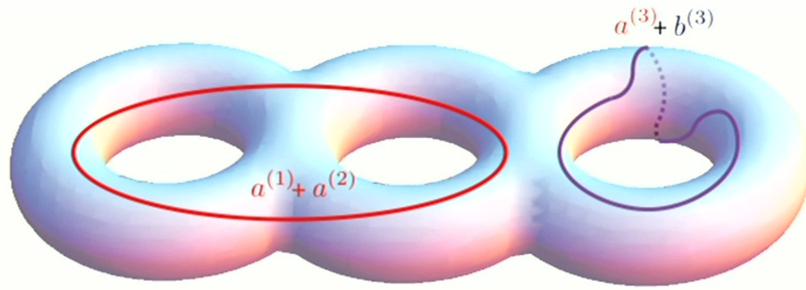

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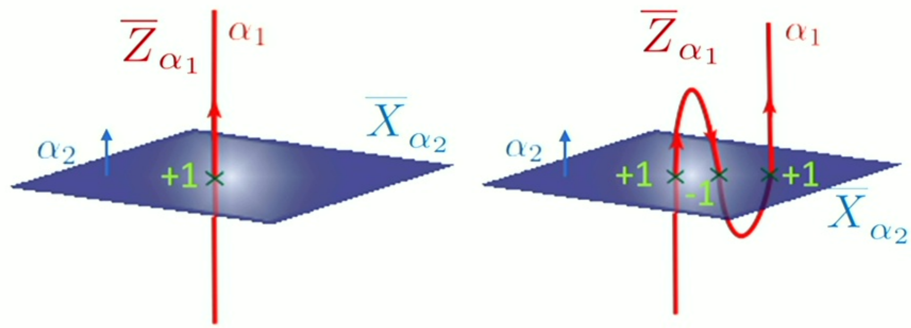
↓ generalize

algebraic intersection number



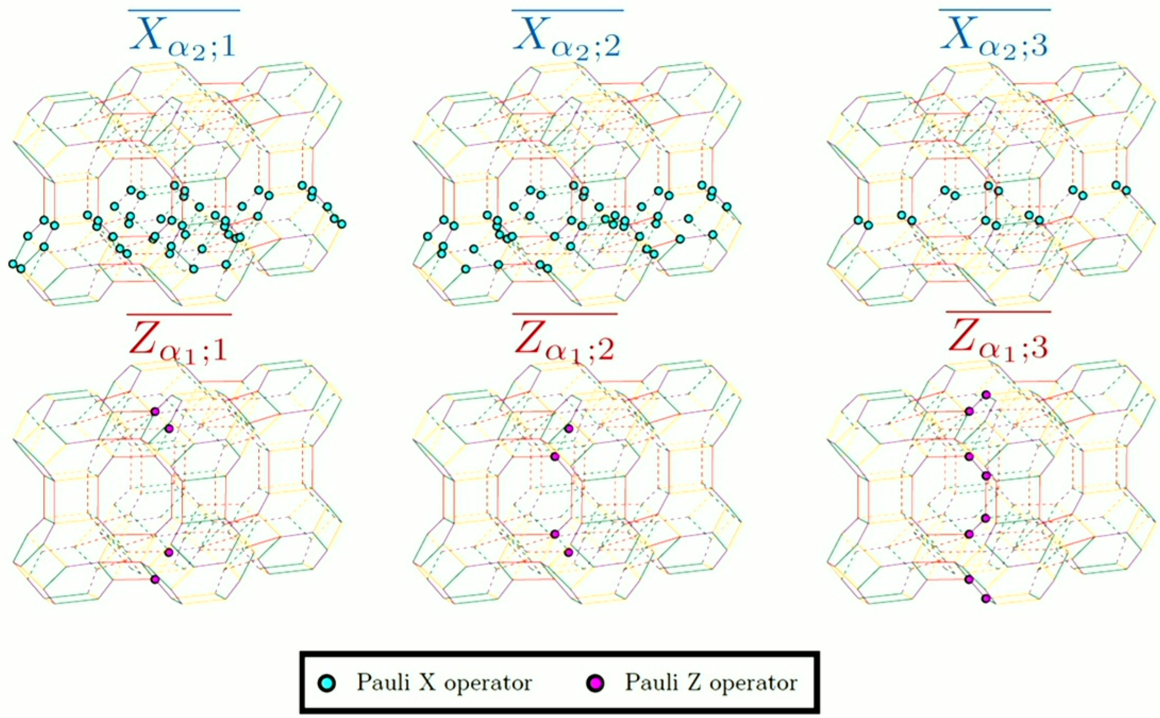
topological invariant

Logical operators and qubit labels

- Color-code logical operators: $\overline{Z_{\alpha_1;1}},\ \overline{Z_{\alpha_1;2}}$ and $\overline{Z_{\alpha_1;3}}$ $\qquad$ $\overline{X_{\alpha_2;1}},\ \overline{X_{\alpha_2;2}}$ and $\overline{X_{\alpha_2;3}}$

  $[\alpha_1] \in H_1(\mathcal{M}^3;\mathbb{Z}_2)$ ← Poincare dual → $[\alpha_2] \in H_2(\mathcal{M}^3;\mathbb{Z}_2)$

- Toric-code logical operators: $V\overline{Z_{\alpha_1;i}}V^\dagger = \overline{Z}^{(i)}_{\alpha_1},\qquad V\overline{X_{\alpha_2;i}}V^\dagger = \overline{X}^{(i)}_{\alpha_2}$



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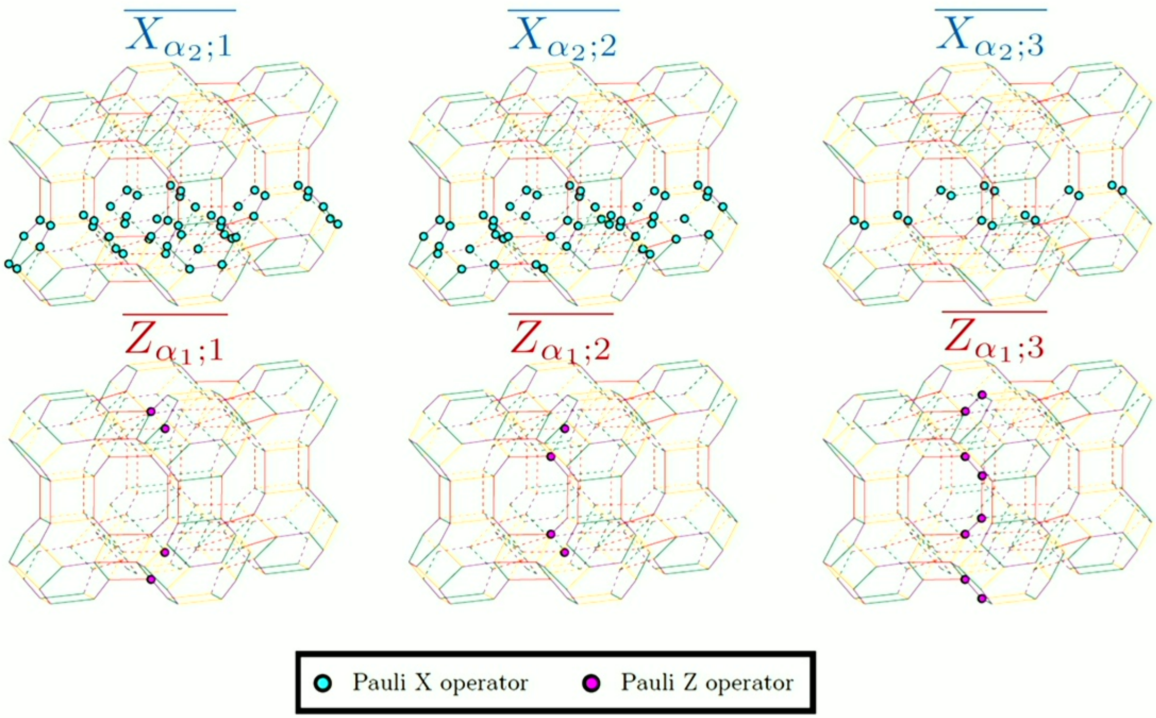
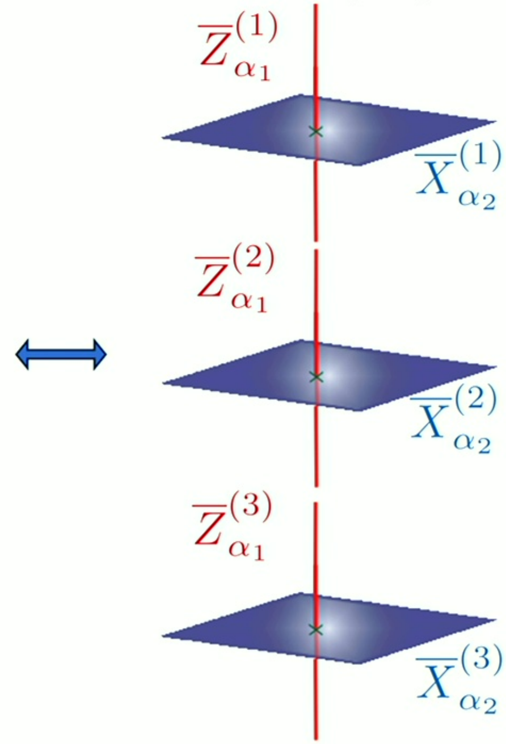
- Intersection and anti-commutation: $|\alpha_1 \cap \alpha_2| = 1$ $\quad$ $\overline{X_{\alpha_2;i}}\,\overline{Z_{\alpha_1;i}} = -\overline{Z_{\alpha_1;i}}\,\overline{X_{\alpha_2;i}}$ $\quad$ $\overline{X}^{(i)}_{\alpha_2}\overline{Z}^{(i)}_{\alpha_1} = -\overline{Z}^{(i)}_{\alpha_1}\overline{X}^{(i)}_{\alpha_2}$





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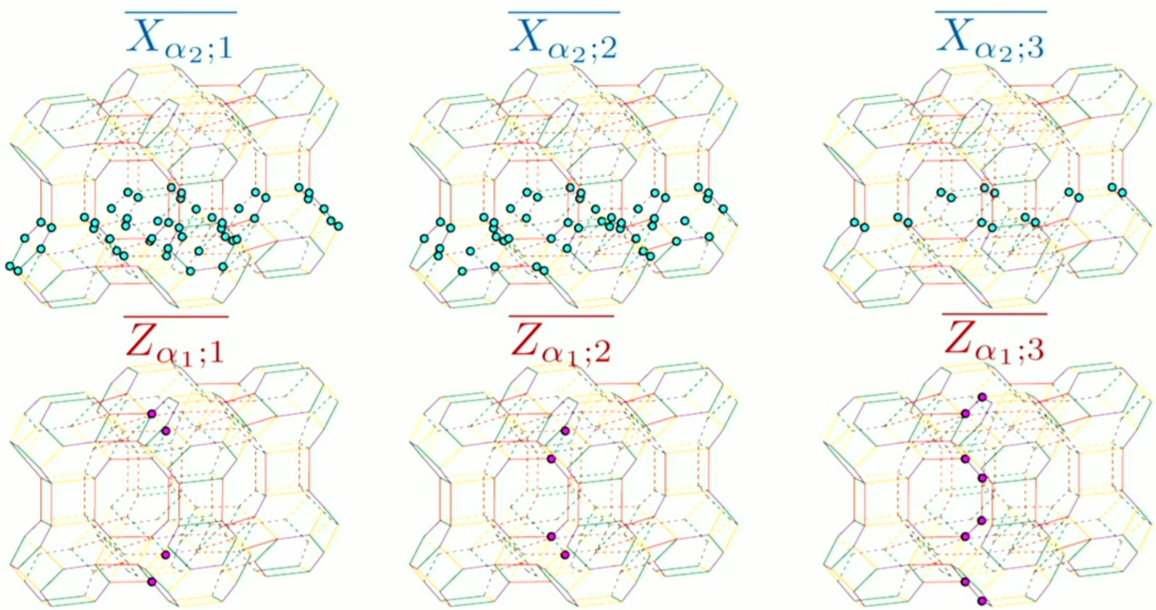
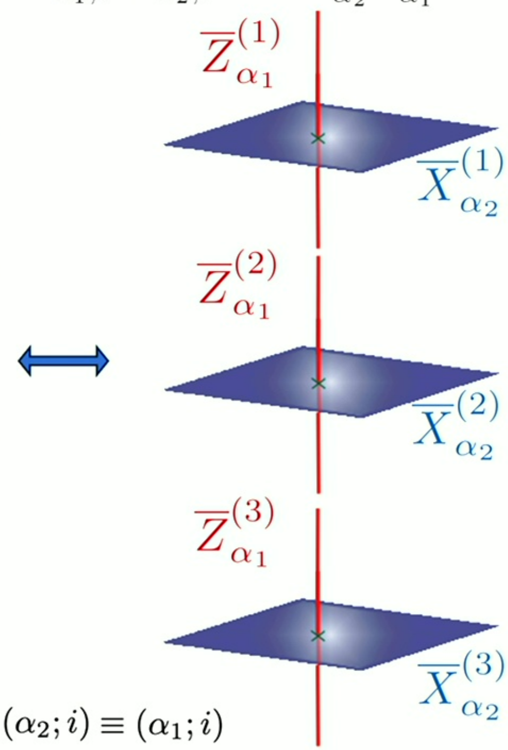
  $[\alpha_1]\in H_1(\mathcal{M}^3;\mathbb{Z}_2)$ ← Poincare dual → $[\alpha_2]\in H_2(\mathcal{M}^3;\mathbb{Z}_2)$

- Toric-code logical operators: $V\overline{Z_{\alpha_1;i}}V^\dagger = \overline{Z}^{(i)}_{\alpha_1}$, $\quad V\overline{X_{\alpha_2;i}}V^\dagger = \overline{X}^{(i)}_{\alpha_2}$

- Intersection and anti-commutation: $|\alpha_1 \cap \alpha_2| = 1$ $\quad \overline{X_{\alpha_2;i}}\,\overline{Z_{\alpha_1;i}} = -\overline{Z_{\alpha_1;i}}\,\overline{X_{\alpha_2;i}}$ $\quad \overline{X}^{(i)}_{\alpha_2}\overline{Z}^{(i)}_{\alpha_1} = -\overline{Z}^{(i)}_{\alpha_1}\overline{X}^{(i)}_{\alpha_2}$





○ Pauli X operator ● Pauli Z operator

- Qubit label: $(\alpha_2; i) \equiv (\alpha_1; i)$

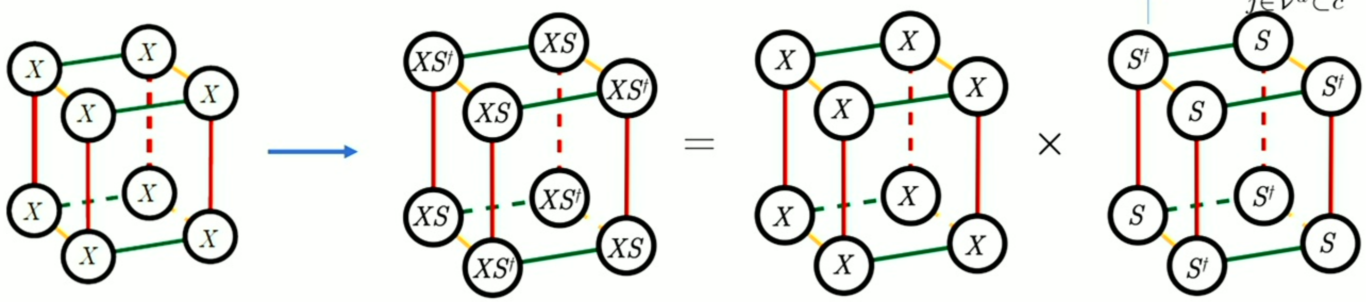
Transversal T gate

- Color code has a Bipartite lattice: $\mathcal{V} = \mathcal{V}^a \cup \mathcal{V}^b$
- Expression: $\widetilde{T} = \bigotimes_{j \in \mathcal{V}^a} T(j) \bigotimes_{j \in \mathcal{V}^b} T^\dagger(j)$
- It is a 0-form global onsite symmetry acting on the entire system.
- It is a logical gate since it maps the code space back to itself $\widetilde{T} : \mathcal{H}_{CC(\mathcal{M}^3)} \rightarrow \mathcal{H}_{CC(\mathcal{M}^3)}$

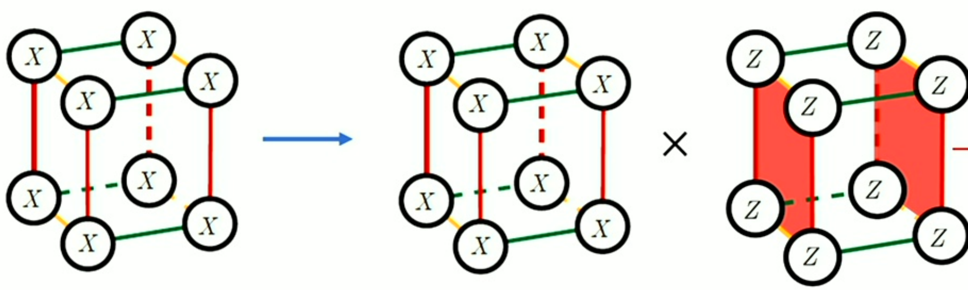
  Criteria: It preserve stabilizers up to logical identity.

Logical identity $\bigotimes_{j \in \mathcal{V}^a \subset c} S(j) \bigotimes_{j \in \mathcal{V}^b \subset c} S^\dagger(j)$

$\widetilde{T}$:



$\bigotimes_{j \in \mathcal{V}^a \subset c} S(j) \bigotimes_{j \in \mathcal{V}^b \subset c} S^\dagger(j)$:



$\rightarrow \prod_f S_f^Z$ (product of Z stabilizer)

Logical non-Clifford gate and triple intersection

- Consider a triplet of noncontractible 2-cycles belonging to the homology basis: $\alpha_2,\ \beta_2,\ \gamma_2 \in B_2$

$$\widetilde{T}\overline{X_{\alpha_2;1}}\widetilde{T}^\dagger = \overline{X_{\alpha_2;1}}\widetilde{S}_{\alpha_2;2,3}$$

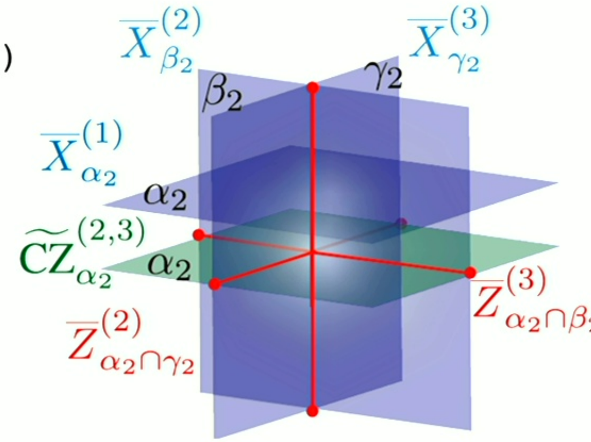
(Note that $\widetilde{S}_{\alpha_2;2,3}$ has the same support as $\overline{X_{\alpha_2;1}}$)

$$V\widetilde{T}\overline{X_{\alpha_2;1}}\widetilde{T}^\dagger V^\dagger = \overline{X}^{(1)}_{\alpha_2}\widetilde{\mathrm{CZ}}^{(2,3)}_{\alpha_2}$$

$$\left(V\widetilde{S}_{\alpha_2;2,3}V^\dagger = \widetilde{\mathrm{CZ}}^{(2,3)}_{\alpha_2}\right)$$

Kubica, Yoshida, Pastawski (2015)

where $\widetilde{\mathrm{CZ}}^{(2,3)}_{\alpha_2} : \overline{X}^{(2)}_{\beta_2} \to \overline{X}^{(2)}_{\beta_2}\overline{Z}^{(3)}_{\alpha_2\cap\beta_2}$



- Now if $\widetilde{\mathrm{CZ}}^{(2,3)}_{\alpha_2}$ is the logical CZ gate acting on logical qubits associated with $\overline{X}^{(2)}_{\beta_2}$ and $\overline{X}^{(3)}_{\gamma_2}$ one must satisfy $\overline{X}^{(3)}_{\gamma_2}\overline{Z}^{(3)}_{\alpha_2\cap\beta_2} = -\overline{Z}^{(3)}_{\alpha_2\cap\beta_2}\overline{X}^{(3)}_{\gamma_2} \longrightarrow |\alpha_2\cap\beta_2\cap\gamma_2| = 1$

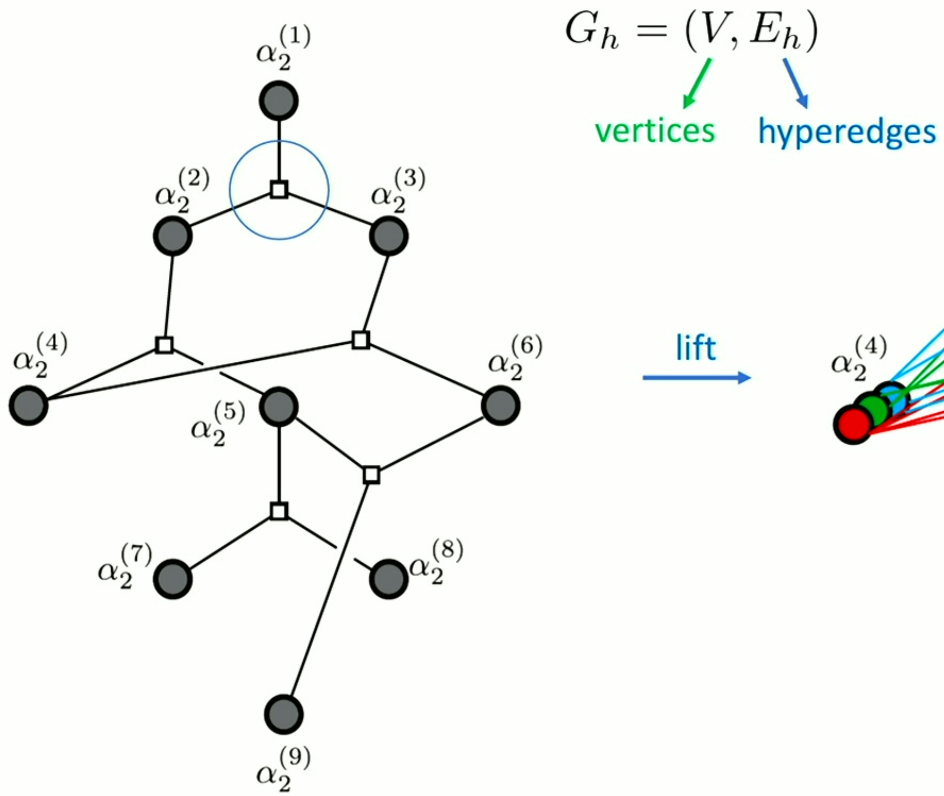
$\mathbb{Z}_2$ triple intersection number

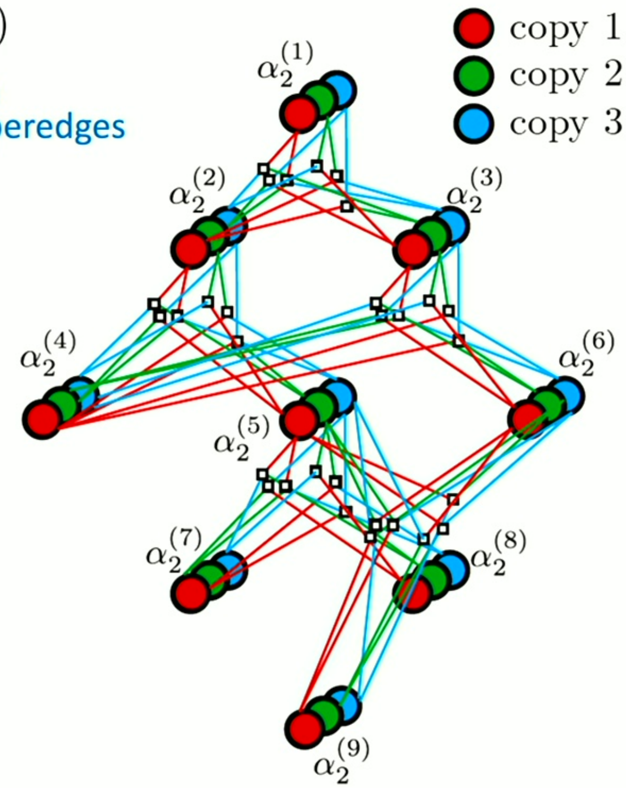
- Repeat the analysis for $\overline{X_{\beta_2;2}}$ and $\overline{X_{\gamma_2;3}}$, we can derive

$$\widetilde{T} = \prod_{\alpha_2,\beta_2,\gamma_2\in B_2} [\overline{\mathrm{CCZ}}((\alpha_2;1),(\beta_2;2),(\gamma_2;3))]^{|\alpha_2\cap\beta_2\cap\gamma_2|}$$

Interaction hypergraph

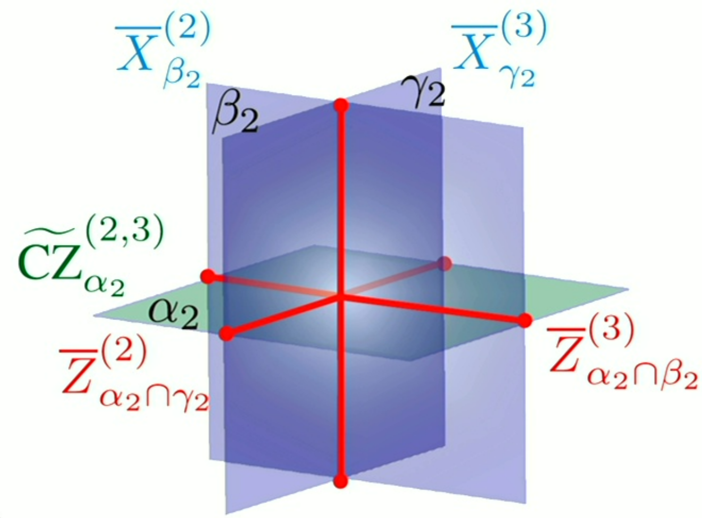
- Base interaction hypergraph (intersection hypergraph)
- Interaction hypergraph for color codes on 3-manifolds (3 copies of toric codes)





Parallelizable logical Clifford gates

$$\widetilde{\mathrm{CZ}}_{\alpha_2}^{(i,j)} \sim \widetilde{S}_{\alpha_2;i,j} = \prod_{\beta_2,\gamma_2 \in B_2} [\overline{\mathrm{CZ}}((\beta_2;i),(\gamma_2;j))]^{|\alpha_2\cap\beta_2\cap\gamma_2|}$$



- It is a 1-form symmetry acting on a codimension-1 (2D) submanifold.
- In general, k-form (higher-form) symmetry acts on a codimension-k submanifold.
- This leads to addressable and parallelizable logical CZ gates.

  Different $\widetilde{\mathrm{CZ}}_{\alpha_2}^{(i,j)}$ commute with each other and can be applied in parallel.
- Number of addressable logical gates scales as $N_g = |H_2(\mathcal{M}^3;\mathbb{Z}_2)| = 2^{b_2(\mathcal{M}^3;\mathbb{Z}_2)} = 2^k = O(2^n)$ (if $k = O(n)$).

Extension of Eq. (1) of topological codes/order: $k = b_1(\mathcal{M}^3;\mathbb{Z}_2) = b_2(\mathcal{M}^3;\mathbb{Z}_2)$

Exponential scaling in contrast to the linear scaling in fold-transversal gates (0-form symmetry)!

Logical gates as topological invariants in TQFT

- **The 3D color code is equivalent to a (3+1)D topological quantum field theory (TQFT): $Z_2 \times Z_2 \times Z_2$ gauge theory**

M. Barkeshli, Y.-A. Chen, S.-J. Huang, R. Kobayashi, N. Tantivasadakarn, and **GZ**, Sci-Post Phys. 14, 065 (2023).

- **TQFT action:** $S_{\mathbb{Z}_2^3} = \pi \int_{\mathcal{M}^4} a^{e_1} \cup \delta b^{m_1} + a^{e_2} \cup \delta b^{m_2} + a^{e_3} \cup \delta b^{m_3}.$ $\qquad \mathcal{M}^4 = \mathcal{M}^3 \times S^1_t$

Electric Z_2 gauge field: $a^{e_1}, a^{e_2}, a^{e_3} \in C^1(\mathcal{M}^4; \mathbb{Z}_2)$

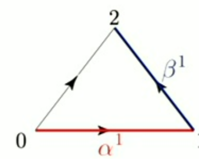
Magnetic Z_2 gauge field: $b^{m_1}, b^{m_2}, b^{m_3} \in C^2(\mathcal{M}^4; \mathbb{Z}_2)$

$$a^{e_i} = \frac{1}{2}(1 - Z^{(i)}) \qquad b^{m_i} = \frac{1}{2}(1 - X^{(i)})$$

Cup product $\cup$ between a *p*-cochain α^p and *q*-cochain β^q on a triangulation:

$$(\alpha^p \cup \beta^q)(0, \cdots, p+q) = \alpha^p(0, 1, \cdots, p)\beta^q(p, p+1, \cdots, p+q)$$

Example:

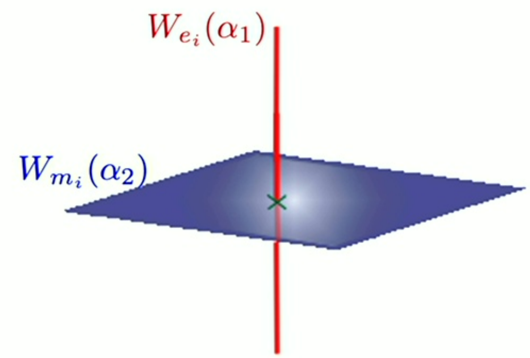


$$(\alpha^1 \cup \beta^1)(012) \equiv \alpha^1(01)\beta^1(12)$$

Geometric meaning: $\int_{\mathcal{M}} \alpha^p \cup \beta^q = |\alpha_p \cap \beta_q|$

- **Electric worldline:** $W_{e_i}(\alpha_1) = \exp\left(\pi \mathrm{i} \int_{\alpha_1} a^{e_i}\right) \equiv \overline{Z}^{(i)}_{\alpha_1}, \quad (i = 1, 2, 3)$

- **Magnetic world-sheet:** $W_{m_i}(\alpha_2) = \exp\left(\pi \mathrm{i} \int_{\alpha_2} b^{m_i}\right) \equiv \overline{X}^{(i)}_{\alpha_2}$



Symmetry operators and defect automorphism

- 0-form symmetry generated by world-volume operators of CCZ defects (gauged 2+1D $Z_2 \times Z_2 \times Z_2$ SPT of type-III cocycle) $s^{(3)}_{1,2,3}$

$$\mathcal{D}_{s^{(3)}_{1,2,3}}(\mathcal{M}^3) = \exp\left(\pi \mathrm{i} \int_{\mathcal{M}^3} a^{e_1} \cup a^{e_2} \cup a^{e_3}\right) = \prod_{\alpha_2,\beta_2,\gamma_2 \in B_2} [\overline{\mathrm{CCZ}}((\alpha_2;1),(\beta_2;2),(\gamma_2;3))]^{|\alpha_2 \cap \beta_2 \cap \gamma_2|}$$

(Note that $\mathrm{CCZ}(1,2,3) = (-1)^{a^{e_1}a^{e_2}a^{e_3}}$)

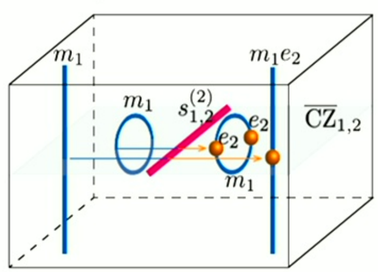
- 1-form symmetry generated by world-sheet operators for CZ defects (gauged 1+1D $Z_2 \times Z_2$ SPT of type-II cocycle) $s^{(2)}_{1,2}$, $s^{(2)}_{2,3}$, $s^{(2)}_{3,1}$

$$\mathcal{D}_{s^{(2)}_{i,j}}(\alpha_2) = \exp\left(\pi \mathrm{i} \int_{\alpha_2} a^{e_i} \cup a^{e_j}\right) = \exp\left(\pi \mathrm{i} \int_{\mathcal{M}^3} a^{e_i} \cup a^{e_j} \cup \alpha^1\right) = \prod_{\beta_2,\gamma_2 \in B_2} [\overline{\mathrm{CZ}}((\beta_2;i),(\gamma_2;j))]^{|\alpha_2 \cap \beta_2 \cap \gamma_2|}, \quad (i \neq j)$$

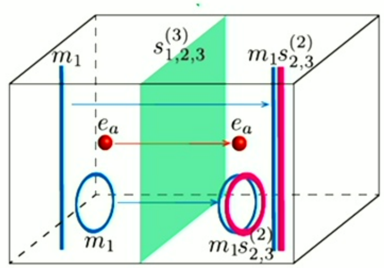
(Note that $\mathrm{CZ}(1,2) = (-1)^{a^{e_1}a^{e_2}}$)

- Defect automorphism:

$s^{(2)}_{i,j} : m_i \to m_i e_j$



$s^{(3)}_{i,j,k} : m_i \to m_i s^{(2)}_{j,k}$



$$\mathcal{D}_{s^{(2)}_{i,j}}(\alpha_2) : W_{m_i}(\beta_2) \to W_{m_i}(\beta_2) W_{e_j}(\alpha_2 \cap \beta_2)$$

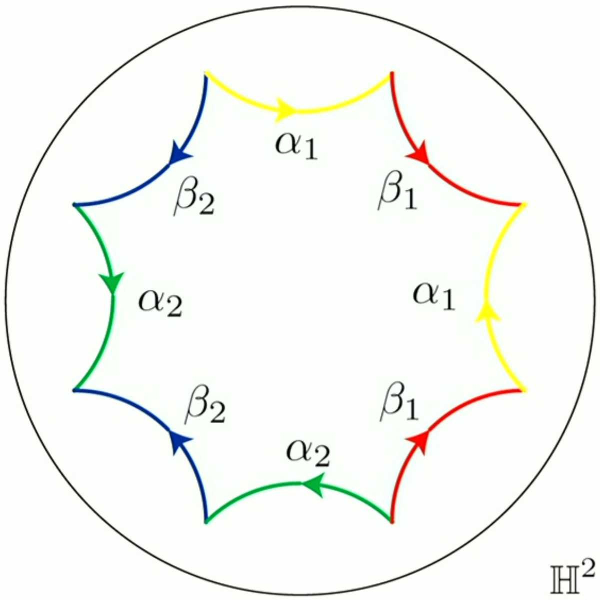
$$\overline{X}^{(i)}_{\beta_2} \to \overline{X}^{(i)}_{\beta_2} \overline{Z}^{(j)}_{\alpha_2 \cap \beta_2}$$

$$\mathcal{D}_{s^{(3)}_{i,j,k}} : W_{m_i}(\alpha_2) \to \mathcal{D}_{s^{(2)}_{j,k}}(\alpha_2) W_{m_i}(\alpha_2)$$

Part III. Construction of 3-manifold geometries and the corresponding codes with constant or almost-constant encoding rate

2D hyperbolic codes: compactify a hyperbolic surface

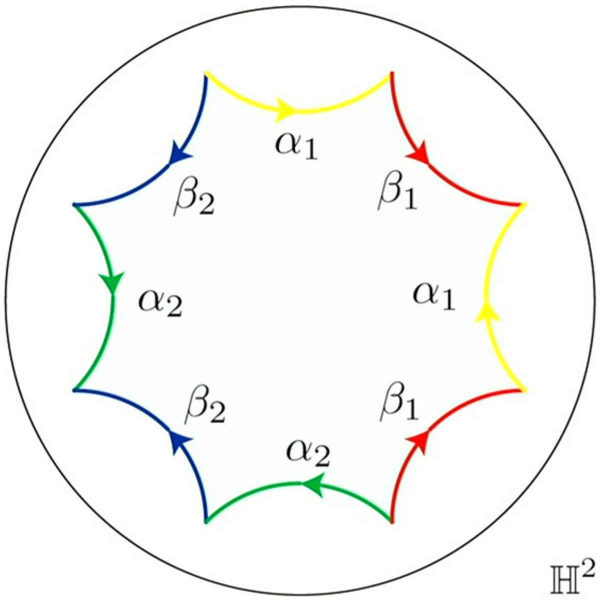
Canonical regular 4*g*-gon



2D hyperbolic codes: compactify a hyperbolic surface

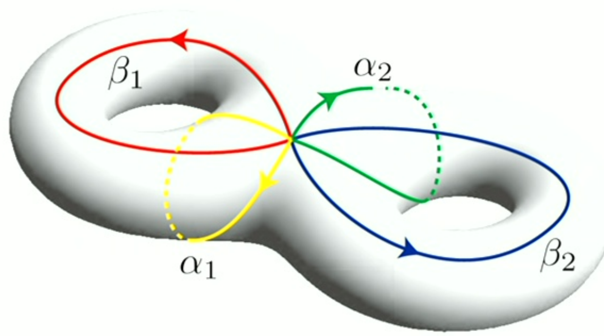
$$\pi_1(\Sigma_2) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid \prod_{i=1}^{2} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \rangle$$

Canonical regular 4*g*-gon



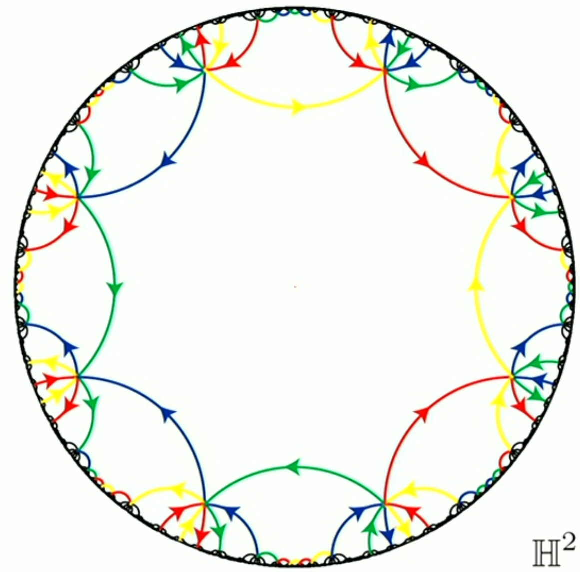
Sum of inner angle is 2π

genus-*g* surface



Example: *g*=2

$p : \mathbb{H}^2 \longrightarrow \Sigma_2$

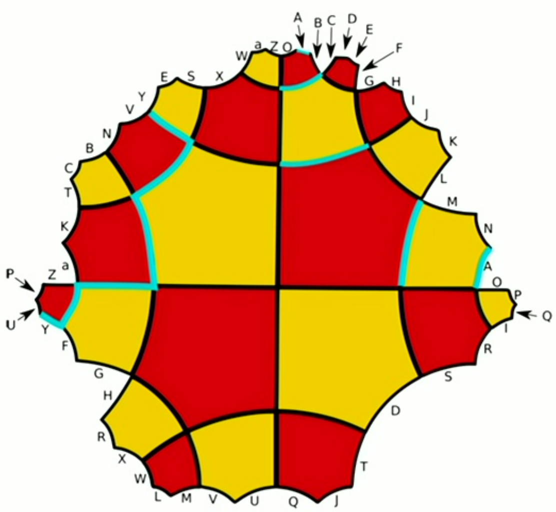


Construct 2D hyperbolic code

1. Use regular tiling of the hyperbolic surface

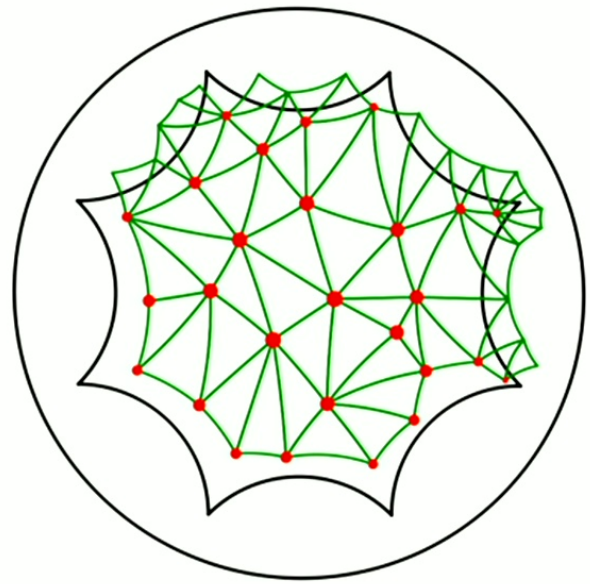
Breuckmann, Vuillot, Campbell, Krishna, Terhal (2017)

(5,4)-tiling



2. Use (random) hyperbolic Delaunay triangulation

e.g. Lavasani, Zhu, Barkeshli (2019)



From CGAL package

Scaling of 2D hyperbolic codes

- Gauss-Bonnet formula:

$$\frac{1}{2\pi}\int \kappa dA = \chi(\mathcal{M}^2) = 2 - 2g$$

Euler characteristic: $\chi = V - E + F$

In the case of hyperbolic manifolds, we have the curvature κ = −1,

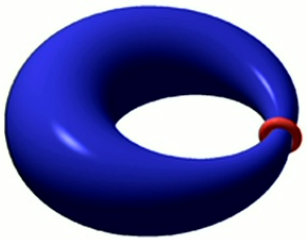
Area: $A = 4\pi(g-1)$

- 1st Betti number: $b_1(\mathcal{M}^2) = 2g$

↓

$$b_1(\mathcal{M}^2) = O(A) = O(vol(\mathcal{M}^2))$$

- Encoding rate: $k = 2g \longrightarrow k/n = \text{const}$



- Code distance d ⟷ Z_2 i-systole : the length of the shortest non-contractible i-cycle

- For arithmetic 2-manifold: $sys_1(\mathcal{M}_h^2; \mathbb{Z}_2) \geq c' \log vol(\mathcal{M}_h^2)$ (Katz, Schaps, Vishne, 2007)

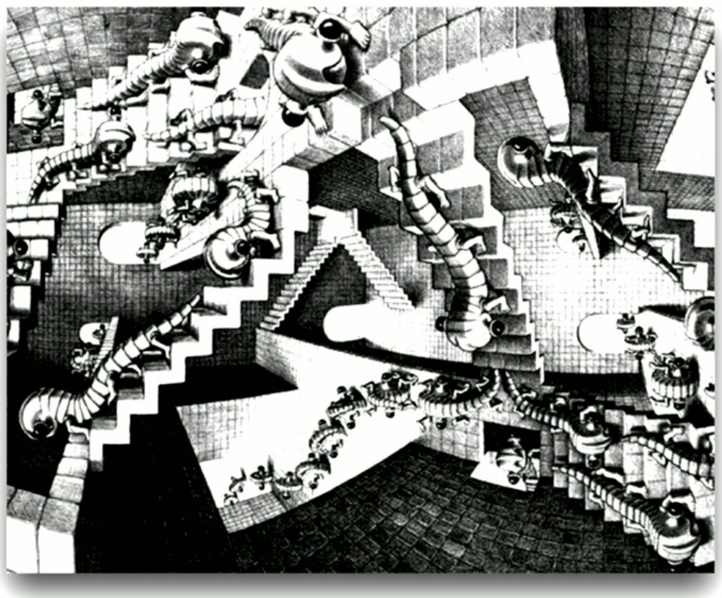
Betti-number scaling on 3-manifolds

Scaling of the code parameters for homological quantum codes defined on 3-manifolds

Essence of the 3-manifold construction:

1. We want as many "holes" as possible ⟷ Betti number – Volume scaling

2. We want the "holes" to be as large as possible ⟷ i-systole – Volume scaling



Betti-number scaling on 3-manifolds

- Gromov's theorem (for *D*-dimensional manifolds):

$$\sum_{i=0}^{D} b_i(\mathcal{M}^D) \leq C_D \cdot vol(\mathcal{M}^D)$$

Information-theoretical perspective: one cannot have more logical qubits than physical qubits!

Three code (manifold) constructions in this paper

I. Quasi-hyperbolic codes:

$$k/n = O(1/\log(n)) \qquad d = O(\log(n))$$

II. Homological fibre-bundle code (based on 3-manifolds by Freedman-Meyer-Luo):

$$k/n = O(1/\log^{\frac{1}{2}}(n)) \qquad d = O(\log^{\frac{1}{2}}(n))$$

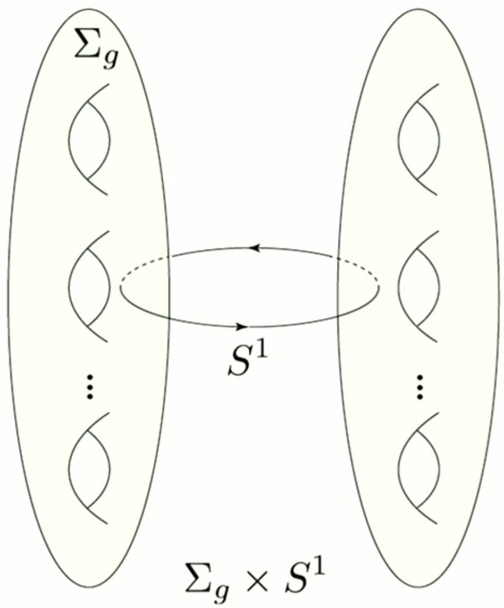
III. Torelli mapping-torus code (based on hyperbolic 3-manifolds by Ian Agol et al.):

$$k/n = \text{const}$$

Distance (systole) lower-bound unkown.
Conjectured to be $\text{poly}(\log(\text{n}))$.

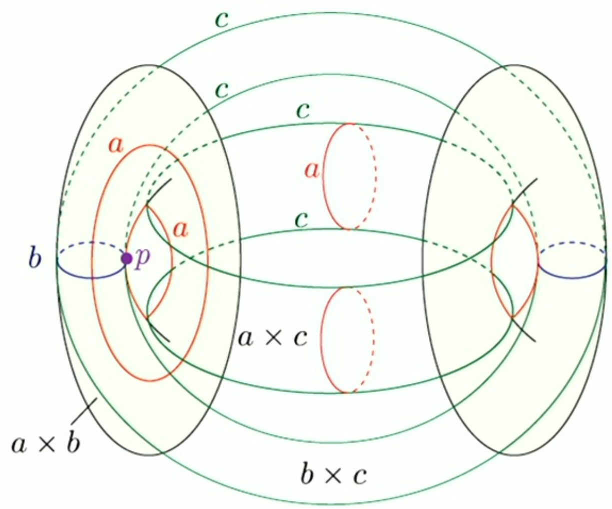
I. Quasi-hyperbolic codes

- Construction: a product of the genus-g hyperbolic surface with area A and a circle of length $\log(A)$



Homology of a 3-torus

$$T^3 = T^2 \times S^1$$



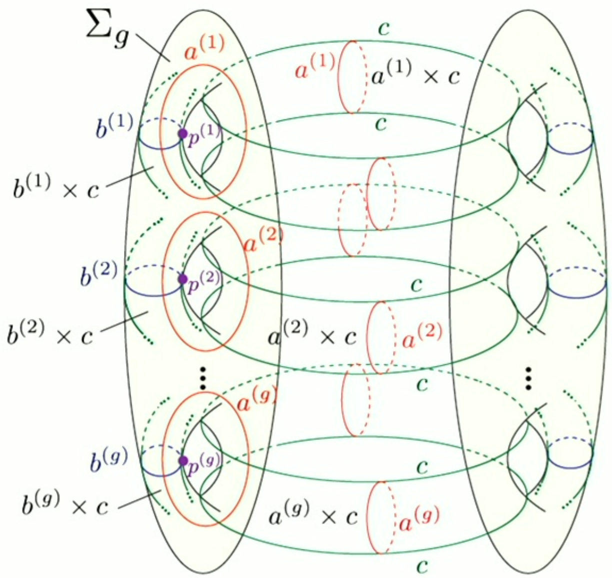
$$H_1(T^3) = H_2(T^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \equiv \mathbb{Z}_2^3$$

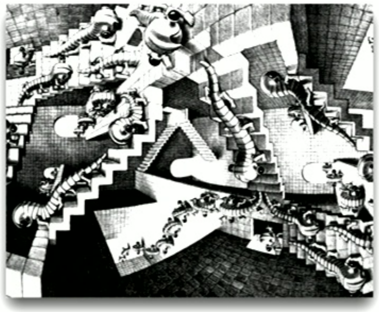
1st homology basis is $B_1 = \{a, b, c\}$

The dual 2nd homology basis is $B_2 = \{b \times c, a \times c, a \times b\}$

$$|a \cap (b \times c)| = |b \cap (a \times c)| = |c \cap (a \times b)| = 1$$

Homology of the quasi-hyperbolic code





- 1st homology group of the genus-g surface:

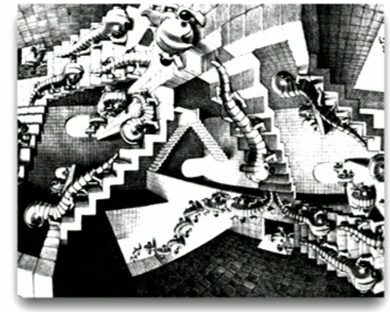
  $H_1(\Sigma_g; \mathbb{Z}_2) = \mathbb{Z}_2^{2g}$

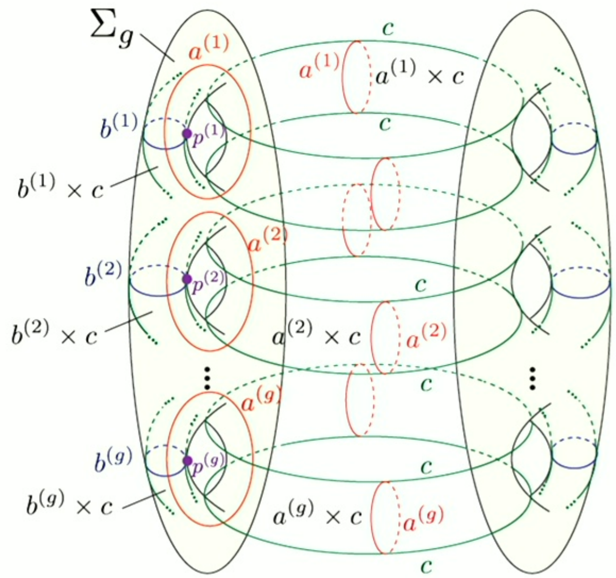
- 1st homology basis on the genus-g surface: $\{a^{(i)}, b^{(i)} | i = 1, 2, \ldots, g\}$

- 1st-homology of the 3-manifold can be obtained from the *Kunneth formula*:

$$
\begin{aligned}
&H_1(\Sigma_g \times S^1; \mathbb{Z}_2) \\
=&[H_1(\Sigma_g; \mathbb{Z}_2) \otimes H_0(S^1; \mathbb{Z}_2)] \oplus [H_0(\Sigma_g; \mathbb{Z}_2) \otimes H_1(S^1; \mathbb{Z}_2)] \\
=&\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_2 = \mathbb{Z}_2^{2g+1}
\end{aligned}
$$

Homology of the quasi-hyperbolic code





- 1st homology group of the genus-g surface:

  $H_1(\Sigma_g; \mathbb{Z}_2) = \mathbb{Z}_2^{2g}$

- 1st homology basis on the genus-g surface: $\{a^{(i)}, b^{(i)} | i = 1, 2, \ldots, g\}$

- 1st-homology of the 3-manifold can be obtained from the *Kunneth formula*:

$$H_1(\Sigma_g \times S^1; \mathbb{Z}_2)$$
$$= [H_1(\Sigma_g; \mathbb{Z}_2) \otimes H_0(S^1; \mathbb{Z}_2)] \oplus [H_0(\Sigma_g; \mathbb{Z}_2) \otimes H_1(S^1; \mathbb{Z}_2)]$$
$$= \mathbb{Z}_2^{2g} \oplus \mathbb{Z}_2 = \mathbb{Z}_2^{2g+1}$$

- Betti number and the number of logical qubits:

$$k = b_1(\Sigma_g \times S^1) = Rank(H_1(\Sigma_g \times S^1; \mathbb{Z}_2)) = 2g + 1$$

- 1st homology basis for the 3-manifold: $B_1 = \{a^{(i)}, b^{(i)}, c | \ i = 1, 2, \ldots, g\}$

Code parameter scaling of the quasi-hyperbolic code

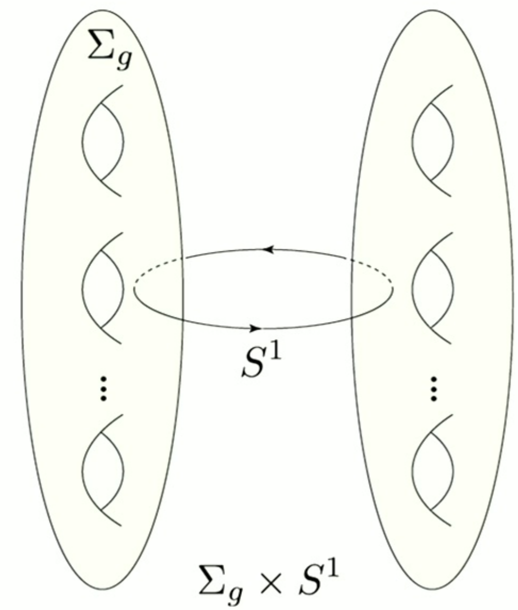
- Gauss-Bonnet formula: $A = 4\pi(g-1)$
- Lower-bound of the 1-sytole on the genus-g surface: $sys_1(\Sigma_g; \mathbb{Z}_2) \geq c' \cdot \log A$
- Choose the circle length as the 1-systole of the surface: $l = O(\log A)$
- Code distance: $d \propto l = O(\log A)$
- The volume V of the quasi-hyperbolic 3-manifold:

$$V = A \cdot O(\log A)$$

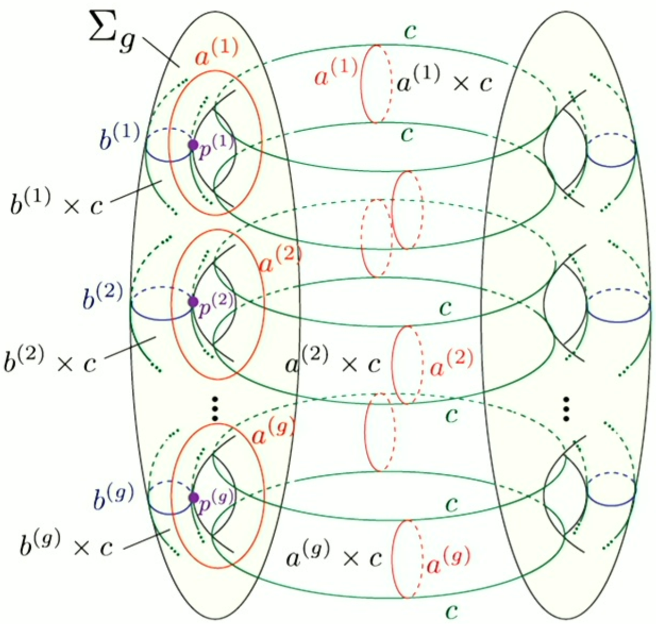
- Betti number scaling and encoding rate:

$$b_1/V = O(g/V) = O\left(1/\log(V)\right) \implies k/n = O(1/\log(n))$$

- Code distance scaling: $l = O(\log(V)) \implies d = O(\log(n))$



Triple intersection and logical gate structure



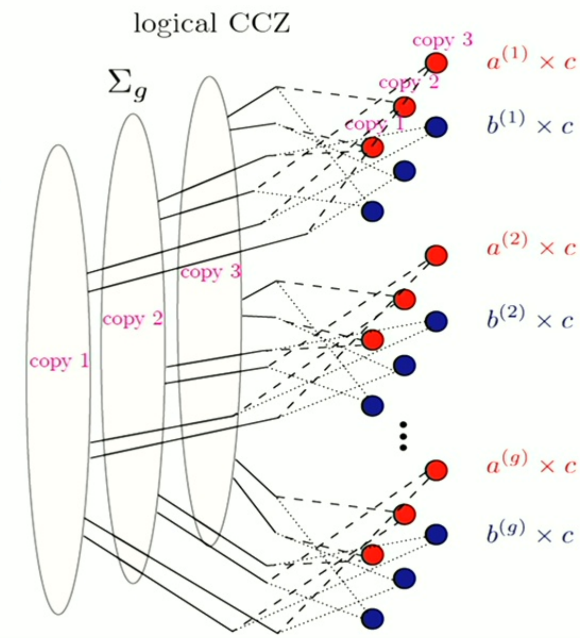
- Triple intersection points: $\Sigma_g \cap (a^{(i)} \times c) \cap (b^{(i)} \times c) = \Sigma_g \cap c = p^{(i)}$

- Transversal T-gates:

$$\widetilde{T} = \prod_{(r,s,l)} \overline{\mathrm{CCZ}}((\Sigma_g; r), (a^{(i)} \times c; s), (b^{(i)} \times c; l))$$

(r, s, l) represent labels of three different toric-code copies with arbitrary possible permutations

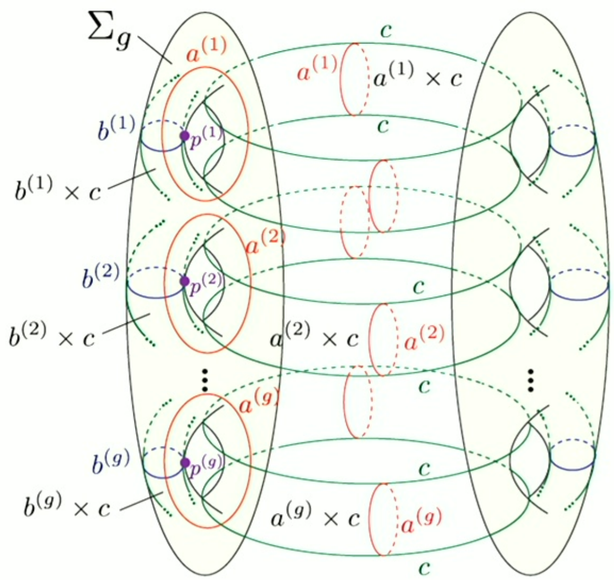
- Interaction hypergraph →



- \# of triple intersection points:

$$n_p = g = O(k) = O(n/\log n)$$

Parallelizable logical CZ gates



1-form symmetries acting on 2-cycles (membranes)

$$\widetilde{\mathrm{CZ}}^{(r,s)}_{a^{(i)}\times c} = \overline{\mathrm{CZ}}((b^{(i)} \times c; r), (\Sigma_g; s))\overline{\mathrm{CZ}}((b^{(i)} \times c; s), (\Sigma_g; r))$$

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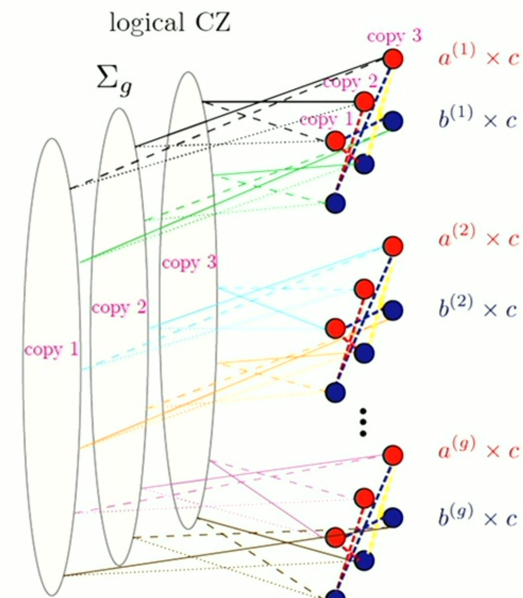
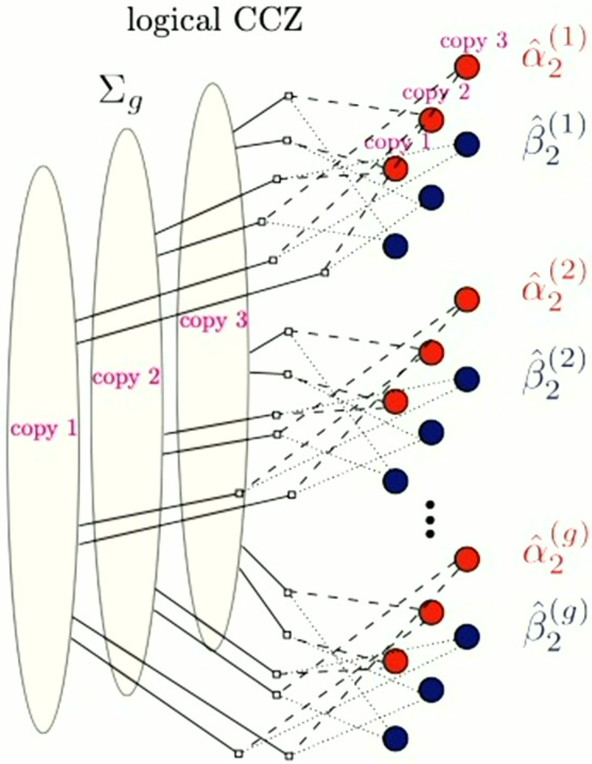
$$\widetilde{\mathrm{CZ}}^{(r,s)}_{\Sigma_g} = \prod_i \overline{\mathrm{CZ}}((a^{(i)} \times c; r), (b^{(i)} \times c; s))$$

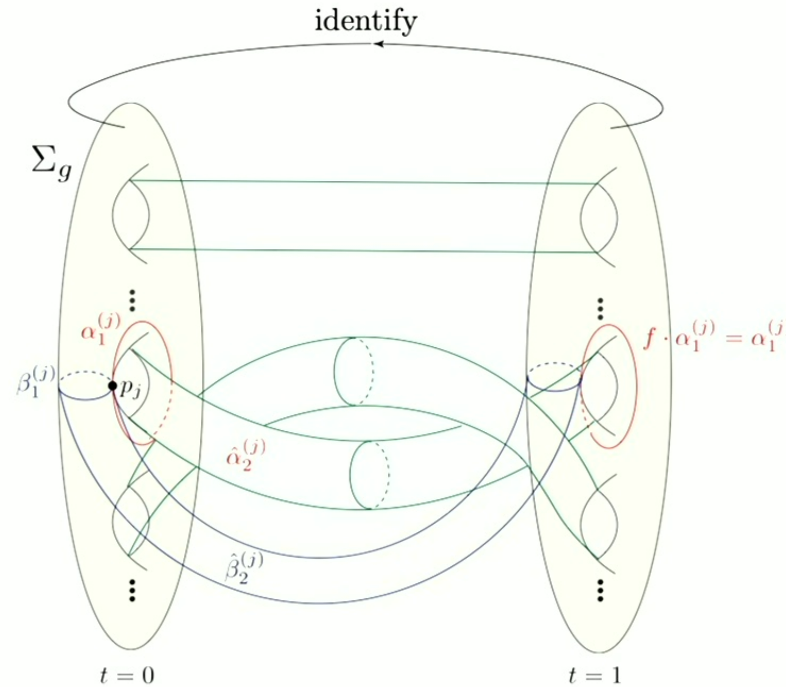
(r, s) represent labels of different toric-code copies

- Interaction graph with colored edges:



Interaction hypergraph





Part IV. A no-Abelian self-correcting memory and connection to symmetry defects

- In D space-time dimension, a Z_2 m-form, n-form, $(D\text{-}m\text{-}n)$-form gauge theories with an additional topological action:

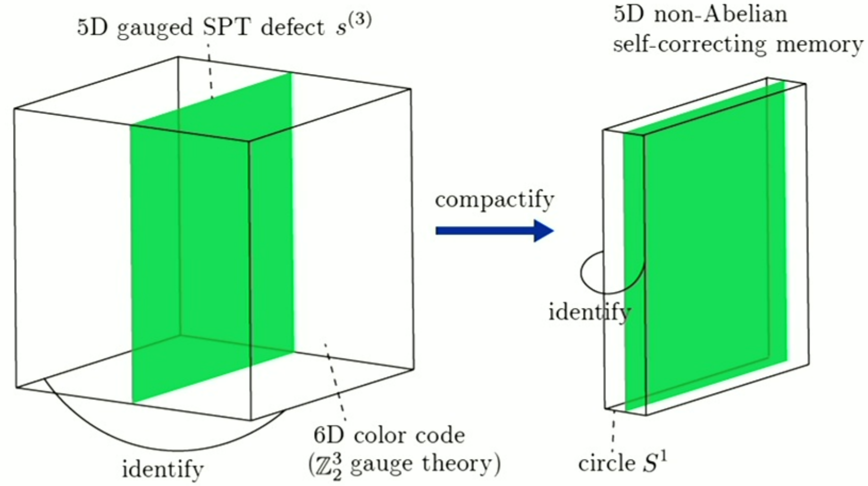
$$\pi \int a^m \cup b^n \cup c^{D-m-n}$$

- $m{=}n{=}1$: D_8 gauge theory

- Minimal dimension without particle excitations: $D = 5{+}1$ $(m = n = D\text{-}m\text{-}n = 2)$
  A 5d non-Abelian self-correcting quantum memory
  (Abelian loop excitation, non-Abelian membrane excitation)

- The 5d memory can be obtained from a twisted compactification of a 6d color code self-correcting memory (Bombin) with a 5d gauged SPT defect. The defect world-sheet operator is:

$$\mathcal{D}_{s^{(3)}}(\mathcal{M}^{5+1}) = \exp\left(\pi \mathrm{i} \int_{\mathcal{M}^{5+1}} a^2 \cup b^2 \cup c^2\right)$$



Po-Shen Hsin, Ryohei Kobayashi, GZ, arXiv:2405.11719

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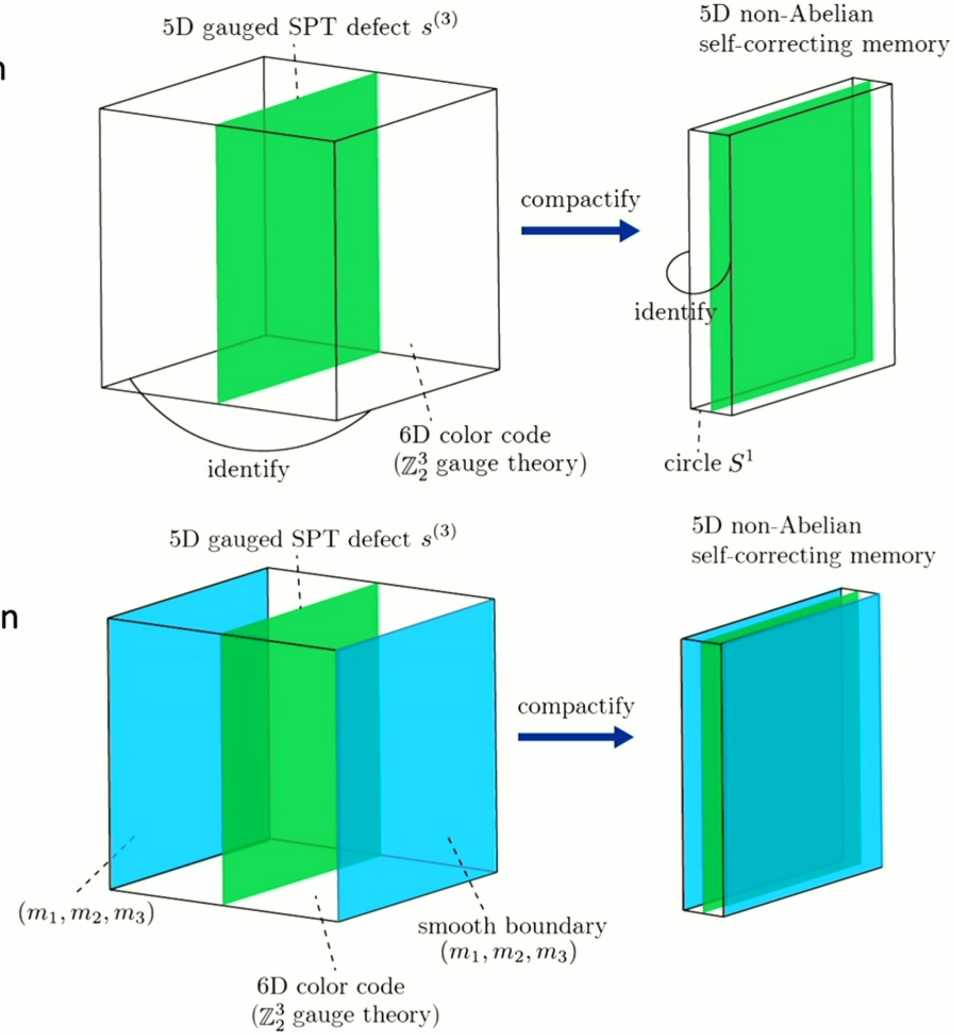

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Summary and Outlook

- Generalized symmetries play an important role in fault-tolerant logical gates in qLDPC cdoes
- Higher-dimensional homological LDPC codes with non-trivial triple-intersection has logical non-Clifford gates.
- Higher-form symmetries give rise to addressable and parallelizable logical gates.
- A class of qLDPC codes defined on qLDPC codes have high rate and logarithmic distance

Future directions:

- Exploring the systole and distance scaling in the Torelli mapping-torus code (likely via numerics).
- Generalize to expander-based qLDPC codes to improve the code parameters.
- Logical gates with non-invertible symmetries and defects.
- TQFT and gauge theories on higher-dimensional manifolds and general chain complexes beyond manifolds.

Thanks for your attention!