Title: Typical eigenstate entanglement entropy as a diagnostic of quantum chaos and integrability

Speakers: Marcos Rigol

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Abstract: Quantum-chaotic systems are known to exhibit eigenstate thermalization and to generically thermalize under unitary dynamics. In contrast, quantum-integrable systems exhibit a generalized form of eigenstate thermalization and need to be described using generalized Gibbs ensembles after equilibration. I will discuss evidence that the entanglement properties of highly excited eigenstates of quantum-chaotic and quantum-integrable systems are fundamentally different. They both exhibit a typical bipartite entanglement entropy whose leading term scales with the volume of the subsystem. However, while the coefficient is constant and maximal in quantum-
chaotic models, in integrable models it depends on the fraction of the system that is traced out. The latter is typical in random Gaussian pure states. I will also discuss the nature of the subleading corrections that emerge as a consequence of the presence of abelian and nonabelian symmetries in such models.

Typical eigenstate entanglement entropy as a diagnostic of quantum chaos and integrability

Marcos Rigol

Department of Physics
The Pennsylvania State University

Physics of Quantum Information
Perimeter Institute for Theoretical Physics

Tutorial: Bianchi, Hackl, Kieburg, MR & Vidmar,
Volume-law entanglement entropy of typical pure quantum states,
PRX Quantum **3**, 030201 (2022).

Outline

1. Introduction
   - ETH, integrability & dynamics
   - Eigenstate entanglement entropy
2. Entanglement entropy
   - Quantum-chaotic interacting Hamiltonians
   - Quadratic fermionic Hamiltonians
3. Summary

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Eigenstate thermalization hypothesis

Eigenstate thermalization hypothesis

Srednicki, J. Phys. A **32**, 1163 (1999), D'Alessio *et al.*, Adv. Phys. **65**, 239 (2016).

$$O_{\alpha\beta} = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

where $E \equiv (E_\alpha + E_\beta)/2$, $\omega \equiv E_\alpha - E_\beta$, $S(E)$ is the thermodynamic entropy at energy E, and $R_{\alpha\beta}$ is a random number with zero mean and unit variance.

Spin-1/2 XXZ chain with nearest- and next-nearest-neighbor interactions

$$\hat{H} = \sum_{i=1}^{L} \left[\frac{1}{2} \left(\hat{S}_i^+ \hat{S}_{i+1}^- + \text{H.c.} \right) + \Delta \hat{S}_i^z \hat{S}_{i+1}^z \right] + \lambda \sum_{i=1}^{L} \left[\frac{1}{2} \left(\hat{S}_i^+ \hat{S}_{i+2}^- + \text{H.c.} \right) + \frac{1}{2} \hat{S}_i^z \hat{S}_{i+2}^z \right].$$

Symmetries: $U(1)$ and translation
(Z_2 at $M^z = 0$ and parity at $k = 0, \pi$).

Integrable for $\lambda = 0$.
(It can be solved exactly using the Bethe ansatz.)

Intensive local operators

$$\hat{U}_n = \frac{1}{L} \sum_{i=1}^{L} \hat{S}_i^z \hat{S}_{i+1}^z \qquad \& \qquad \hat{K}_{nn} = \frac{1}{L} \sum_{i=1}^{L} \left(\hat{S}_i^+ \hat{S}_{i+2}^- + \text{H.c.} \right).$$

LeBlond, Mallayya, Vidmar & MR, PRE **100**, 062134 (2019).

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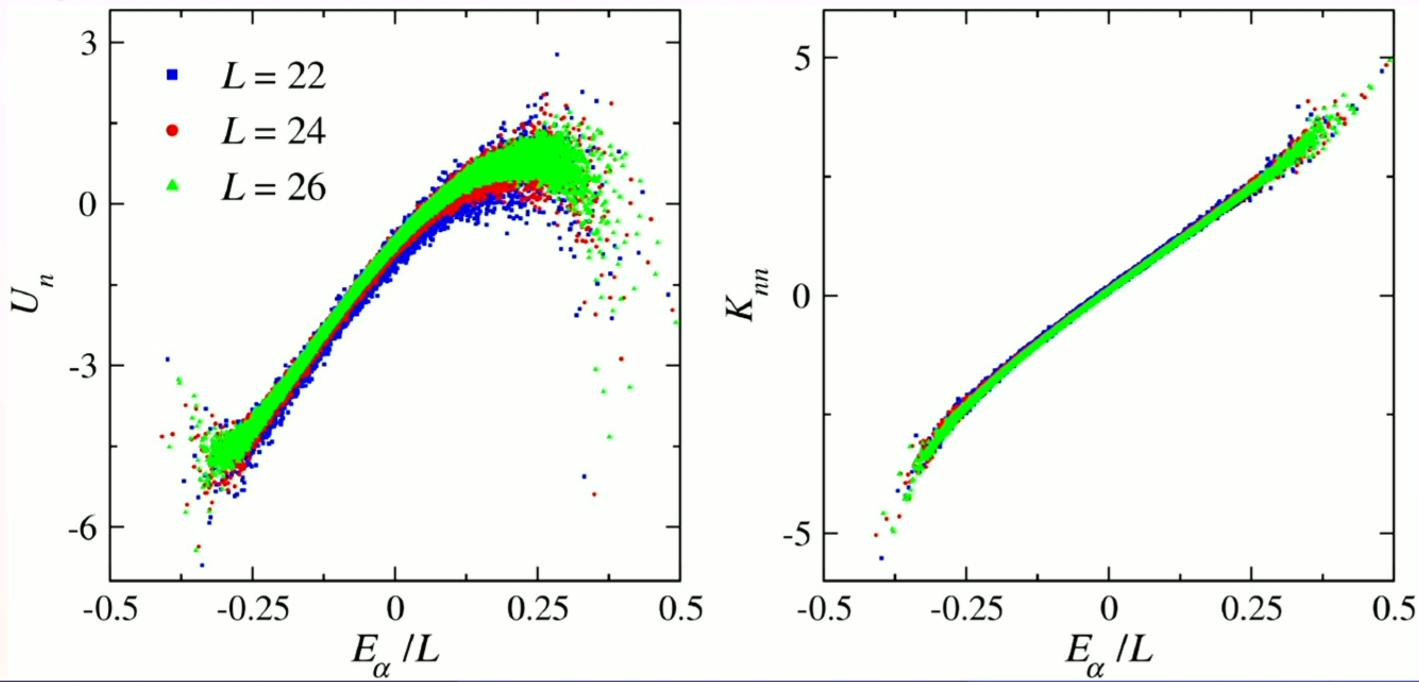
Diagonal matrix elements ($\Delta = 0.55$ and $\lambda = 1.0$)



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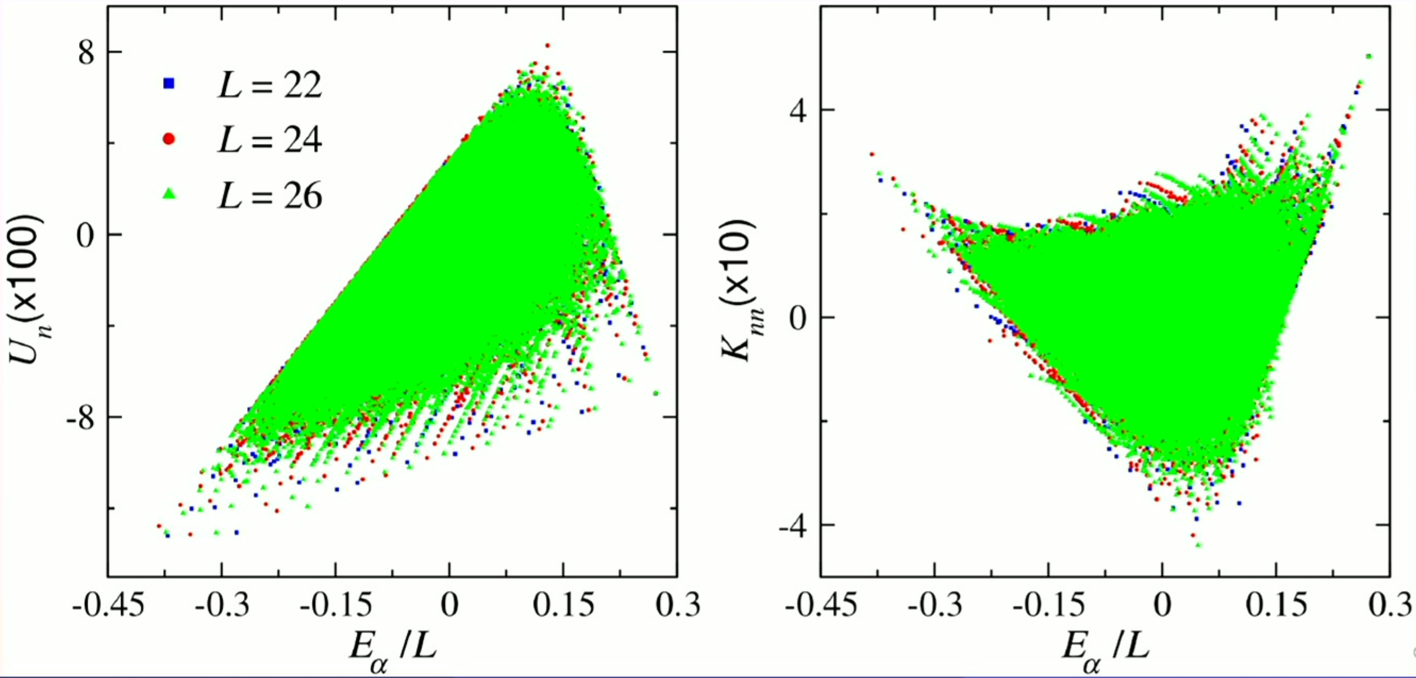

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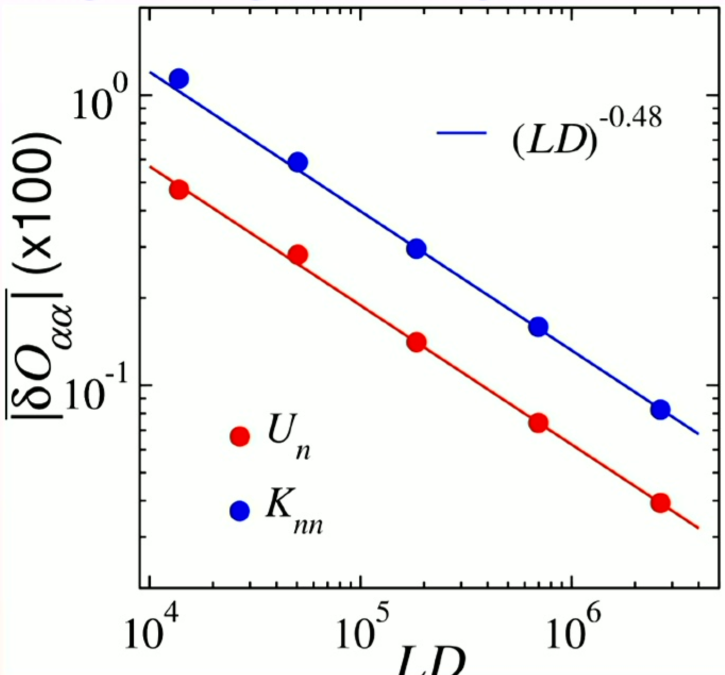
Scaling of the eigenstate to eigenstate fluctuations ($\delta O_{\alpha\alpha} = O_{\alpha\alpha} - O_{\alpha+1\alpha+1}$)



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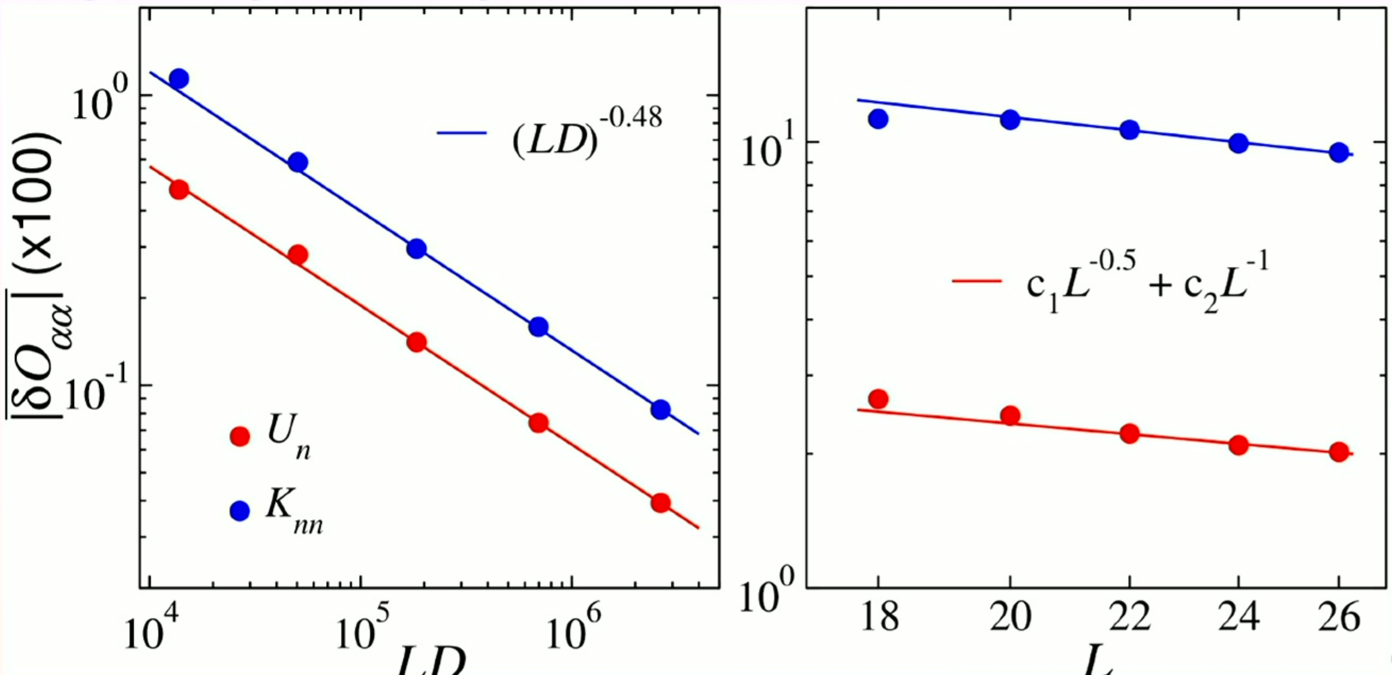


Bose-Fermi mapping for the $\lambda = \Delta = 0$ case

Hard-core boson Hamiltonian

$$\hat{H} = -\sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right).$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0.$$

$$\Downarrow$$

Map to spinless fermions (Jordan-Wigner transformation)

$$\hat{b}_i^\dagger = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \hat{b}_i = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

$$\Downarrow$$

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right).$$

MR and Muramatsu, PRA **70**, 031603(R) (2004); PRL **93**, 230404 (2004).

Bose-Fermi mapping for the $\lambda = \Delta = 0$ case

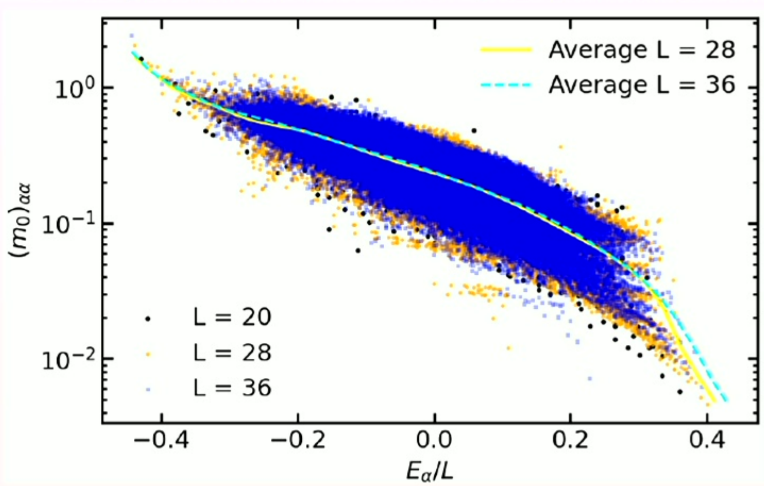
Hard-core boson Hamiltonian

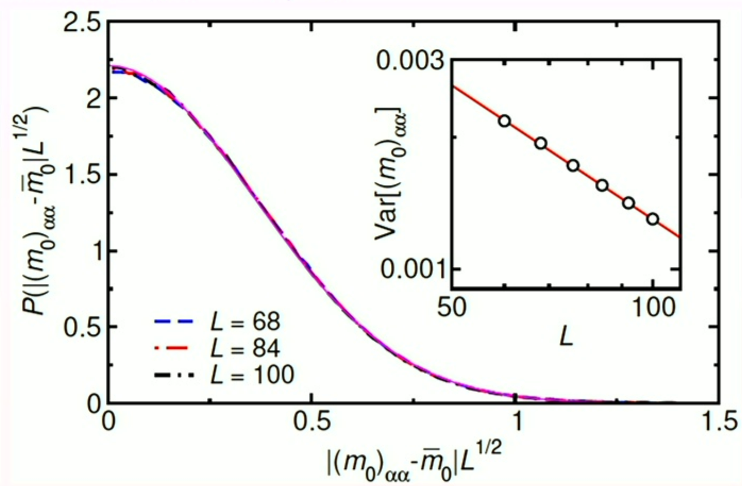
$$\hat{H} = -\sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right).$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0.$$

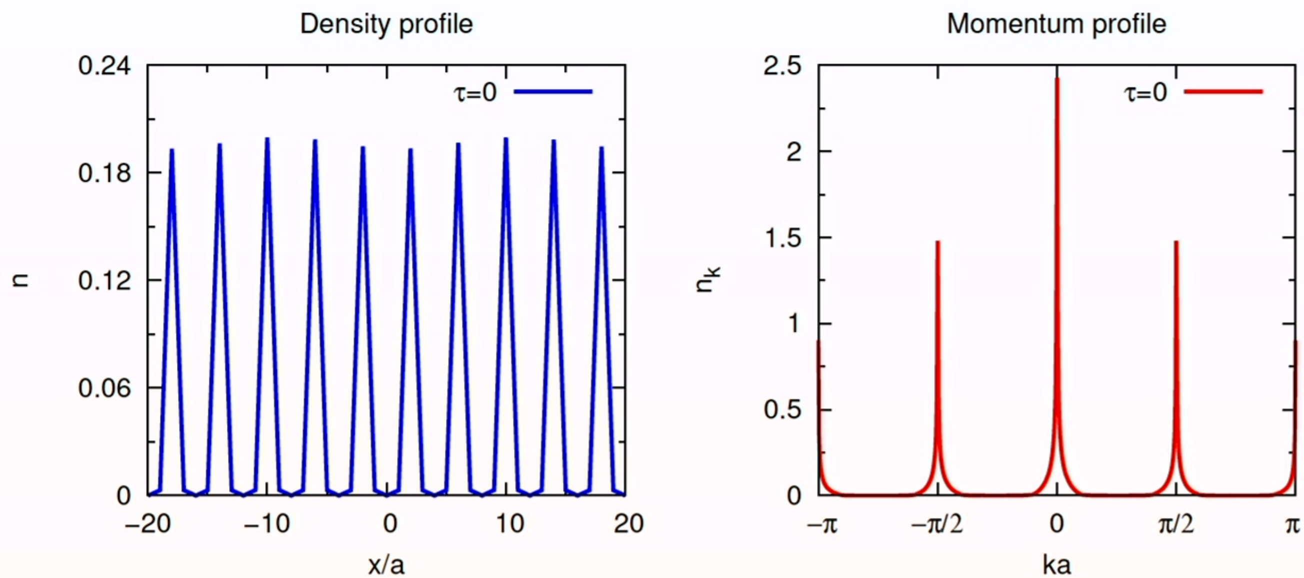
Occupation of the zero quasi-momentum mode ($N = L/4$)



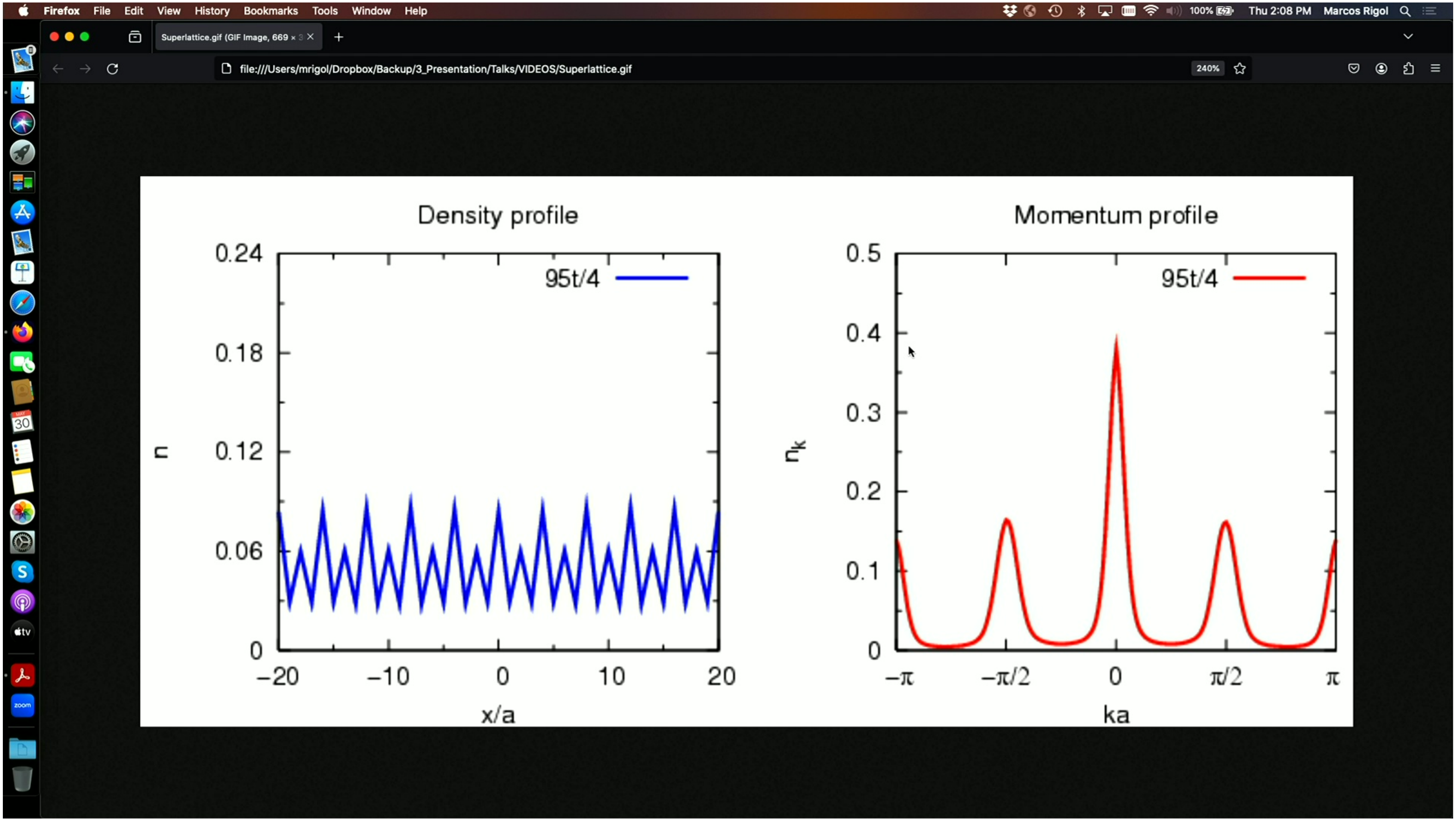


Zhang, Vidmar & MR, PRE **106**, 014132 (2022).

Relaxation dynamics in an integrable system



MR, Dunjko, Yurovsky & Olshanii, PRL **98**, 050405 (2007).


Density profile
95t/4
0.24
0.18
0.12
0.06
0
n
−20
−10
0
10
20
x/a
Momentum profile
95t/4
0.5
0.4
0.3
0.2
0.1
0
n_k
−π
−π/2
0
π/2
π
ka

Bose-Fermi mapping for the $\lambda = \Delta = 0$ case

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \, \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

Set of conserved quantitites

(Occupations of the single-particle energy eigenstates of the noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^\dagger |0\rangle = E_m \hat{\gamma}_m^\dagger |0\rangle$$
$$\left\{ \hat{I}_m \right\} = \{ \hat{\gamma}_m^\dagger \hat{\gamma}_m \}$$

Bose-Fermi mapping for the $\lambda = \Delta = 0$ case

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \, \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

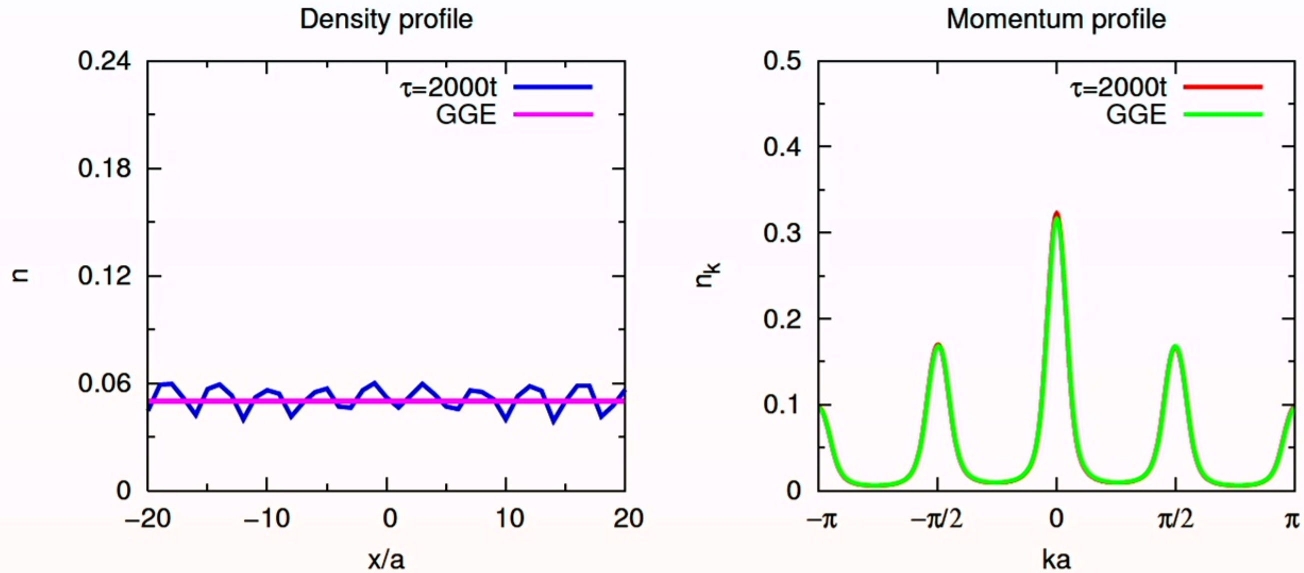
Generalized Gibbs ensemble

$$\hat{\rho}_{\text{GGE}} = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m \right], \quad \text{where} \quad Z_c = \text{Tr}\left\{ \exp\left[-\sum_m \lambda_m \hat{I}_m \right] \right\}$$

The Lagrange multipliers are determined from

$$\text{Tr}\left\{ \hat{I}_m \hat{\rho}_{\text{GGE}} \right\} = \langle \hat{I}_m \rangle_{\tau=0} \quad \Longrightarrow \quad \lambda_m = \ln\left[\frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}} \right]$$

Generalized Gibbs ensemble (GGE)



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von Neumann and Renyi entanglement entropies

Let $|\psi\rangle$ be a state ket, with density matrix $\rho^\psi = |\psi\rangle\langle\psi|$. The reduced density matrix

$\hat{\rho}^\psi_A = \mathrm{Tr}_{\bar{A}}(\rho^\psi)$, A is the subsystem of interest and $\bar{A}$ is its complement

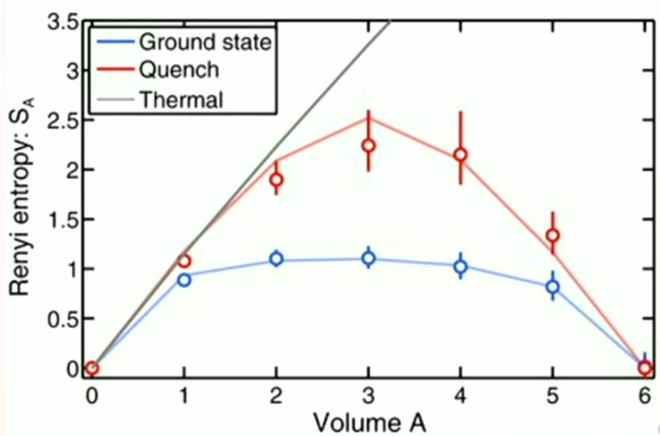
The von Neumann and n-Renyi entanglement entropies are defined as:

$$S^\psi_{\text{vN}} = -\mathrm{Tr}[\hat{\rho}^\psi_A \ln \hat{\rho}^\psi_A] \quad \text{and} \quad S^\psi_n = \frac{1}{1-n} \ln \mathrm{Tr}[(\hat{\rho}^\psi_A)^n]$$

Ground-state entanglement entropies (area laws, quantum phase transitions)

Srednicki'93; Osterloh, Amico, Falci & Fazio'02; Osborne & Nielsen'02;
Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04, . . . , Eisert, Cramer & Plenio, RMP'10

S_2 has been measured in experiments with ultracold quantum gases



Kaufman et al. (Greiner's group), Science **353**, 794 (2016).

Typicality (uniform distribution in the unit sphere)

Average (vN) entanglement entropy of subsystems of random pure states

$$\bar{S}_{\rm ran} \simeq \ln \mathcal{D}_{\rm A} - (1/2)\mathcal{D}_{\rm A}^2/\mathcal{D},$$

for $1 \ll \mathcal{D}_{\rm A} \leq \sqrt{\mathcal{D}}$.
Page, PRL **71**, 1291 (1993).

Since $\ln \mathcal{D}_{\rm A} = V_{\rm A} \ln 2$, and the variance is $\propto e^{-cV_{\rm A}}$, we say that typical pure states exhibit a volume law entanglement. They are (nearly) maximally entangled!
(the correction is exponentially small for $V_{\rm A} < V/2$. It is $1/2$ for $V_{\rm A} = V/2$).

Spin-1/2 XYZ and XXZ chains with integrability breaking terms:

$$\hat{H}_{\rm XYZ} = \hat{H}_{\rm n} + J'\,\hat{H}_{\rm nn} + h\,\hat{H}_f,$$

$$\hat{H}_{\rm n} = \sum_\ell [(1-\eta)\hat{s}^x_\ell \hat{s}^x_{\ell+1} + (1+\eta)\hat{s}^y_\ell \hat{s}^y_{\ell+1} + \Delta \hat{s}^z_\ell \hat{s}^z_{\ell+1}],$$

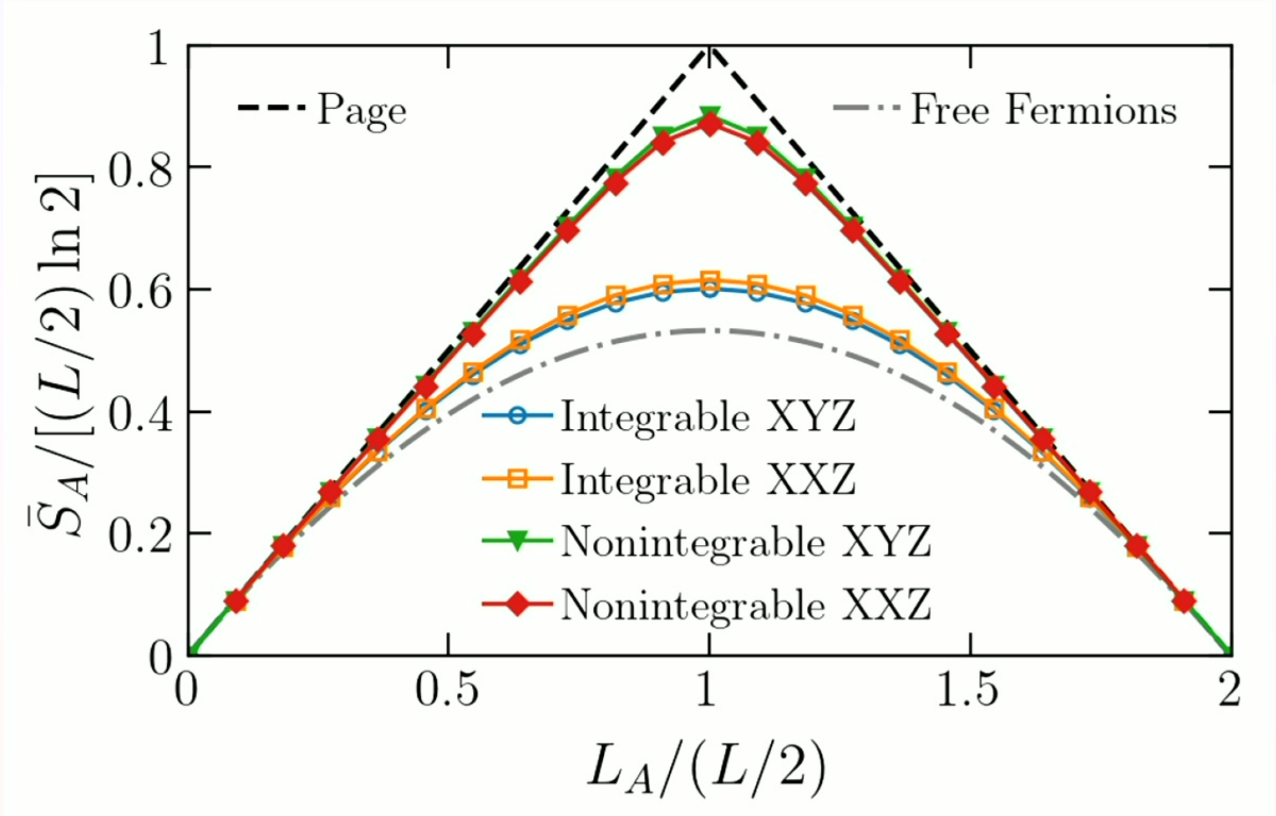
$$\hat{H}_{\rm nn} = \sum_\ell [(1-\eta)\hat{s}^x_\ell \hat{s}^x_{\ell+2} + (1+\eta)\hat{s}^y_\ell \hat{s}^y_{\ell+2} + \Delta' \hat{s}^z_\ell \hat{s}^z_{\ell+2}],$$

$$\hat{H}_f = \sum_\ell (\hat{s}^z_\ell + h' \hat{s}^x_\ell),$$

$J' = h = 0$: Integrable ($\Delta = 0$ mappable to a quadratic fermionic model)
$\eta = h = 0$: $U(1)$ symmetry

Entanglement entropy: Integrability vs quantum chaos

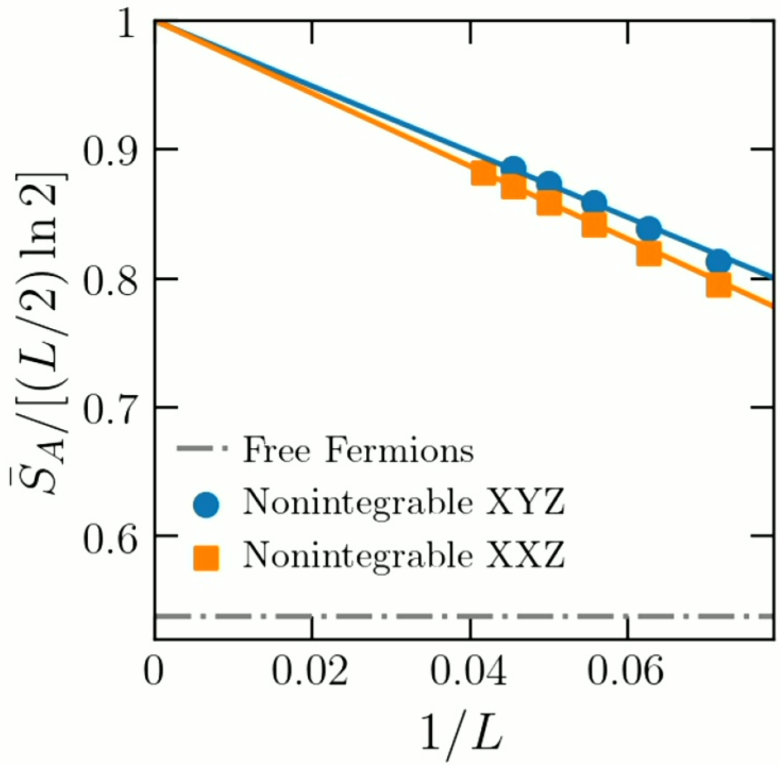
Average entanglement entropy ($L = 22$)



Kliczkowski, Świętek, Vidmar & MR, PRE **107**, 064119 (2023).
Świętek, Kliczkowski, Vidmar & MR, PRE **109**, 024117 (2024).

Entanglement entropy: Integrability vs quantum chaos

Scaling of the average entanglement entropy at $L_A = L/2$



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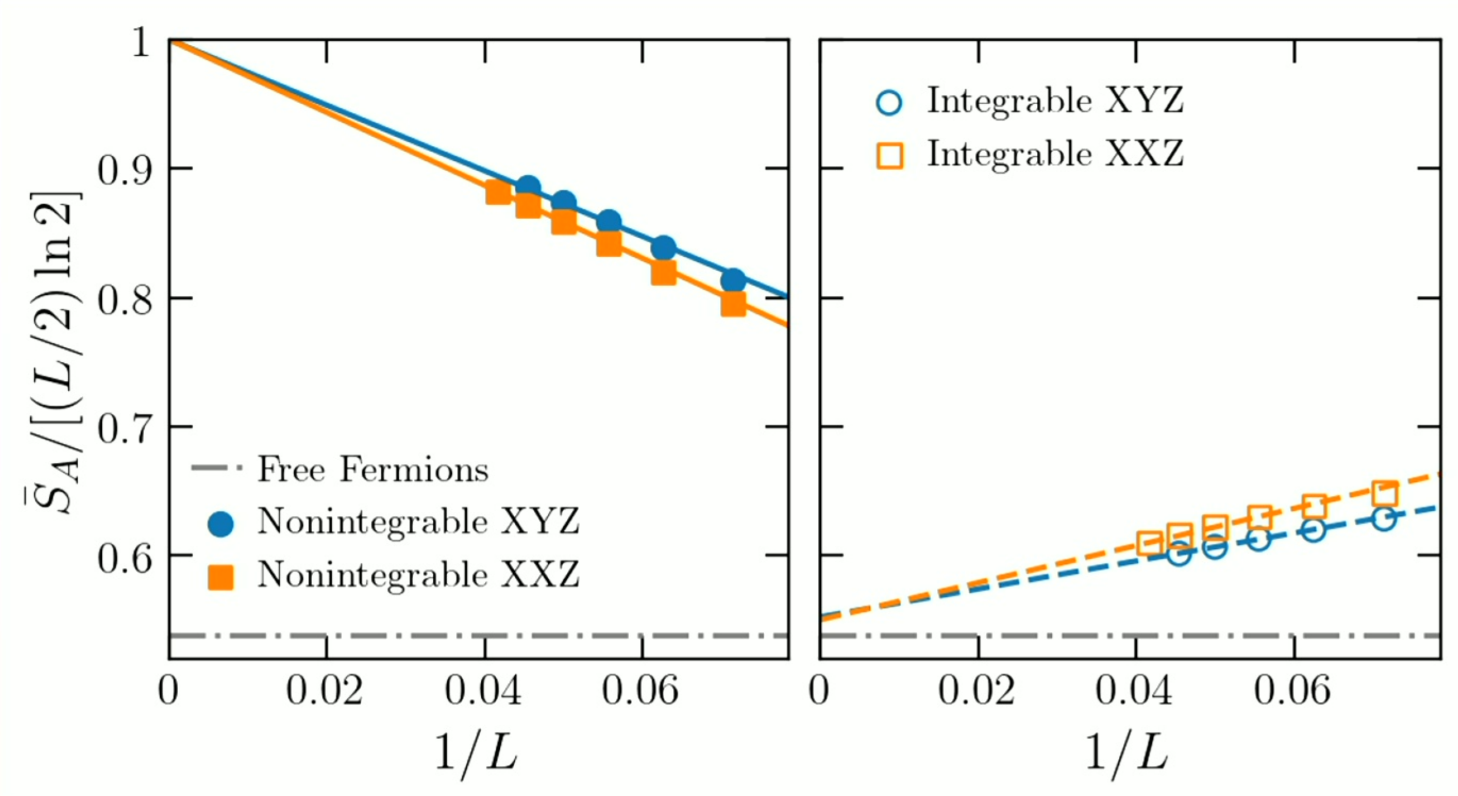

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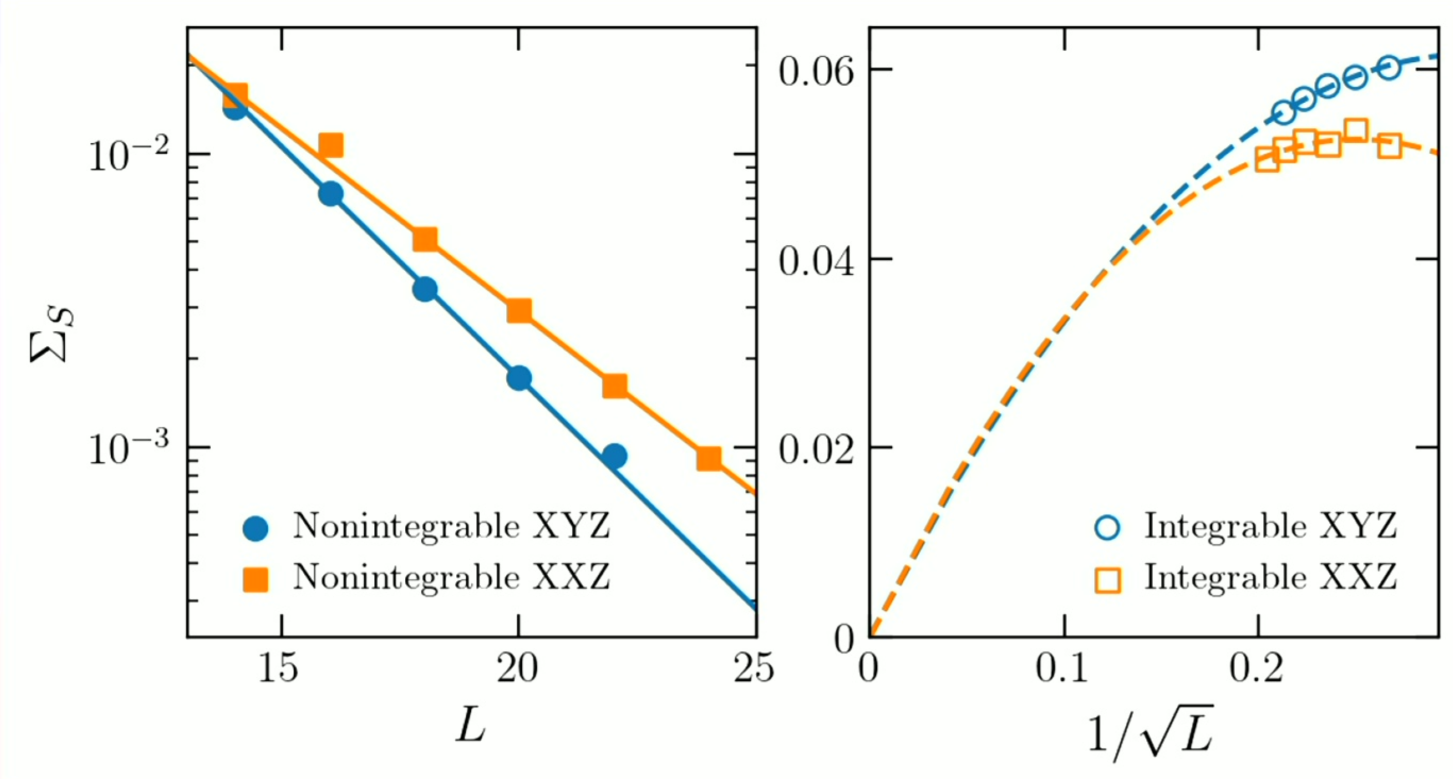
Entanglement entropy: Integrability vs quantum chaos

Scaling of $\Sigma_S = \sqrt{\overline{(S_A^\alpha - \bar{S}_A^\alpha)^2}}/(L_A \ln 2)$ at $L_A = L/2$



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S_{vN} in eigenstates of quantum chaotic Hamiltonians

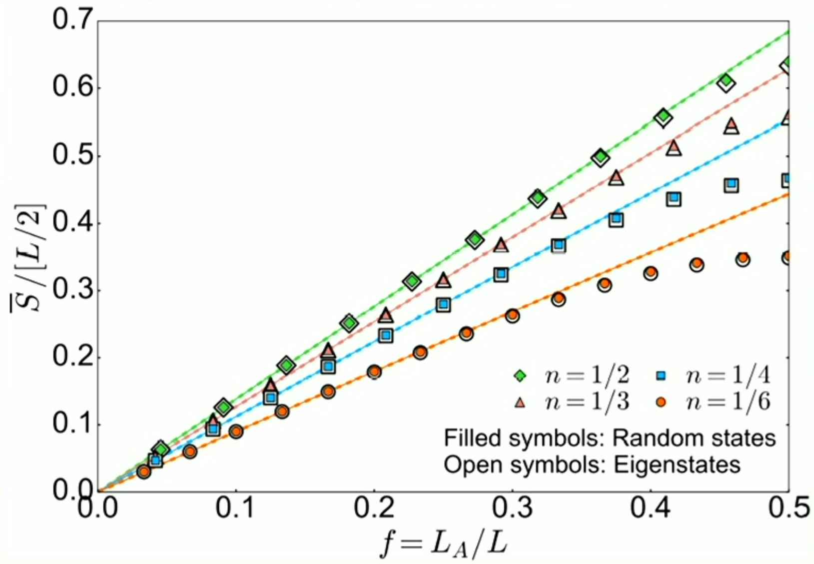
Hard-core bosons in one dimension ($t = t' = 1$ and $V = V' = 1.1$)

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

Random canonical states (z_j: normally distributed real random number)

$$|\psi_N\rangle = \frac{1}{\sqrt{\mathcal{D}_N}} \sum_{j=1}^{\mathcal{D}_N} z_j |j\rangle, \qquad |j\rangle \text{ are base kets for } N \text{ particles in the site basis}$$

$\bar{S}_{\text{vN}}$ vs subsystem fraction $f = L_A/L$ [$L = 22$ (n=1/2), 24 (n=1/3,1/4) and 30 (n=1/6)]



Vidmar & MR, PRL **119**, 220603 (2017).

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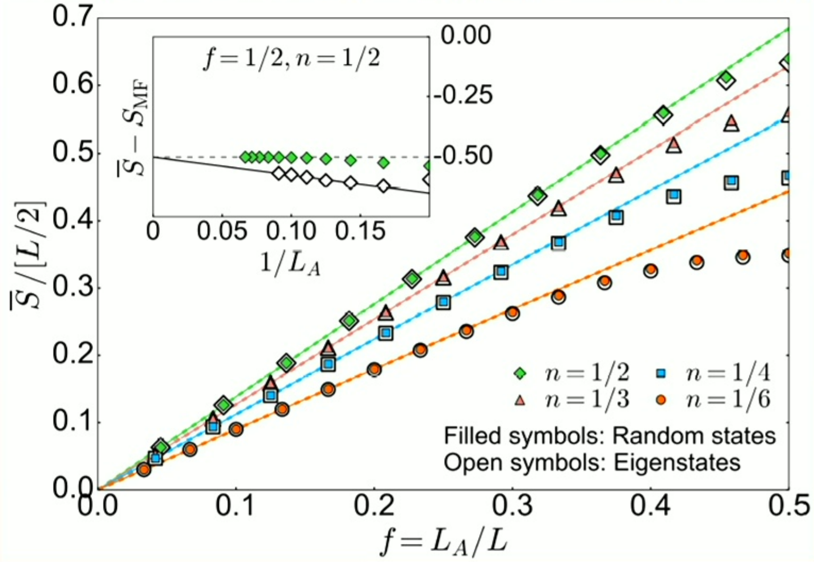

Vidmar & MR,
PRL **119**, 220603 (2017).

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To calculate $\bar{S} = -\overline{\mathrm{Tr}\{\hat{\rho}_A \ln(\hat{\rho}_A)\}}$, we define

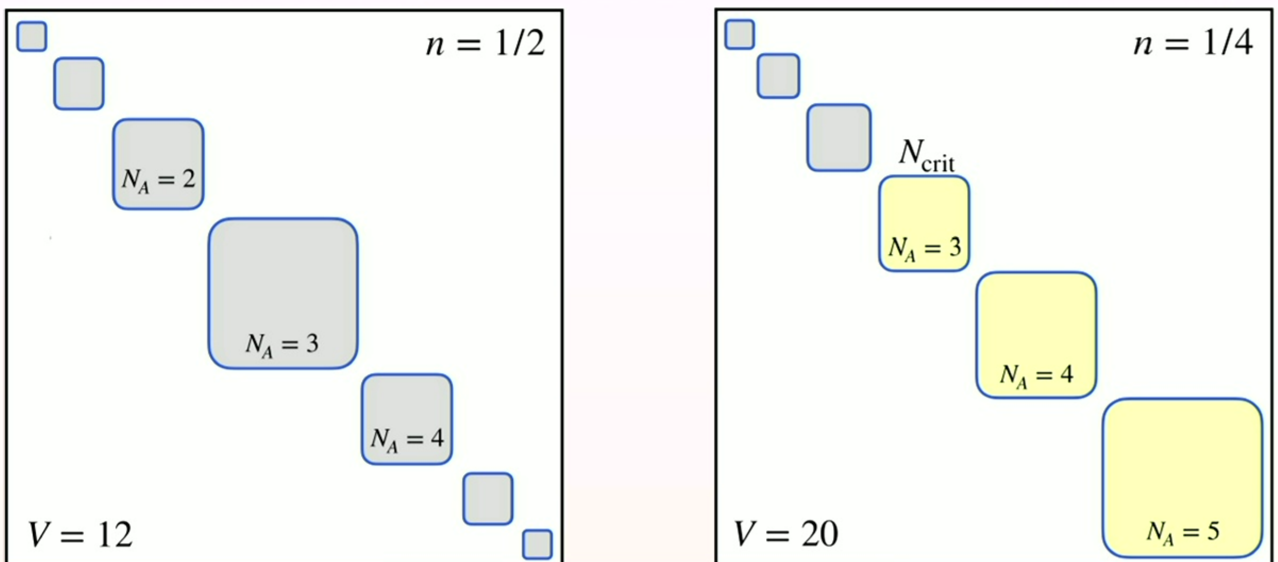
$$\hat{M} = (\hat{\bar{\rho}}_A)^{-1}(\hat{\rho}_A - \hat{\bar{\rho}}_A)\,,$$

so that

$$\bar{S} = -\overline{\mathrm{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\ln\left[\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\right]\right\}} = S_{\mathrm{MF}} + \bar{S}_0 + \bar{S}_{\mathrm{fluct}}$$

$$S_{\mathrm{MF}} = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A \ln \hat{\bar{\rho}}_A\right\},\ S_0 = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A \hat{M} \ln \hat{\bar{\rho}}_A\right\},\ S_{\mathrm{fluct}} = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\ln(\hat{I}+\hat{M})\right\}$$

What is different for fixed N? [tutorial discussion in PRX Quantum **3**, 030201 (2022)]



$S_{\rm vN}$ in eigenstates of quantum chaotic Hamiltonians

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$$\bar{S} = -\overline{\mathrm{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\ln\left[\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\right]\right\}} = S_{\rm MF} + \bar{S}_0 + \bar{S}_{\rm fluct}$$

$$S_{\rm MF} = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A \ln \hat{\bar{\rho}}_A\right\},\ S_0 = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A \hat{M} \ln \hat{\bar{\rho}}_A\right\},\ S_{\rm fluct} = -\mathrm{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I}+\hat{M})\ln(\hat{I}+\hat{M})\right\}$$

For large systems, using Stirling's approximation and replacing $\sum_{N_A} \to \int \mathrm{d}n$:

$$S^{*}_{\rm MF} = -L_A\left[n\ln n + (1-n)\ln(1-n)\right] + \frac{f+\ln(1-f)}{2}$$

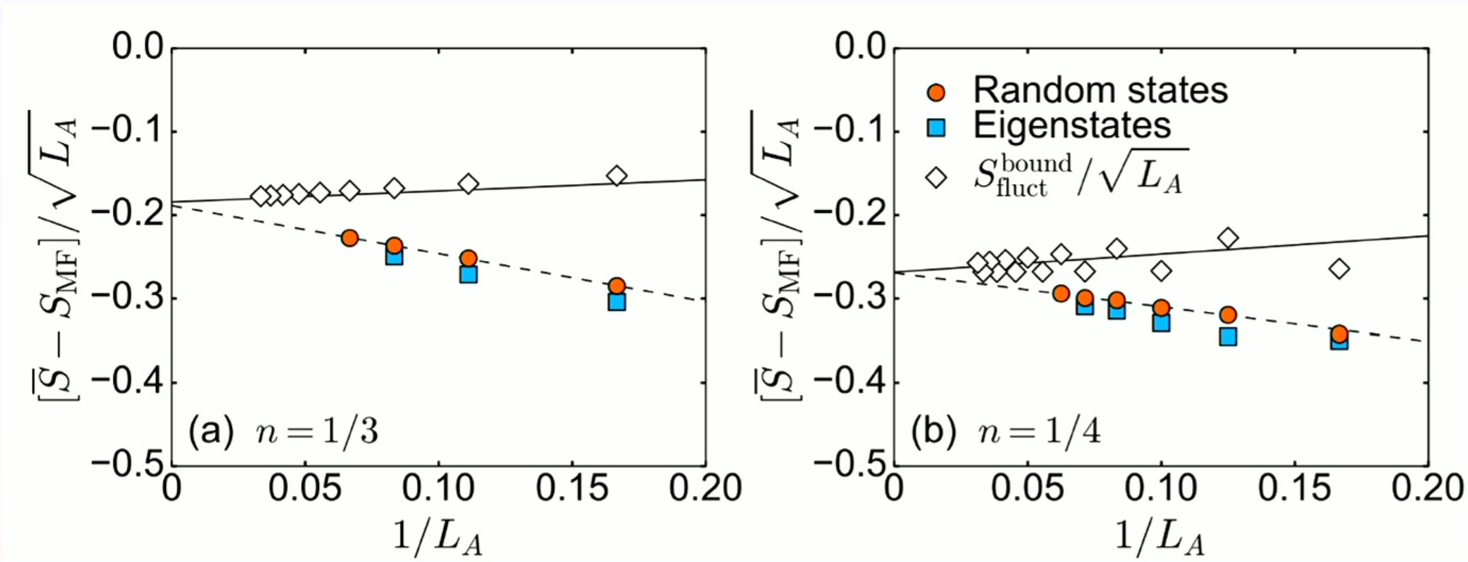
Leading order: Garrison & Grover, PRX **8**, 021026 (2018).

One can prove that $\bar{S}_{\rm fluct} \le S^{\rm bound}_{\rm fluct}$, where for large systems and $f=1/2$

$$S^{\rm bound*}_{\rm fluct} \simeq -\sqrt{L_A}\,\ln\left(\frac{1-n}{n}\right)\sqrt{\frac{n(1-n)}{\pi}} + \frac{1}{\sqrt{L_A}}\,\frac{(1-2n)}{3\sqrt{\pi n(1-n)}}$$

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Fluctuation contribution to the average entanglement entropy for $f = 1/2$



Vidmar & MR, PRL **119**, 220603 (2017).

Energy conservation also introduces a $\sqrt{N}$ correction at $f = 1/2$

Murthy & Srednicki, PRE **100**, 022131 (2019).

S_{vN} for the average over pure states with N particles

One needs to evaluate

$$\langle S_A \rangle_N = \sum_{N_A=0}^{\min(N,V_A)} \frac{d_A d_B}{d_N}\Big(\langle S_A \rangle + \Psi(d_N+1) - \Psi(d_A d_B+1)\Big),$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and $\langle S_A \rangle$ is Page's result.
Bianchi & Donà, PRD **100**, 105010 (2019).

Leading order and first subleading corrections:
[tutorial derivation in PRX Quantum **3**, 030201 (2022)]

$$\langle S_A \rangle_N = f\,V[(n-1)\ln(1-n) - n\ln(n)] - \sqrt{V}\sqrt{\frac{n(1-n)}{2\pi}}\left|\ln\left(\frac{1-n}{n}\right)\right|\delta_{f,\frac{1}{2}} + \frac{f+\ln(1-f)}{2} - \frac{1}{2}\delta_{f,\frac{1}{2}}\delta_{n,\frac{1}{2}} + o(1),$$

for $f \leq \frac{1}{2}$. ($f > \frac{1}{2}$ can be obtained using $f \to 1-f$ symmetry.)

For Hamiltonians, the $O(1)$ term depends on the fraction of midspectrum eigenstates used to compute the average entanglement entropy:
Kliczkowski, Świętek, Vidmar & MR, PRE **107**, 064119 (2023).

S_{vN} for the average over pure states with N particles

One needs to evaluate

$$\langle S_A \rangle_N = \sum_{N_A=0}^{\min(N,V_A)} \frac{d_A d_B}{d_N} \big(\langle S_A \rangle + \Psi(d_N+1) - \Psi(d_A d_B + 1) \big),$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and $\langle S_A \rangle$ is Page's result.
Bianchi & Donà, PRD **100**, 105010 (2019).

Leading order and first subleading corrections:
[tutorial derivation in PRX Quantum **3**, 030201 (2022)]

$$\langle S_A \rangle_N = f\,V[(n-1)\ln(1-n) - n\ln(n)] - \sqrt{V}\sqrt{\frac{n(1-n)}{2\pi}}\left|\ln\left(\frac{1-n}{n}\right)\right|\delta_{f,\frac{1}{2}} + \frac{f+\ln(1-f)}{2} - \frac{1}{2}\delta_{f,\frac{1}{2}}\delta_{n,\frac{1}{2}} + o(1),$$

for $f \leq \frac{1}{2}$. ($f > \frac{1}{2}$ can be obtained using $f \to 1-f$ symmetry.)

For Hamiltonians, the $O(1)$ term depends on the fraction of midspectrum eigenstates used to compute the average entanglement entropy:
Kliczkowski, Świętek, Vidmar & MR, PRE **107**, 064119 (2023).

Generalization to arbitrary models with U(1) symmetry:
Cheng, Patil, Zhang, MR & Hackl, arXiv:2310.19862. (Cheng's poster)

Outline

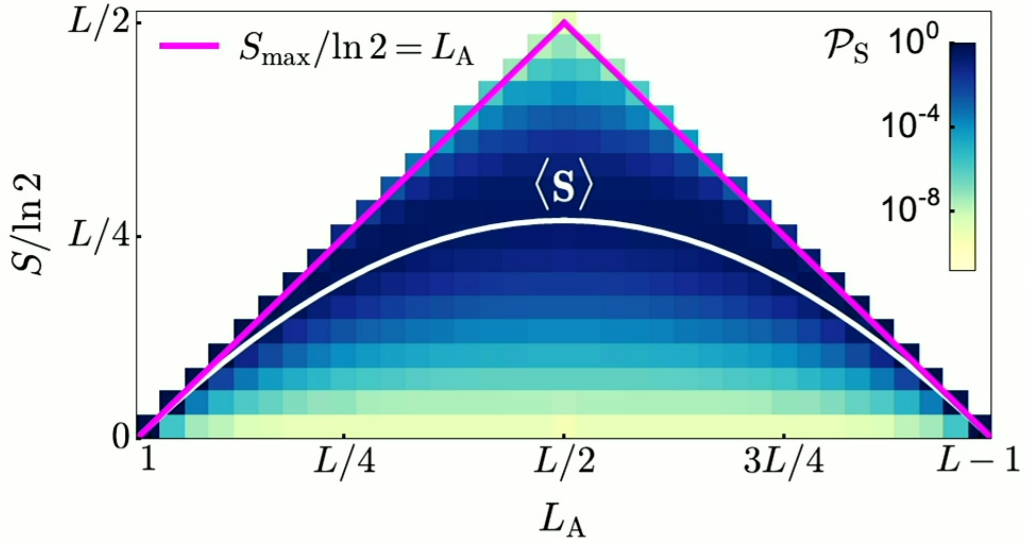
1. Introduction
   - ETH, integrability & dynamics
   - Eigenstate entanglement entropy

2. Entanglement entropy
   - Quantum-chaotic interacting Hamiltonians
   - Quadratic fermionic Hamiltonians

3. Summary

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free fermions in 1D ($\hat{H} = -\sum_{i=1}^{L}(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.})$, $L = 36$ sites)



Vidmar, Hackl, Bianchi & MR, PRL **119**, 020601 (2017).
Earlier results in 2D: Storms & Singh, PRE **89**, 012125 (2014).

The entanglement entropy of an eigenstate of any quadratic fermionic Hamiltonian

$$S_{\text{vN}} = -\text{Tr}\left\{\left(\frac{\mathbb{1} + [F]_A}{2}\right)\ln\left(\frac{\mathbb{1} + [F]_A}{2}\right)\right\},$$

where $[F]_A$ is the $2V_A \times 2V_A$ matrix with $i, j \in A$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

We are interested in the average over all eigenstates

$$\langle S_{\text{vN}} \rangle = V_{\text{A}} \ln 2 - \sum_{n=1}^{\infty} \frac{\langle \text{Tr}[F]_{\text{A}}^{2n} \rangle}{4n(2n-1)}$$

$\langle \text{Tr}[F]_{\text{A}}^{2n} \rangle$ can be computed using correlation functions of the binomial distribution. Truncating the sum at $n = 1$ gives an upper bound, substituting $\text{Tr}[F]_{\text{A}}^{2n} \to \text{Tr}[F]_{\text{A}}^{2}$ gives a lower bound:

$$V_{\text{A}} \ln 2 - \frac{\langle \text{Tr}[F]_{\text{A}}^{2} \rangle}{2} \ln 2 \leq \langle S_{\text{vN}} \rangle \leq V_{\text{A}} \ln 2 - \frac{\langle \text{Tr}[F]_{\text{A}}^{2} \rangle}{4}$$

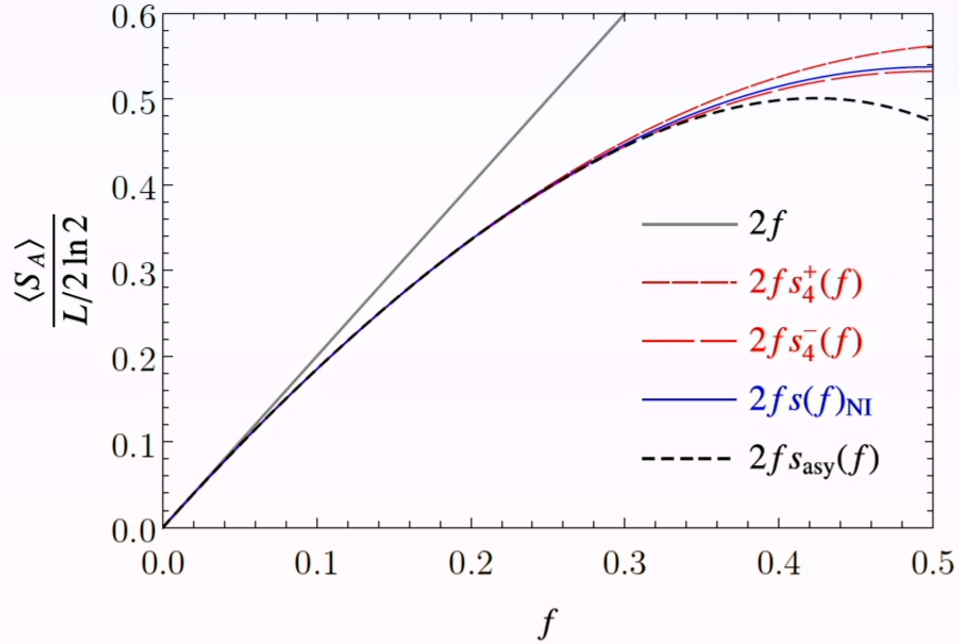
The “first order” (results for $n = 1$) bounds are universal in the thermodynamic limit

$$V_{\text{A}} \left(\ln 2 - \ln 2 \frac{V_{\text{A}}}{V} \right) \leq \langle S_{\text{vN}} \rangle \leq V_{\text{A}} \left(\ln 2 - \frac{1}{2} \frac{V_{\text{A}}}{V} \right)$$

Typical eigenstates have maximal entanglement entropy for $V_{\text{A}}/V \to 0$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free fermions in 1D ($\hat{H} = -\sum_{i=1}^{L}(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.})$, $L = 36$ sites)



Hackl, Vidmar, MR & Bianchi, PRB **99**, 075123 (2019).

Normalized width of the distribution of S_{vN} for eigenstates

$$\Delta = \frac{\sqrt{\langle S^2 \rangle - \langle S \rangle^2}}{L_{\text{A}} \ln 2}$$

vanishes as $1/\sqrt{L}$ or faster.

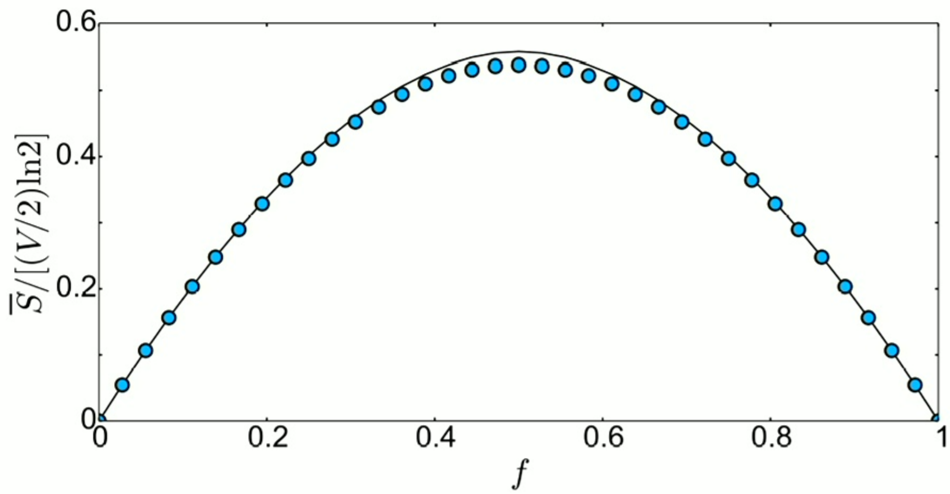
S_{vN} in quantum-chaotic quadratic Hamiltonians

Quadratic Hamiltonians: $\hat{H} = -\sum_{i,j=1}^{V} A_{ij} \hat{f}_i^\dagger \hat{f}_j$, where A_{ij} is drawn from GUE

$$\bar{S} = \left(1 - \frac{1 + f^{-1}(1-f)\ln(1-f)}{\ln 2}\right) V_A \ln 2 \,,$$

Łydżba, MR & Vidmar, PRL **125**, 180604 (2020).

Random ($V = \infty$) vs translationally invariant in 1D ($V \equiv L = 36$):



At $f = 1/2$, $\bar{S}/([V/2]\ln 2) = 0.5573$, while $\bar{S}_{\text{TI}}/([V/2]\ln 2) = 0.5378$

Bianchi, Hackl, Kieburg, MR & Vidmar, PRX Quantum **3**, 030201 (2022).

S_{vN} for the average over fermionic Gaussian states

Closed-form expression for the average over all Gaussian states

$$\langle S_A\rangle_{\mathrm{G}} = (V-\tfrac{1}{2})\Psi(2V) + (\tfrac{1}{2}+V_A-V)\Psi(2V-2V_A) + (\tfrac{1}{4}-V_A)\Psi(V) - \tfrac{1}{4}\Psi(V-V_A) - V_A\,,$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

Same leading order as for quantum-chaotic quadratic Hamiltonians, first sub. is $O(1)$.

Bianchi, Hackl & Kieburg, PRB **103**, L241118 (2021).

Closed-form expression for the verage over Gaussian states with fixed N

$$\langle S_A\rangle_{\mathrm{G},N} = 1-\tfrac{V_A}{V}(1+V)-\tfrac{NV_A}{V}\Psi(N)+V\Psi(V)+\tfrac{V_A(N-V)}{V}\Psi(V-N)+(V_A-V)\Psi(V-V_A+1),$$

for $V_A \le N \le V/2$. All other $\langle S_A\rangle_{\mathrm{G},N}$ can be obtained using symmetries.

Leading order and first subleading correction

$$\langle S_A\rangle_{\mathrm{G},N} \simeq V\,((f-1)\ln(1-f)+f\,[(n-1)\ln(1-n)-n\ln n-1]) + \frac{f[1-f+n(1-n)]}{12(1-f)(1-n)n}\,\frac{1}{V}.$$

Same leading order as for quantum-chaotic quadratic Hamiltonians for $n = 1/2$.

Bianchi, Hackl, Kieburg, MR & Vidmar, PRX Quantum **3**, 030201 (2022).

Summary

- The entanglement properties of typical eigenstates of integrable models are different from those of typical eigenstates of nonintegrable ones
  - Need a nonvanishing subsytem fraction to detect them!
- The leading and first subleading [when $O(\sqrt{N})$] terms in the von Neumann entanglement entropy of typical eigenstates of quantum chaotic Hamiltonians are universal (same as for the average over pure states).

Summary of the analytic results [PRX Quantum **3**, 030201 (2022)]

	General pure states	**Pure fermionic Gaussian states**
no particle number	$\langle S_A\rangle = aV - c + O(2^{-V})$ & **exact** $(\Delta S_A)^2 = \alpha e^{-\beta V} + o(e^{-\beta V})$	$\langle S_A\rangle_{\mathrm{G}} = aV + c + O(\frac{1}{V})$ & **exact** $(\Delta S_A)^2_{\mathrm{G}} = a + o(1)$
fixed particle number	$\langle S_A\rangle_N = aV - b\sqrt{V} - c + o(1)$ $(\Delta S_A)^2_N = \alpha V^{3/2} e^{-\beta V} + o(e^{-\beta V})$	$\langle S_A\rangle_{\mathrm{G},N} = aV - \frac{e}{V} + O(\frac{1}{V^2})$ & **exact** $(\Delta S_A)^2_{\mathrm{G},N} = a + o(1)$
fixed weight	$\langle S_A\rangle_w = aV + b\sqrt{V} + c + o(1)$ $(\Delta S_A)^2_w = \alpha V + o(V)$	$\langle S_A\rangle_{\mathrm{G},w} = aV + c + \frac{d}{\sqrt{V}} + \frac{e}{V} + o(\frac{1}{V})$ $(\Delta S_A)^2_{\mathrm{G},w} = \alpha V + o(V)$

Thank you!

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