Title: Universal quantum computation in two dimensions by converting between the toric code and a non-abelian quantum double

Speakers: Margarita Davydova

Collection: Physics of Quantum Information

Date: May 31, 2024 - 1:00 PM

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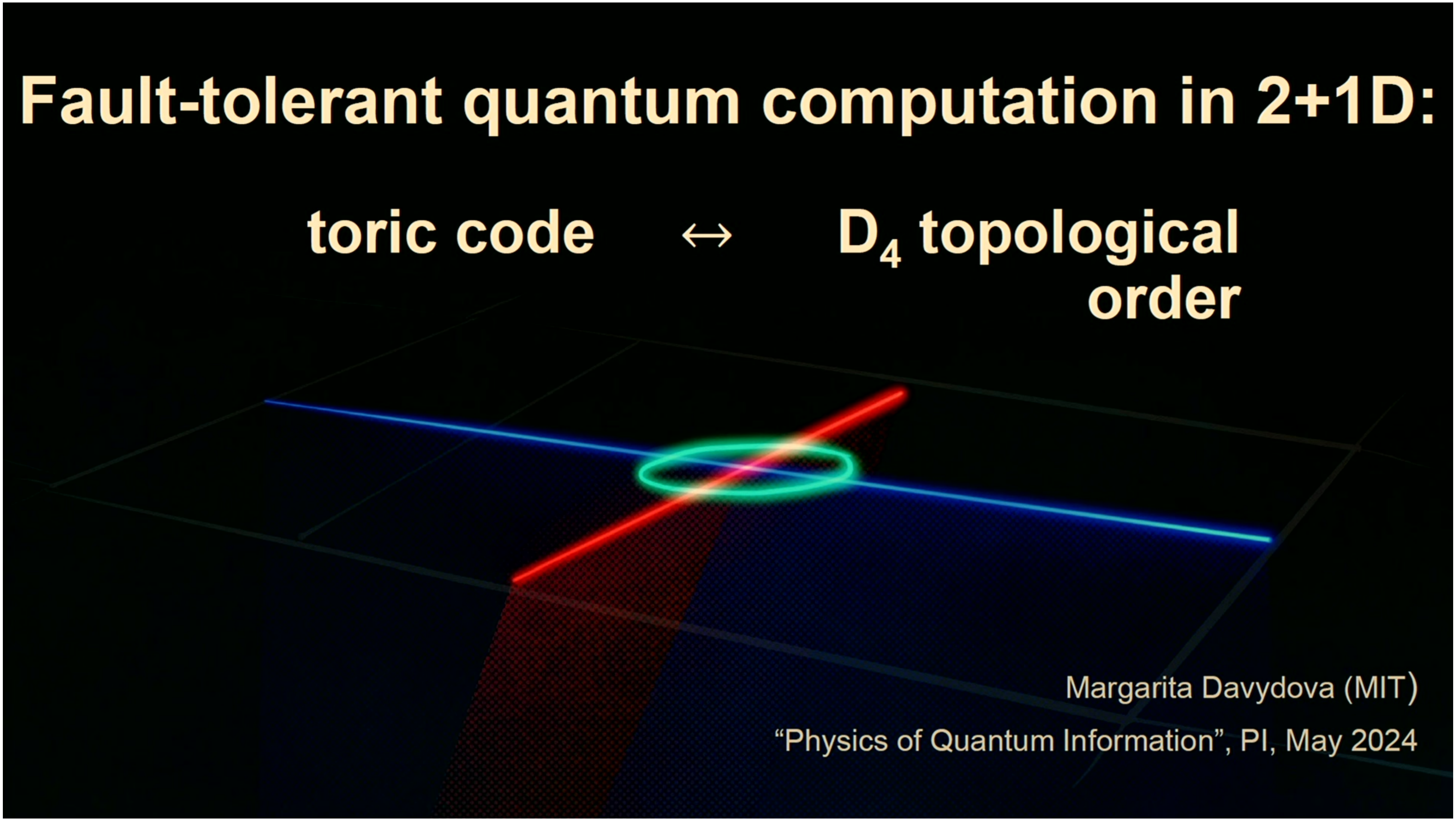
Abstract: In this talk, I will explain how to implement fault-tolerant non-Clifford gates in copies of toric code in two dimensions achieved by transiently switching to a non-Abelian topologically ordered phase by expanding earlier results by Bombin [arXiv.1810.09571] and Brown [SciAdv.aay4929]. This addresses the challenge of performing universal fault-tolerant quantum computation in purely two spatial dimensions and shows a new approach to quantum computation using non-Abelian phases.
This talk is based on upcoming work in collaboration with A. Bauer, B.Brown, J. Magdalena de la Fuente, M. Webster and D. Williamson.


Fault-tolerant quantum computation in 2+1D:
toric code ↔ D_4 topological order
Margarita Davydova (MIT)
“Physics of Quantum Information”, PI, May 2024

Fault-tolerant quantum computation in 2+1D:

toric code ↔ D_4 topological order

Collaboration(s) with:

Ben Brown (IBM), Andi Bauer (MIT),

Jacob Bridgeman (Ghent),

Julio Magdalena de la Fuente (FU Berlin),

Mark Webster (USydney),

Dom Williamson (IBM)

Margarita Davydova (MIT)

"Physics of Quantum Information", PI, May 2024

Outline

- **Introduction:**
  - **Quantum computation with 2D toric code**
  - **Dynamical framework**
- Non-abelian phase
- Dynamical protocols
  - CCZ gate [based on Brown'18; Bombin'17]
  - T gate
  - Magic state preparation [T-state; CZ-state]
- Fault-tolerance
- Outlook

Premise

- **Universal** quantum computation is **hard to do fault-tolerantly.**
  [Eastin, Knill '08, Campbell et al nature23460 etc]

- **2D toric code** (+variations) is the gold standard for fault-tolerant performance.
  - Metrics: noise threshold; spacetime overhead; best studied.

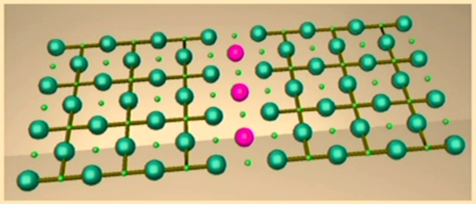
It is also **a good proxy** for working out new approaches.

- For example, many recent EC constructions still involve TC in some form, e.g. layer codes [Baspin, Williamson '23], hierarchical codes [Pattison et al '23], Floquet codes [Hastings Haah '21]

Quantum computation with 2D toric code

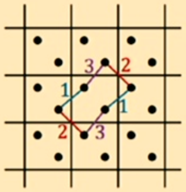
Using copies of 2D TC, how does one perform universal quantum computation?

- full Clifford group from lattice surgery and transversal gates
  [Horsman et al '12; Campbell et al nature23460]



Less known:

- full Clifford group from 'transversal' gates
  [Barkeshli et al '23 SciPostPhys.14.4.065]



For universal gateset, one non-Clifford gate is needed.

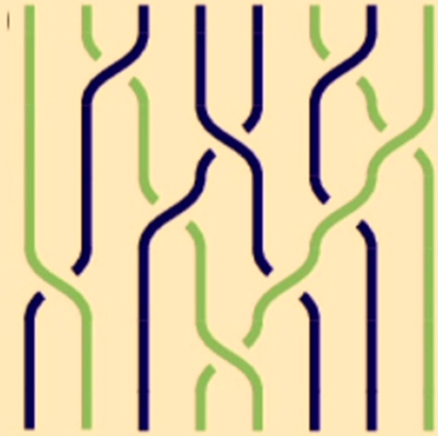
- cannot be done transversally in 2D [Eastin-Knill '08; Bravyi-Koenig '13]
- existing solutions: magic state distillation; [Bravyi-Haah]: spacetime overhead Cd^3, w/ $C\sim 200$;
  code switching to 3D (also $O(d^3)$) overhead) [Fowler et al '12,'13, O'Gorman Campbell'13; Bombin '14]

Quantum computation with non-abelian phases

Usually:

Logical states = fusion states of pairs of non-abelian anyons. Examples:

- Universal by braiding only: fib, $SU(2)_{k>2,k\neq 4}$, …
- Universal by braiding + topological charge measurements: $SU(2)_4$, S_3, …
- Simplest but (likely) non-universal: A_5, D_4, …



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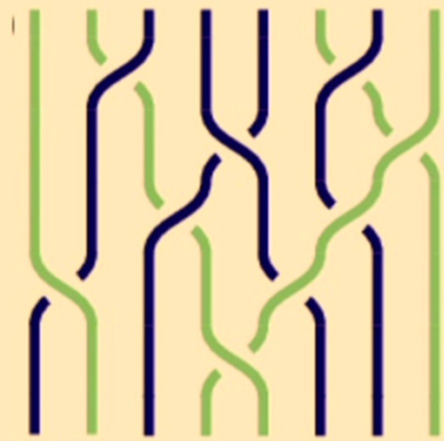
Not as common (at all):

Logical states = ground state (+ odd-parity-charge sector)

Previously [Cong, Cheng, Wang '17], [Laubscher, Loss, Wootton' 18]



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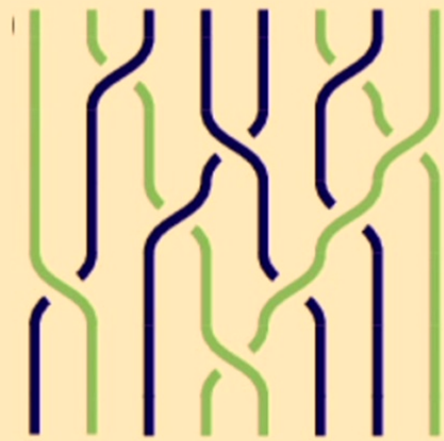
22 ground states on a torus



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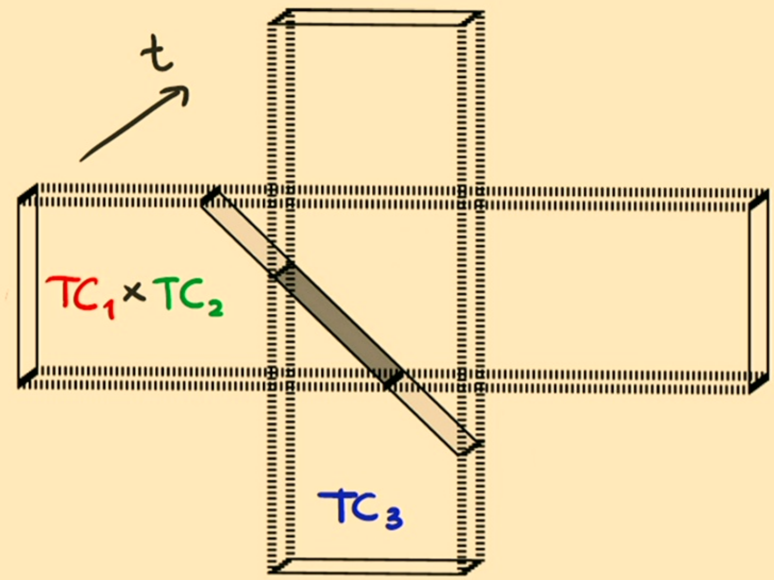
Alternatives? (toric code, again)

[Brown '18 sciadv.aay4929, Bombin '17 arXiv:1810.0957]

Copies of 3D toric code have a transversal CCZ (non-Clifford) gate.

Trade space for time!



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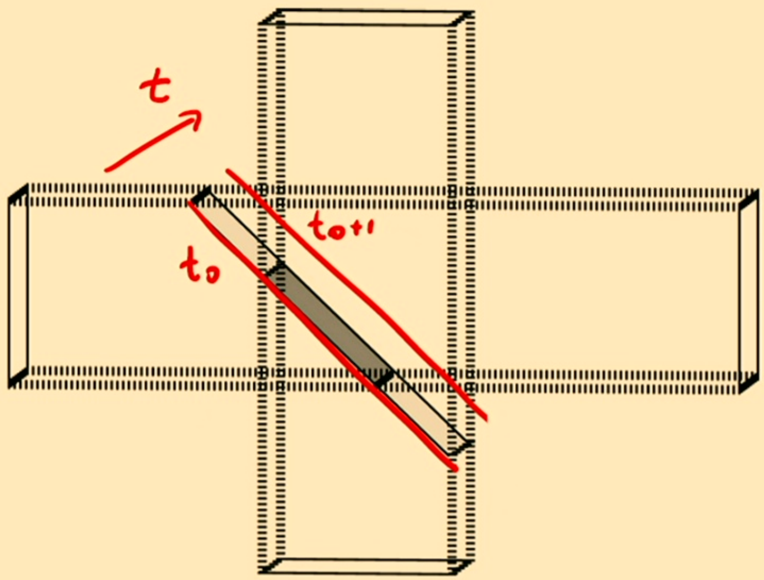

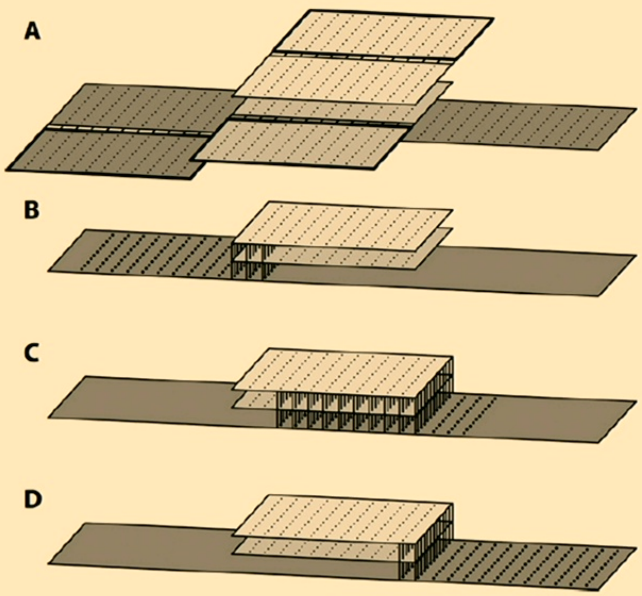

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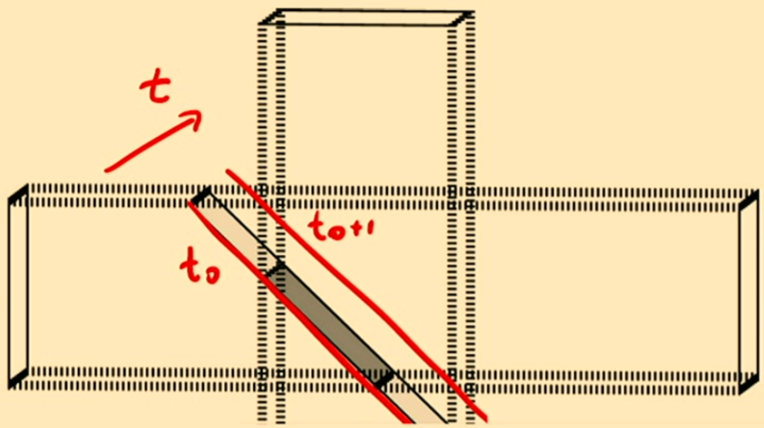
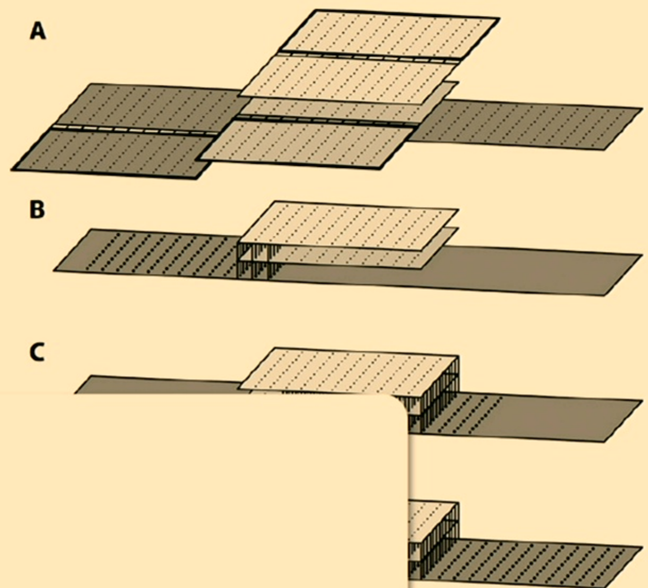


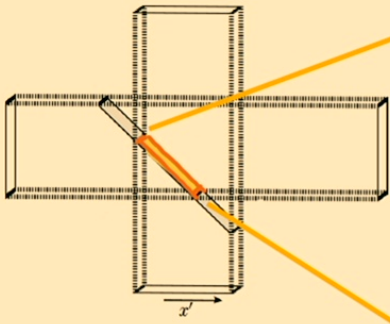


Does this actually work?

How? Is it just a trick or is there a deeper mechanism?

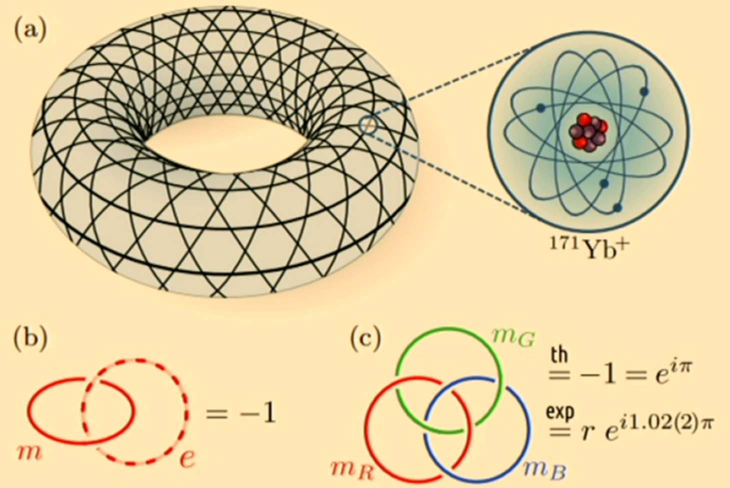
CCZ causes twisted quantum double

$$\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathcal{D}(D_4)$$



Non-abelian topological order used for **ground state** topological quantum computation (without braiding).

Previously [Cong, Cheng, Wang '17], [Laubscher, Loss, Wootton' 18]



[Iqbal et al (Quantinuum) Non-abelian topological order and anyons on a trapped-ion processor. Nature 626, (2024)]

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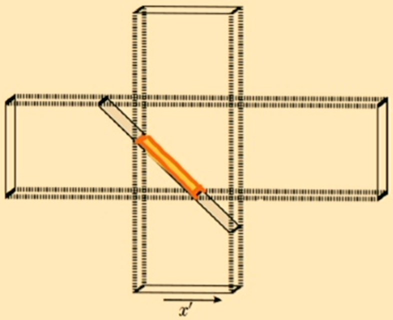

Non-abelian topological order used for **ground state** topological quantum computation (without braiding).

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Tools:

- Pauli stabilizer measurements
- single-qubit Pauli measurements
- local non-Clifford gates
- just-in-time decoding

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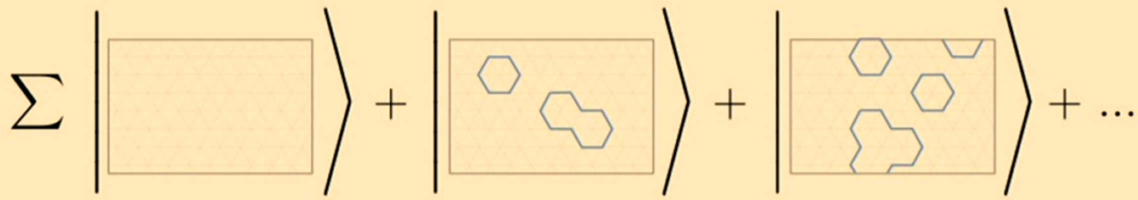
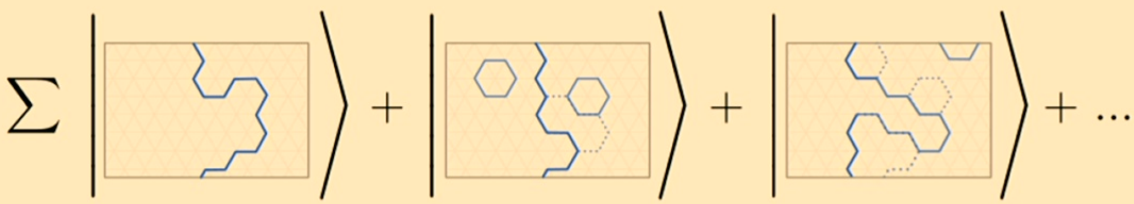
Toric code and twisted quantum doubles

Toric code = quantum double $\mathcal{D}(\mathbb{Z}_2)$

Ground state:



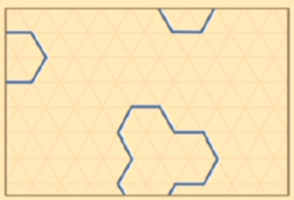
A different ground state:



Toric code and twisted quantum doubles

1 copy of the toric code = quantum double $\mathcal{D}(\mathbb{Z}_2)$

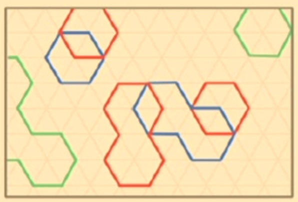
Ground state:

$\sum_{\substack{\text{loop} \\ \text{conf.}}} \Big|$  $\Big\rangle$

Toric code and twisted quantum doubles

3 copies of the toric code = quantum double $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Ground state:

$\sum_{\substack{\text{loop}\\\text{conf.}}} \Big|$  $\Big\rangle$

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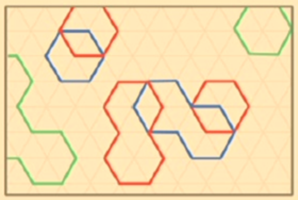
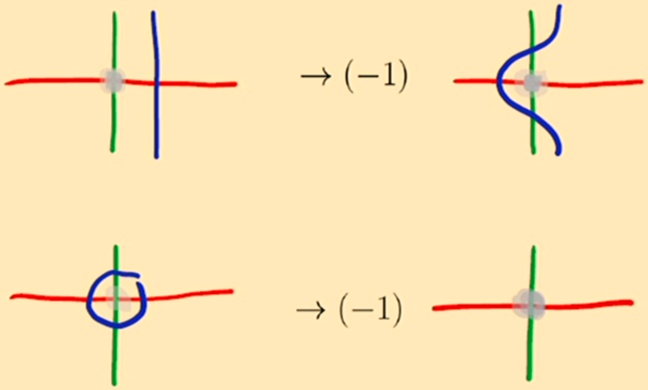
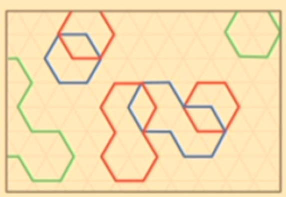
$$\sum_{\substack{\text{loop}\\\text{conf.}}} \Bigg|$$  $$\Bigg\rangle$$

$$f(\text{conf.}) = \omega_{III}(a,b,c) = (-1)^{\int a \cup b \cup c}$$



Twisted quantum double $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathcal{D}(D_4)$

$$\sum_{\substack{\text{loop}\\\text{conf.}}} (-1)^{\text{f(conf.)}} \Bigg|$$  $$\Bigg\rangle$$

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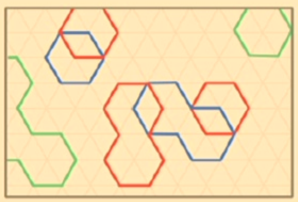

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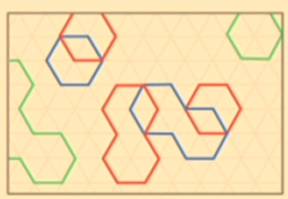

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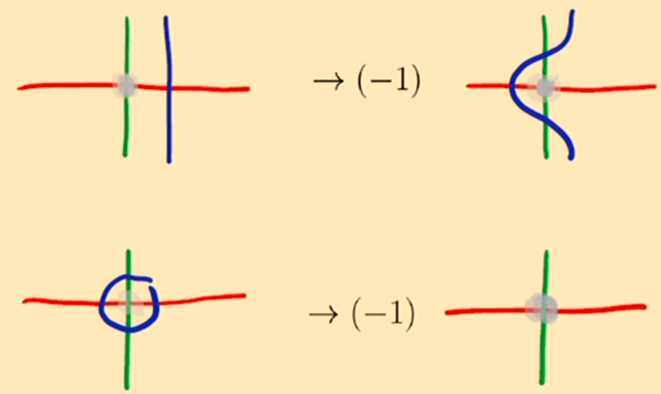
$$\sim (-1)^{a_1 b_2 c_3} \sim CCZ$$



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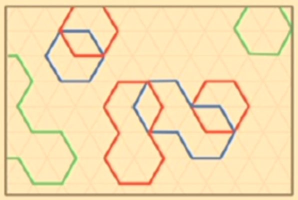
$$\sum_{\substack{\text{loop}\\\text{conf.}}} \Big| \quad \Big\rangle$$



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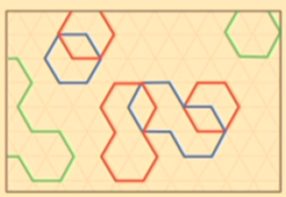

This topological phase is very "compatible" with copies of the TC

Compatibility between TC and TQD: anyons

	Copies of the toric code $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$		Twisted quantum double $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
charges	$\{1, \mathrm{e}_{\mathtt{r}}, \mathrm{e}_{\mathtt{g}}, \mathrm{e}_{\mathtt{b}}\}$ $(\# = 8)$	(abelian) charges	$\{1, \mathrm{e}_{\mathtt{r}}, \mathrm{e}_{\mathtt{g}}, \mathrm{e}_{\mathtt{b}}\}$ $(\# = 8)$

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charges	$\{1, e_r, e_g, e_b\}$ $(\# = 8)$	(abelian) charges	$\{1, e_r, e_g, e_b\}$ $(\# = 8)$
fluxes	$\{m_r, m_g, m_b\}$ $(\# = 7)$	"fluxes" $d = 2$	$\{m_r, m_g, m_b\}$ $(\# = 6)$ $m_r \times e_{g,b} = m_r$ $m_r \times m_r = 1 + e_g + e_b + e_g e_b$

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fermions	$\{f_r, f_g, f_b\}$ $(\# = 7)$ $m_r \times e_r = f_r$	fermions $d = 2$	$\{f_r, f_g, f_b\}$ $(\# = 6)$ $m_r \times e_r = f_r$
		semions $d = 2$	$\{s, \bar{s}\}$ $(\# = 2)$ $s \in m_r \times m_g \times m_b$

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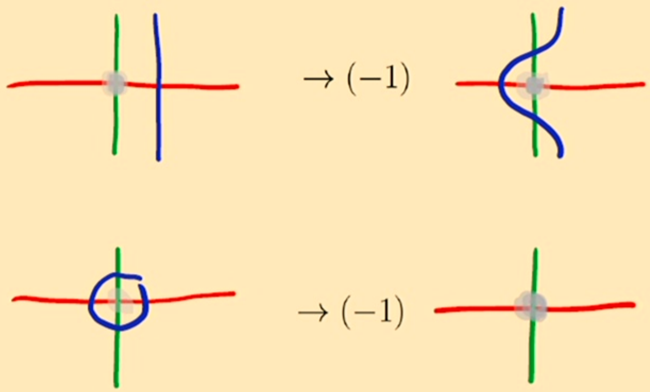
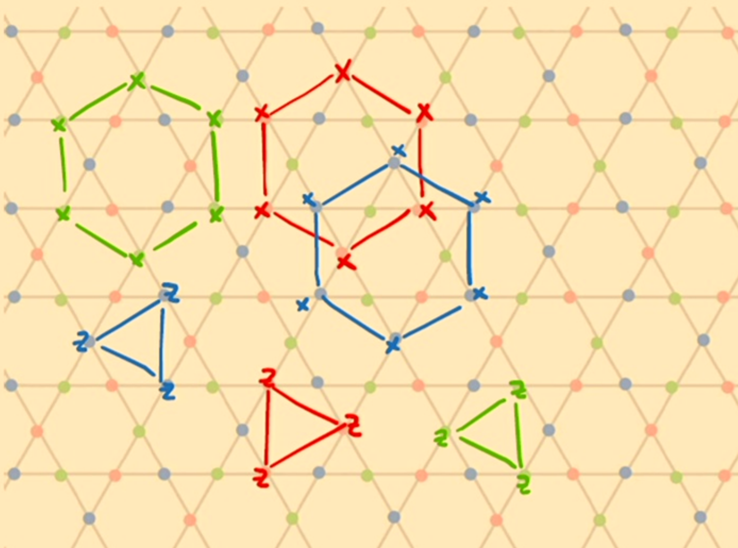
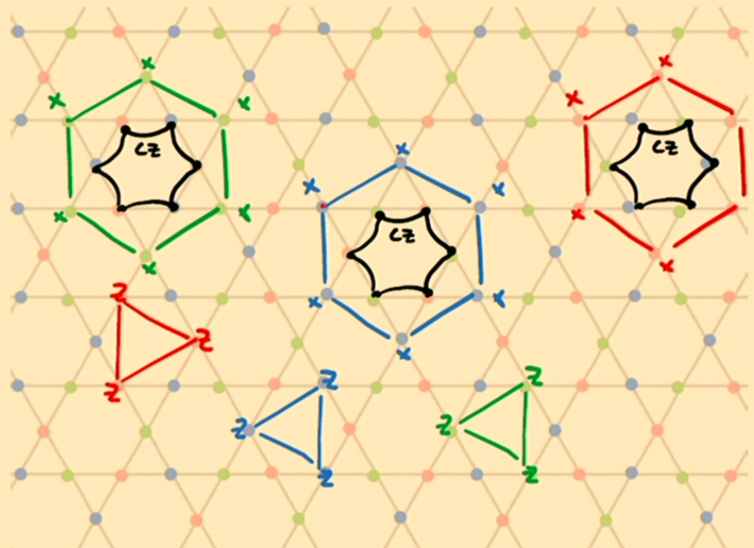


$\mathbb{Z}_2)$	Twisted quantum double $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$
charg	(abelian) charges $\{1, \mathtt{e_r}, \mathtt{e_g}, \mathtt{e_b}\}$ $(\# = 8)$
fluxe	"fluxes" $\mathrm{d} = 2$ $\{\mathtt{m_r}, \mathtt{m_g}, \mathtt{m_b}\}$ $(\# = 6)$ $\mathtt{m_r} \times \mathtt{e_{g,b}} = \mathtt{m_r}$ $\mathtt{m_r} \times \mathtt{m_r} = 1 + \mathtt{e_g} + \mathtt{e_b} + \mathtt{e_g}\mathtt{e_b}$
fermi	fermions $\mathrm{d} = 2$ $\{\mathtt{f_r}, \mathtt{f_g}, \mathtt{f_b}\}$ $(\# = 6)$ $\mathtt{m_r} \times \mathtt{e_r} = \mathtt{f_r}$
	semions $\mathrm{d} = 2$ $\{\mathtt{s}, \bar{\mathtt{s}}\}$ $(\# = 2)$ $\mathtt{s} \in \mathtt{m_r} \times \mathtt{m_g} \times \mathtt{m_b}$

Compatibility between TC and TQD: “stabilizers”

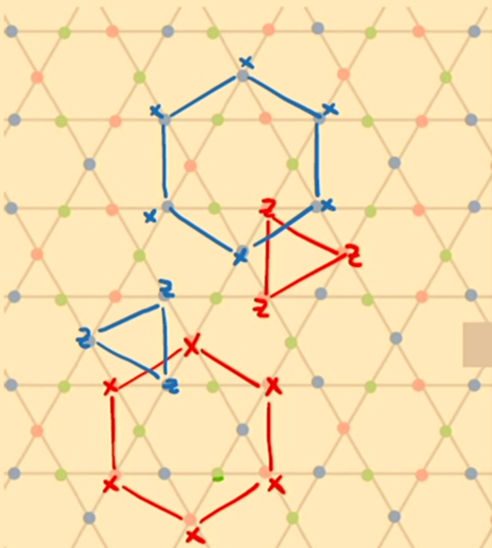
TC × TC × TC $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	Twisted quantum double $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
	

Yoshida arXiv:1508.03468
Iqbal et al ‘24

Compatibility between TC and TQD: gauging

$TC_r \times TC_b$

$\mathcal{D}(\mathbb{Z}_2^2)$



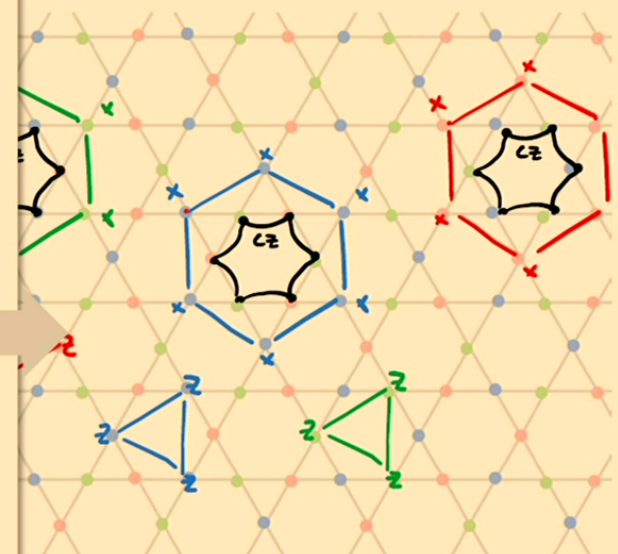
Global symmetries of **CC** $\sim TC_r \times TC_b$

$(S_3 \times S_3) \rtimes \mathbb{Z}_2$

1m	em	e1
mm	ff	ee
m1	me	1e

Twisted quantum double

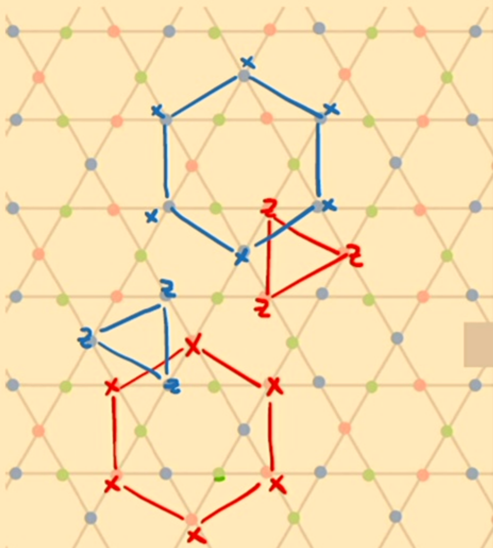
$\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$



Compatibility between TC and TQD: gauging

TC$_r$ × TC$_b$

$\mathcal{D}(\mathbb{Z}_2^2)$



Global symmetries of **CC** ~ TC$_r$ × TC$_b$

$(S_3 \times S_3) \rtimes \mathbb{Z}_2$

1m	em	e1
mm	ff	ee
m1	me	1e

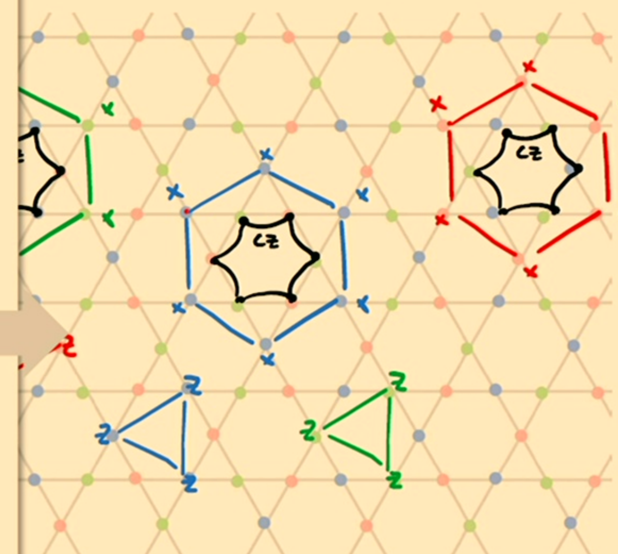
Gauge one of the $\mathbb{Z}_2$ automorphisms, i.e.

- TC×TC: cup product/CZ $\mathbb{Z}_2$
- color code: Hadamard $\mathbb{Z}_2$

Prem, Williamson arXiv 1905.06309
Chen et al arXiv:1303.4301
Tantivasakarn et al arXiv:2112.01519

Twisted quantum double

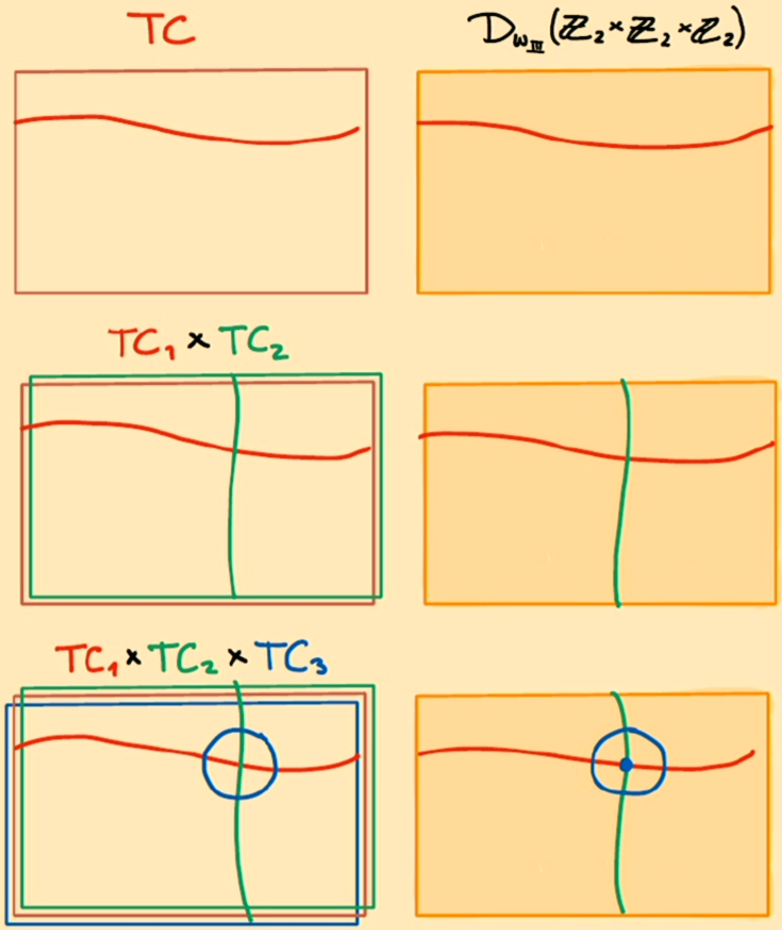
$\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$



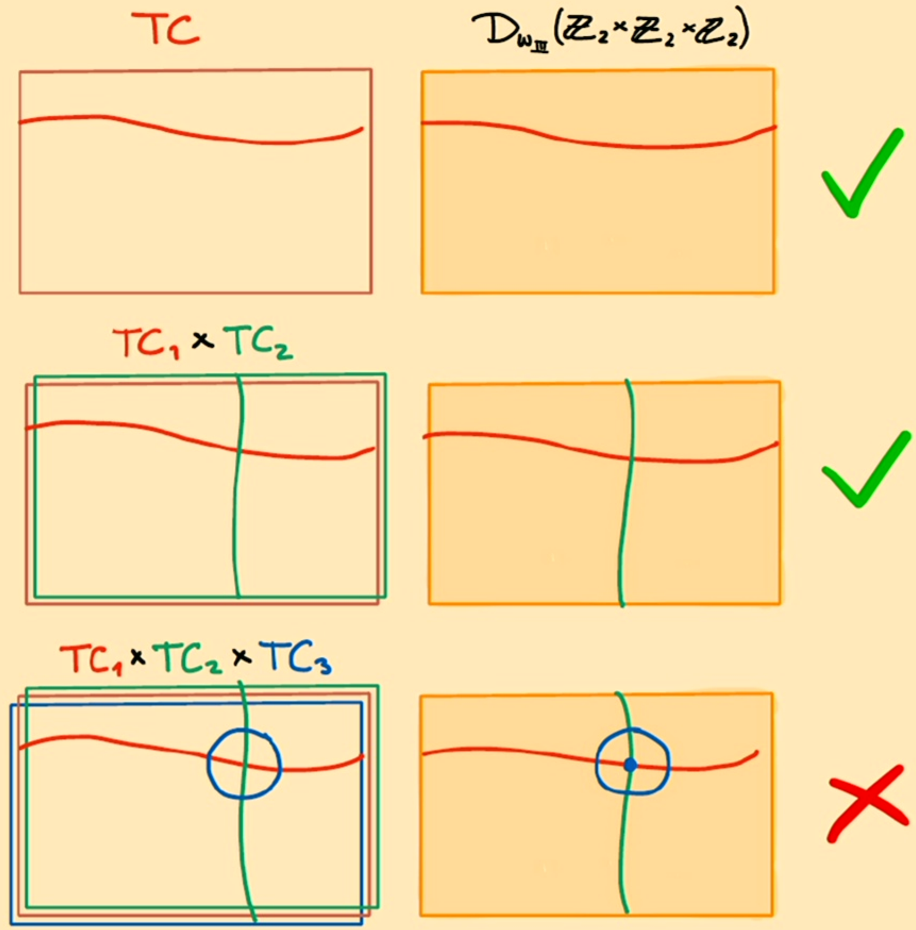
Compatibility between TC and TQD: GSD

TC × TC × TC $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	**Twisted quantum double** $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
GSD(T^2) = 64	GSD(T^2) = 22
GSD(open b.c.) = 8	GSD(open b.c.) = 8

Logical information transfer in a nutshell



Logical information transfer in a nutshell



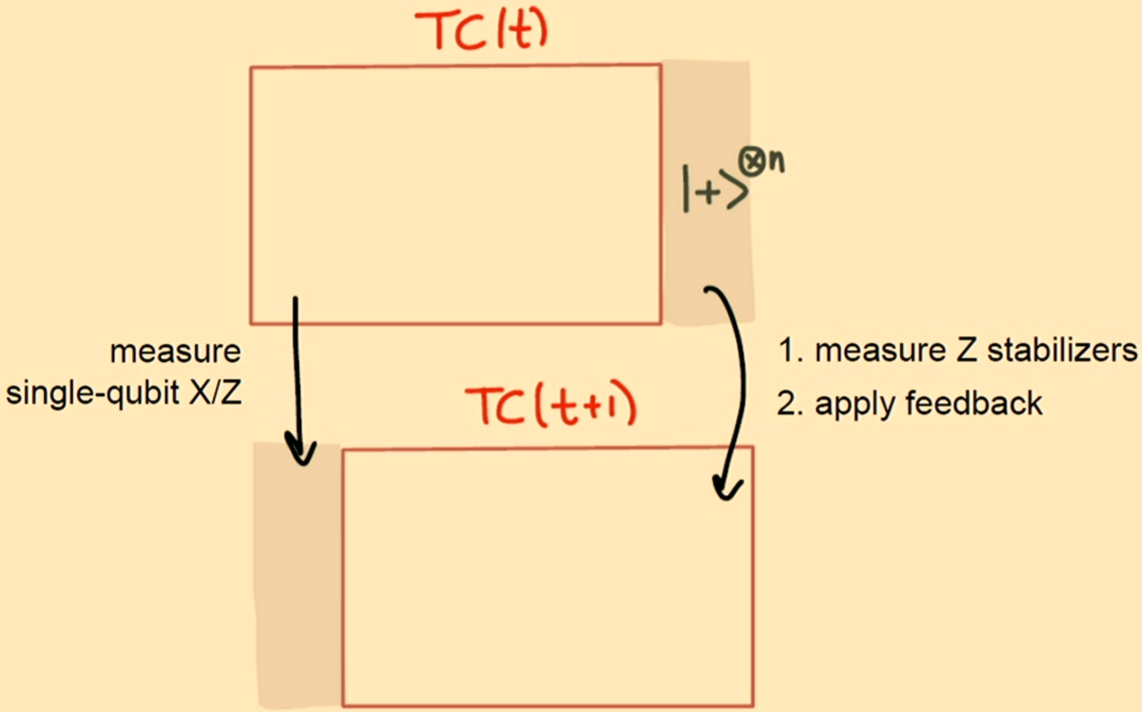
Compatibility between TC and TQD: GSD

TC × TC × TC $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	**Twisted quantum double** $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
GSD(T^2) = 64	GSD(T^2) = 22
GSD(open b.c.) = 8	GSD(open b.c.) = 8

CCZ gate

Assume a planar lattice of qubits with a patch of toric code.

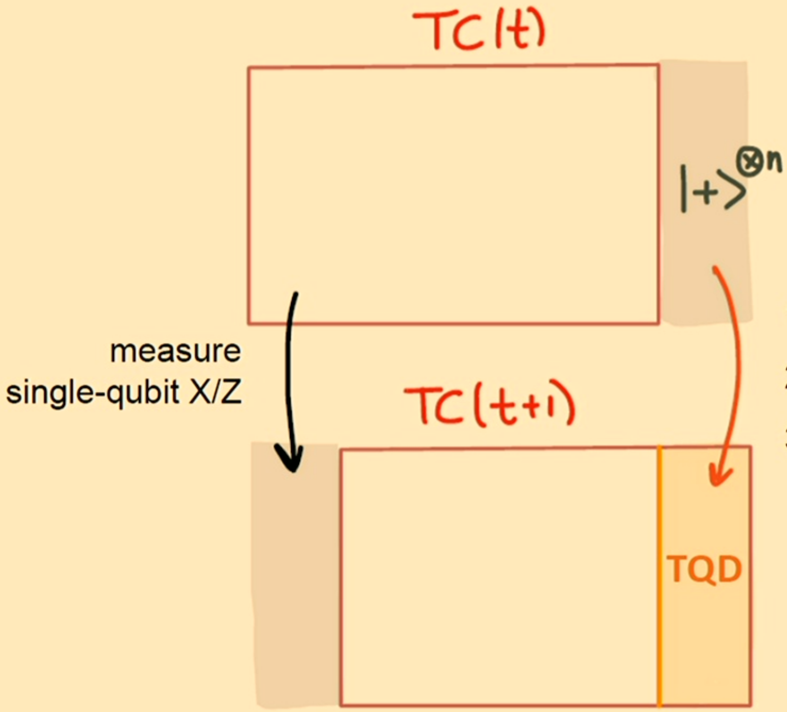
We can move this patch!



CCZ gate

Assume a planar lattice of qubits [3 qubits per unit cell].

We can move copies of the toric code and turn them into twisted quantum double (TQD)

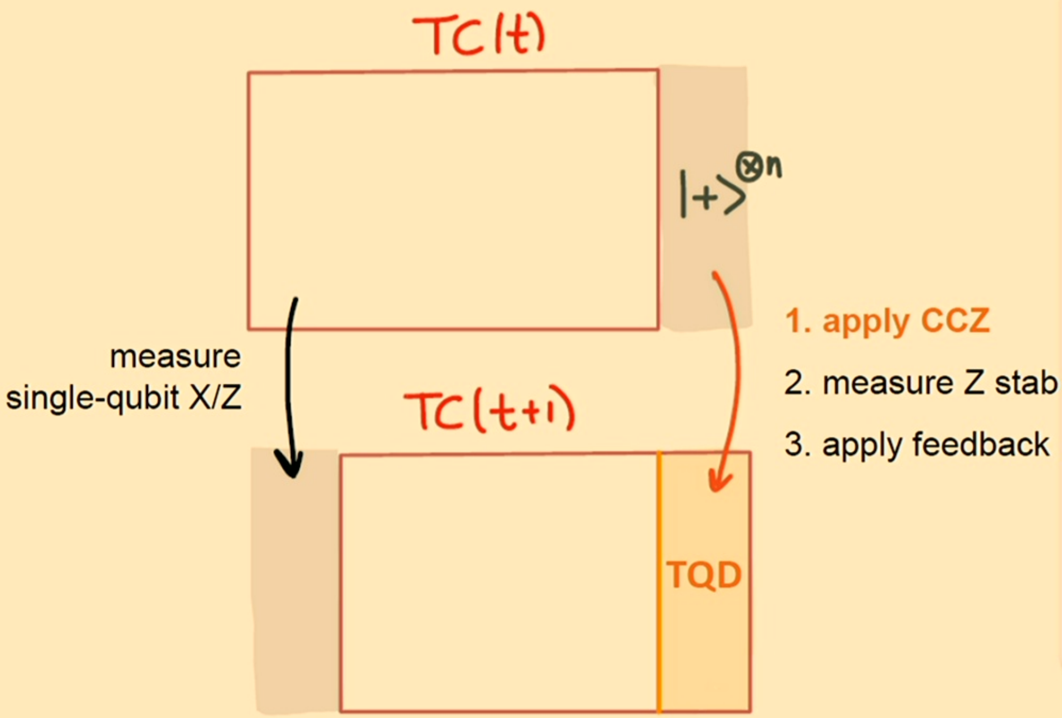


1. apply CCZ (in a particular way)
2. measure Z stabilizers
3. apply feedback

CCZ gate

Assume a planar lattice of qubits [3 qubits per unit cell].

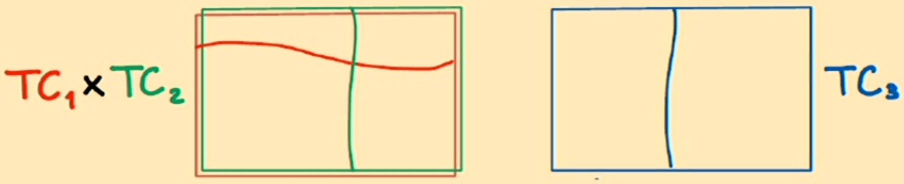
We can move copies of the toric code and turn them into twisted quantum double (TQD)



Tools:

- **Pauli stabilizer measurements**
- **single-qubit Pauli measurements**
- **local non-Clifford gates**
- just-in-time decoding

CCZ gate



$CCZ|abc\rangle = (-1)^{abc}|abc\rangle$

i.e.

$CCZ|001\rangle = |001\rangle$

$CCZ|111\rangle = -|111\rangle$

CCZ gate

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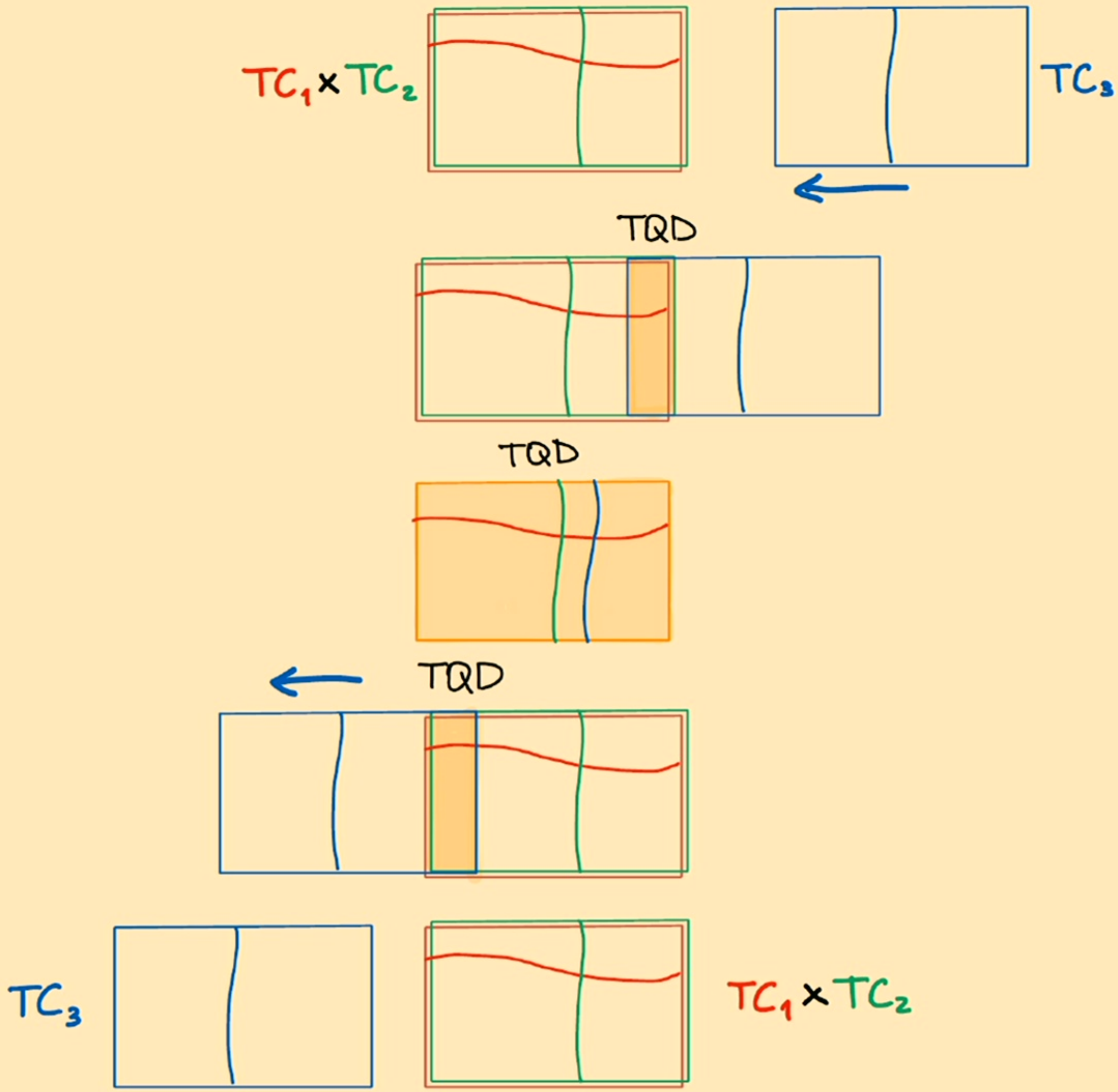
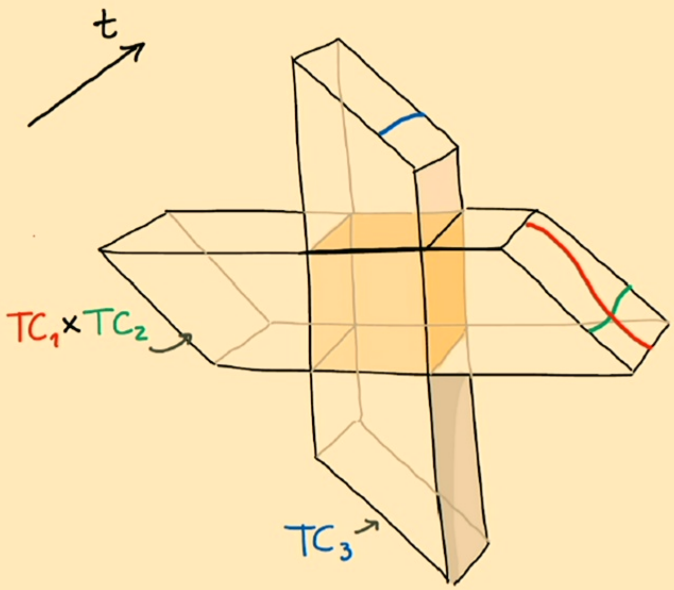
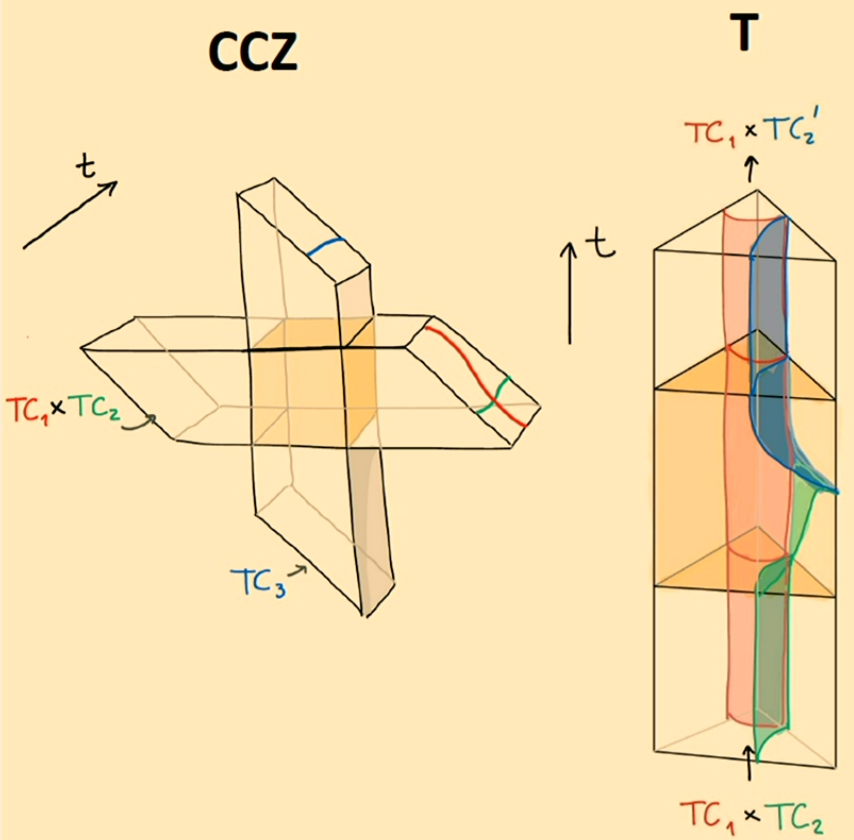


Earlier proposed protocol

Spacetime picture → 3D slice-by-slice version [Brown '18]



Other gates



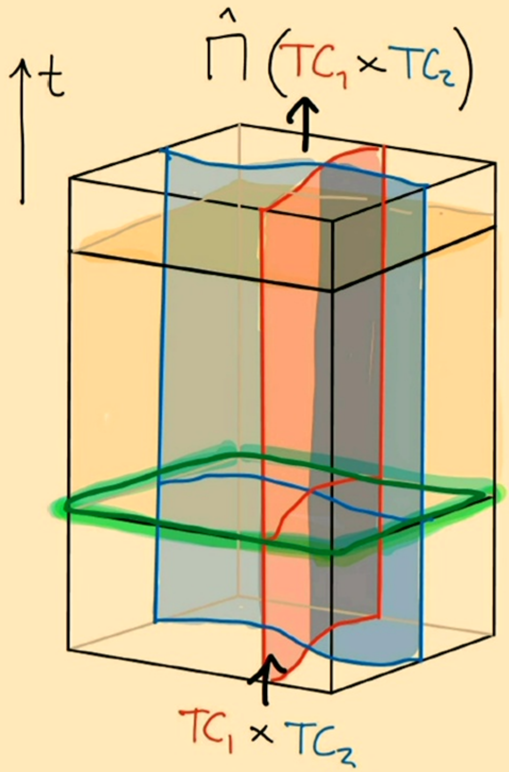
Use the spacetime picture to:

- determine compatible configurations of DWs/boundaries/corner defects
- compatible w/ inputs from TC phase
- arrange resulting logical action

One can get:

- **Phase gates of the 3rd level of Clifford hierarchy (T, CS, CCZ)**

Fault-tolerant state preparation



Use the spacetime picture to:

- determine compatible configurations of DWs/boundaries/corner defects
- compatible w/ inputs from TC phase
- arrange resulting logical action

One can get:

- Phase gates of the 3rd level of Clifford hierarchy (T, CS, CCZ)
- **respective magic states**

Fault-tolerance

Recall that $\mathcal{D}_{\omega_{III}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathcal{D}(D_4)$

D_4 is a nilpotent group ⇒ non-abelian anyons are as close to abelian as it gets.
[Dauphinais, Poulin '16]

Can be error-corrected using Pauli measurements and feedback only!

$\mathrm{m_r}$: string of $X - CZ_{gb}$

- measure plaquette stabilizers (Z)
- apply X correction

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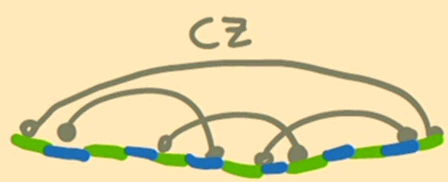
"fluctuating" green/blue charges (CZ/S) left
[Yoshida '15]



- measure star stabilizers (X-CZ)

Fault-tolerance

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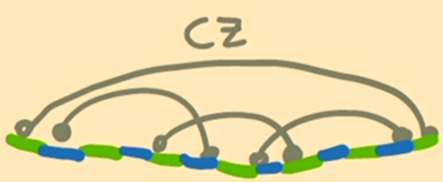

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'collapsed' leftover configuration of abelian g, b charges

- record locations; correct at the end (abelian!)

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just-in-time decoding

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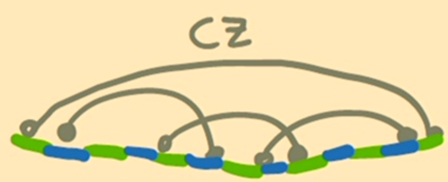

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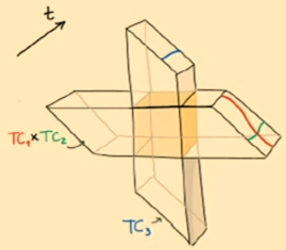
Outlook

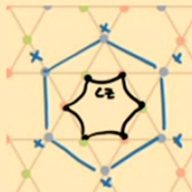
Spacetime approach to error correction / fault tolerance

[also: Bauer ‘23, ’24; Delfosse Paetznick ‘23]



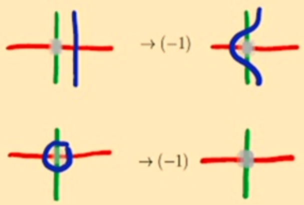
Dynamic codes/protocols beyond Pauli

[also: Davydova et al ’23, Bauer ‘23,‘24]



New approaches to quantum computation with non-abelian phases

[also: Cong, Cheng, Wang ’17, Laubscher, Loss, Wootton’ 18]



$$\omega_{III}(a,b,c) = (-1)^{\int a \cup b \cup c}$$

toric code `borrows’ non-abelian statistics

Outlook

Spacetime approach to error correction / fault tolerance

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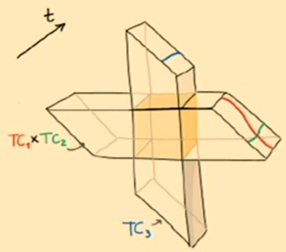

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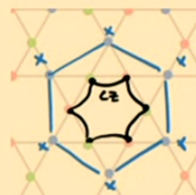

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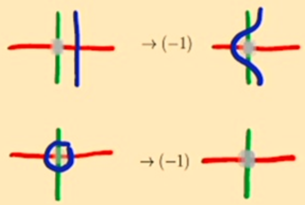

$\omega_{III}(a,b,c) = (-1)^{\int a \cup b \cup c}$

toric code `borrows' non-abelian statistics

Future:

- More protocols, beyond $\mathbb{Z}_2^n$
- More non-Abelian dynamic/Floquet codes

- General picture involving non-invertible symmetries; non-unitary logical action

- General theory for spacetime fault-tolerance. Further bounds on gates (aka Bravyi-Koenig)

- Beyond manifolds; qLDPC?

Thank you for your attention!