Title: Approximate Quantum Codes From Long Wormholes

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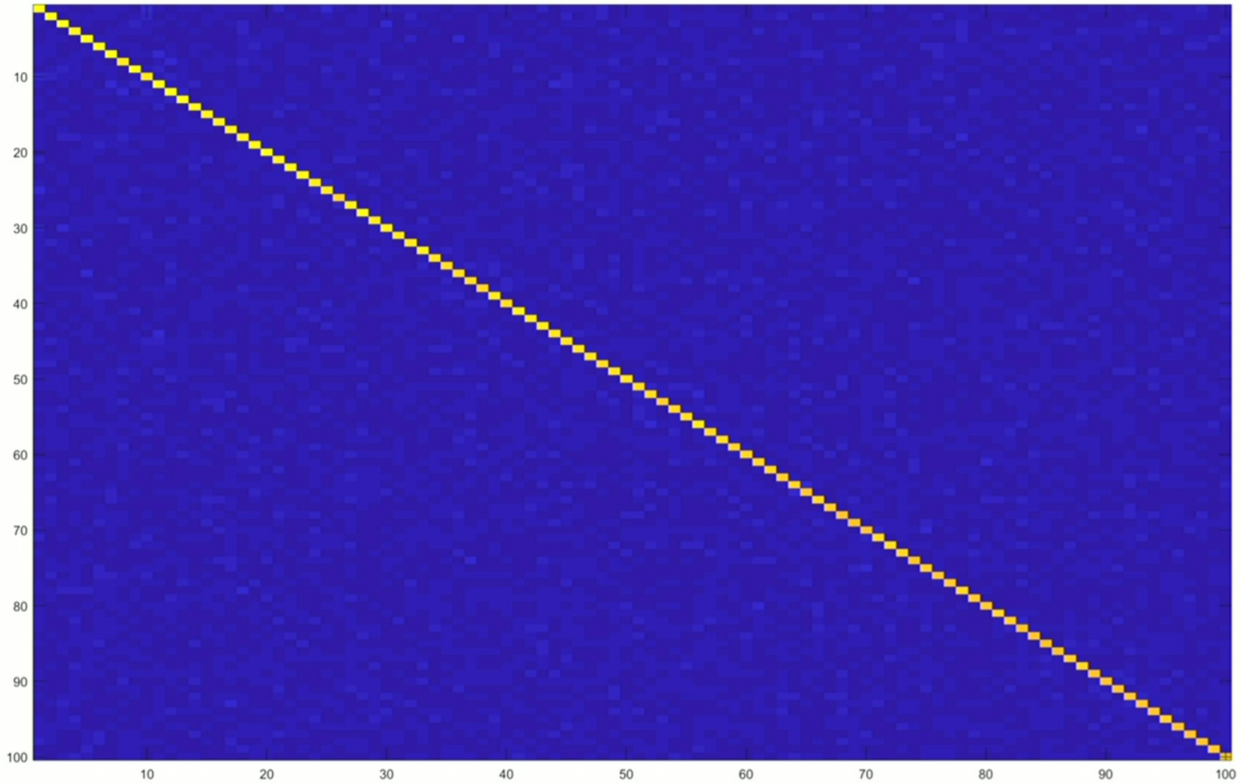
Abstract: We discuss families of approximate quantum error correcting codes which arise as the nearly-degenerate ground states of certain quantum many-body Hamiltonians composed of non-commuting terms. For exact codes, the conditions for error correction can be formulated in terms of the vanishing of a two-sided mutual information in a low-temperature thermofield double state. We consider a notion of distance for approximate codes obtained by demanding that this mutual information instead be small, and we evaluate this mutual information for the Sachdev-Ye-Kitaev (SYK) model and for a family of low-rank SYK models. After an extrapolation to nearly zero temperature, we find that both kinds of models produce fermionic codes with constant rate as the number, N, of fermions goes to infinity. For SYK, the distance scales as N^1/2, and for low-rank SYK, the distance can be arbitrarily close to linear scaling, e.g. N^.99, while maintaining a constant rate. We also consider an analog of the no low-energy trivial states property and show that these models do have trivial low-energy states in the sense of adiabatic continuity. We discuss a holographic model of these codes in which the large code distance is a consequence of the emergence of a long wormhole geometry in a simple model of quantum gravity

Ground states as an approximate code?



N=2 SUSY SYK

$$\langle i|O|j\rangle \stackrel{?}{=} \delta_{ij}\langle O\rangle$$

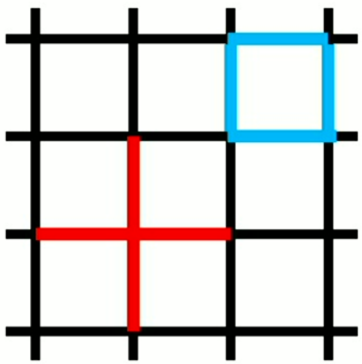
[S unpublished, 5ish years ago]

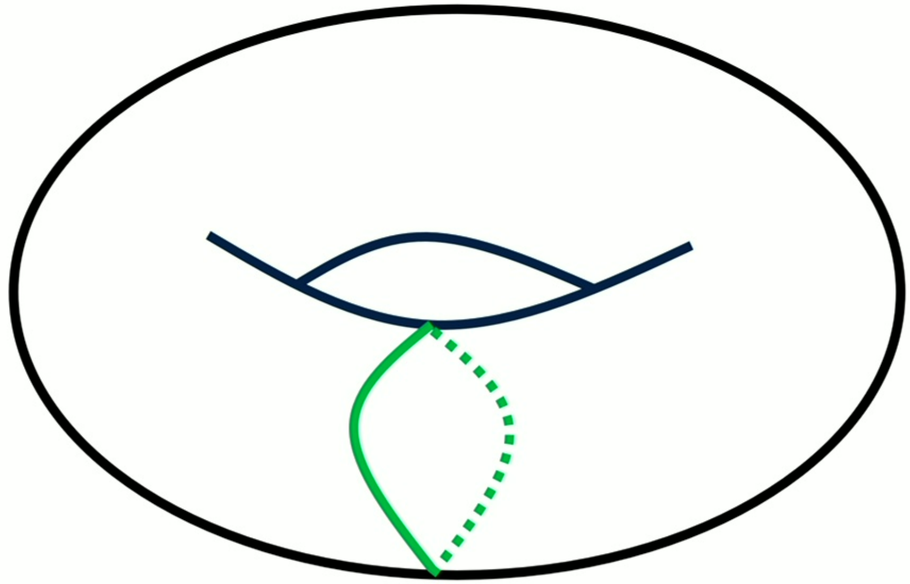
Today: SYK, JT, and codes

- All low-weight stabilizer codes (a big class) can be viewed as degenerate ground spaces of special local* Hamiltonians (*few-body)
- Other examples of interesting ground spaces?
  - Fractional quantum Hall states
  - Certain kinds of frustrated magnets
  - SYK, SUSY SYK
  - Extremal black holes
- If we consider such sets of states as approximate codes, what are their properties?

★ With Greg Bentsen and Phuc Nguyen: [Nguyen-Bentsen-S 2310.07770] ★

Review: toric code





Hamiltonian:
- A sum of commuting terms
- Code space = ground space

Logical operators:
- Non-contractible curves
- Big torus = big protection

Review: Sachdev-Ye-Kitaev model

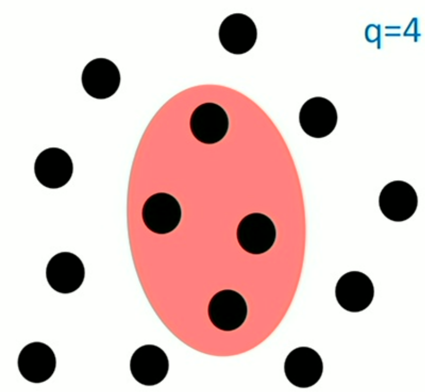
$$\chi_a^\dagger = \chi_a,\ a = 1, \cdots, N$$

$$\{\chi_a, \chi_b\} = \delta_{a,b}$$

$$\dim(\mathcal{H}) = 2^{N/2}$$

$$H = \frac{1}{q!} \sum_A J_{a_1 \cdots a_q} \chi_{a_1} \cdots \chi_{a_q}$$

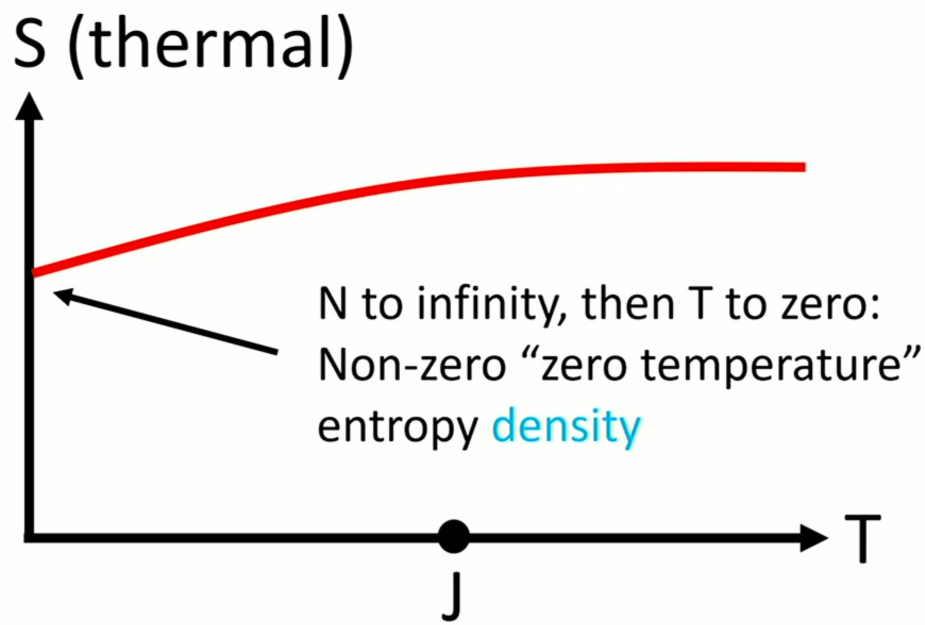
$$\overline{J_A} = 0,\ \overline{J_A^2} = \frac{(q-1)! J^2}{N^{q-1}}$$



[random 2-body ensemble, Sachdev-Ye, Kitaev, Rosenhaus-Polchinski, Maldacena-Stanford, Garcia-Garcia-Verbaarschot, Bagrets-Altland-Kamenev, …]

Review: SYK physics

$$\frac{S(T)}{N} = s_0 + cT + \cdots$$



The holographic description of SYK emerges at low temperature, T << J, where it can be equally well described by matter fields coupled to 2d dilaton gravity (Jackiw-Teitelboim "JT"); this is the regime where our approximate ground space code resides

Why?

- I'm interested in spin glasses and mean-field classical spin glasses have "TAP states" which function like classical codewords; is there a quantum generalization?
- I'm interested in holography and SYK provides a simple holographic model of quantum gravity; understanding these ground states amounts to understanding black hole microstates; our work also touches on bulk reconstruction in the quantum fluctuating regime
- I'm also interested in the field of Hamiltonian complexity, including relations between codes and robust entanglement; we of course would also be delighted to find new useful quantum codes

Back to the story …

Let's define the code space as some band of nearly degenerate approximate ground states in a single instance of SYK

- Rate of the code = ground state entropy per particle

For stabilizer codes, we can think of the maximally mixed state on the code as one side of a zero temperature "thermofield double" (TFD) state; we'll use a very low temperature TFD for the SYK calculation

$$|TFD, T\rangle = \sum_n \sqrt{\frac{e^{-E_n/T}}{Z}} |E_n\rangle_L |E_n\rangle_R$$

What to calculate?

Let's define the code space as some band of nearly degenerate approximate ground states in a single instance of SYK

- Rate of the code = ground state entropy per particle
- Distance of the code = ???

What else might we want to know?

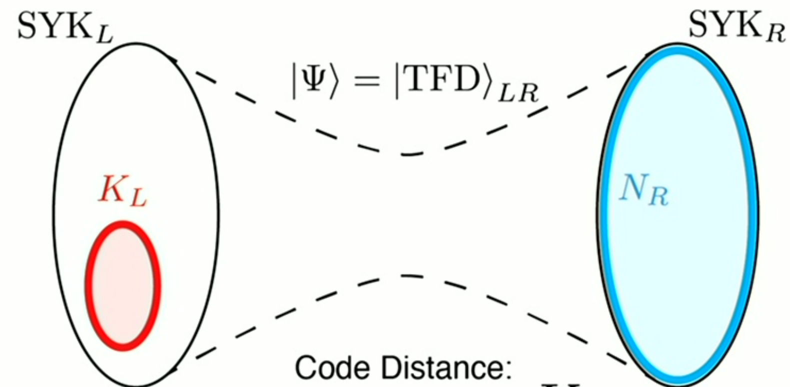
- Encoding?
- Decoding?

Our notion of distance

- For an exact code, the standard Knill-Laflamme conditions for error correction can be reformulated in terms of a vanishing two-sided mutual information
- For an approximate code, we will compute the same mutual information and use its smallness to provide a notion of distance*

*[in progress] compare to other recent works [Yi-Ye-Gottesman-Liu], [Zheng-He-Lee]

Stabilizer Code:	This Work:
$H_{\text{Sta}} = \sum_a h_a$	$H_{\text{SYK}} = \sum_a h_a$
$[h_a, h_b] = 0$	$[h_a, h_b] = \mathcal{O}(1/N)$
Logical subspace = exact ground space	Logical subspace = approx. ground space



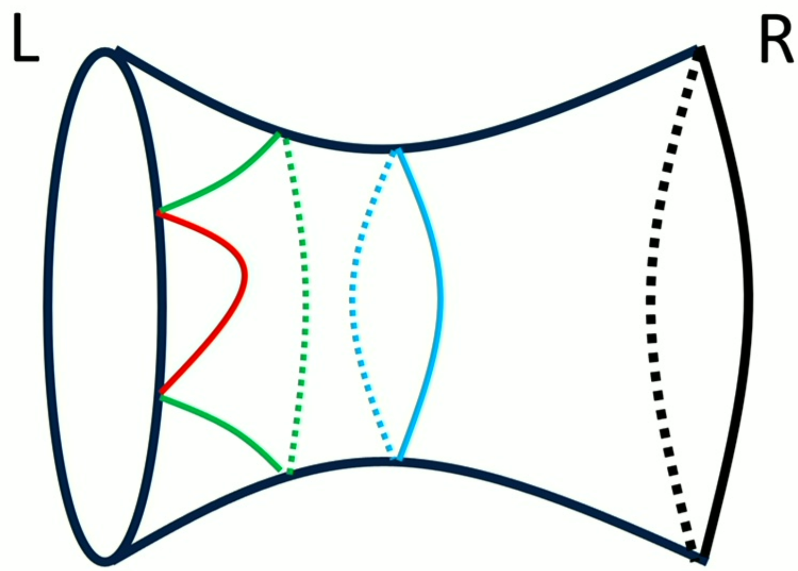
Code Distance:

$$I(K_L : N_R) \propto \frac{K}{N^{2\Delta}}$$

$$= S(K_L) + S(N_R) - S(K_L N_R)$$

Holographic intuition

Let's think about a higher dimensional holographic setup and the powerful Ryu-Takayanagi formula for entropy



$$S = \frac{\text{(area of minimal "surface")}}{4G}$$

$S(K_L)$: red

$S(N_R)$: blue

$S(K_L N_R)$: red+blue OR green

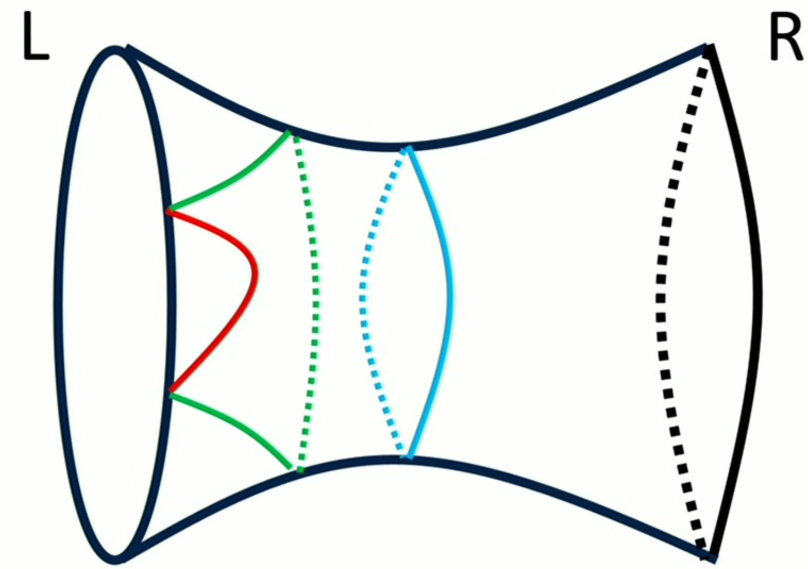
$$I(K_L : N_R) = S(K_L) + S(N_R) - S(K_L N_R)$$

TFD is dual to a geometry with spatial connectivity between L and R, a wormhole!

Aside on non-Abelian gauge codes

- We can arrange our wormhole geometry so that the mutual information is zero even for quite big L regions!
- Are we getting awesome codes?
- Not quite: There are small corrections to the RT calculation coming from matter that give non-zero mutual information
- Maybe yes: One can nevertheless design codes that use continuous non-Abelian gauge invariance to protect information; these can even be thermally long-lived

  [early steps: Cao-Cheng-S 2211.08448]



Main calculation $I(K_L : N_R) = S(K_L) + S(N_R) - S(K_L N_R)$

- Outline of the calculation:
  - Prepare TFD state using Euclidean path integral
  - Compute the individual entropy terms using replica trick
  - Combine the terms and extrapolate to the “ground space”, $T \sim 1/N$
- But there are many subtleties:
  - Saddle point calculation becomes uncontrolled if T is too small
  - Saddle points can shift depending on K if K is too large
  - Renyi entropies are easier to access than von Neumann entropies
- We make some non-rigorous arguments that we can ignore shifts of saddle due to K and breakdown of the saddle point at low T

The main technical work of our paper is doing this calculation; we did it both directly in SYK and using the dual gravity picture

I'll discuss the gravity picture here

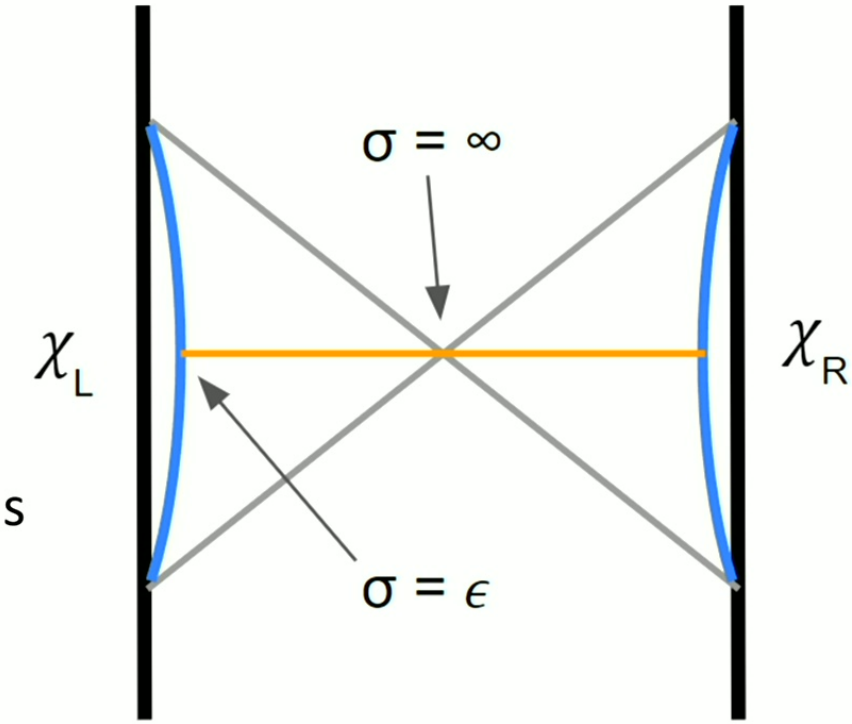
Gravitational picture

$$ds^2 = \frac{4\pi^2}{\beta^2} \frac{-dt^2 + d\sigma^2}{\sinh^2 \frac{2\pi\sigma}{\beta}}$$

Consider a holographic model:

- JT gravity +
- N bulk fields (dimension → mass)

We'll discuss correlations and a model of entanglement of subsets of fermions (this is a more complex low-d version of RT) [QGLabII, Chandrasekaran-Levine, Antonini-Jian-Grado-White-S]

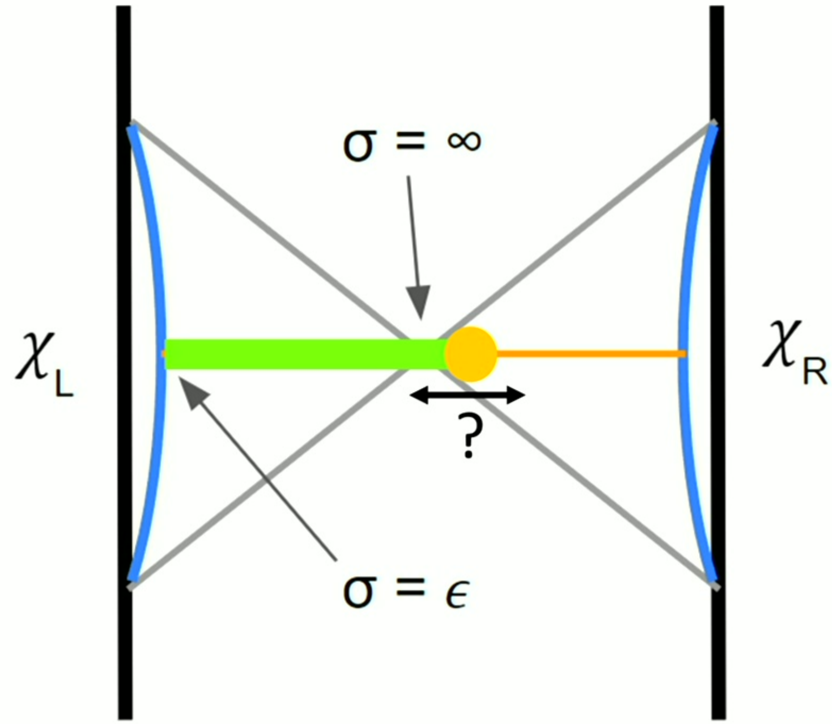


Entropy has two pieces:

- Dilaton contribution (similar to RT, order N)
- Bulk entanglement contribution (large due to N bulk fields)

For example, suppose we want to compute entropy of entire left:

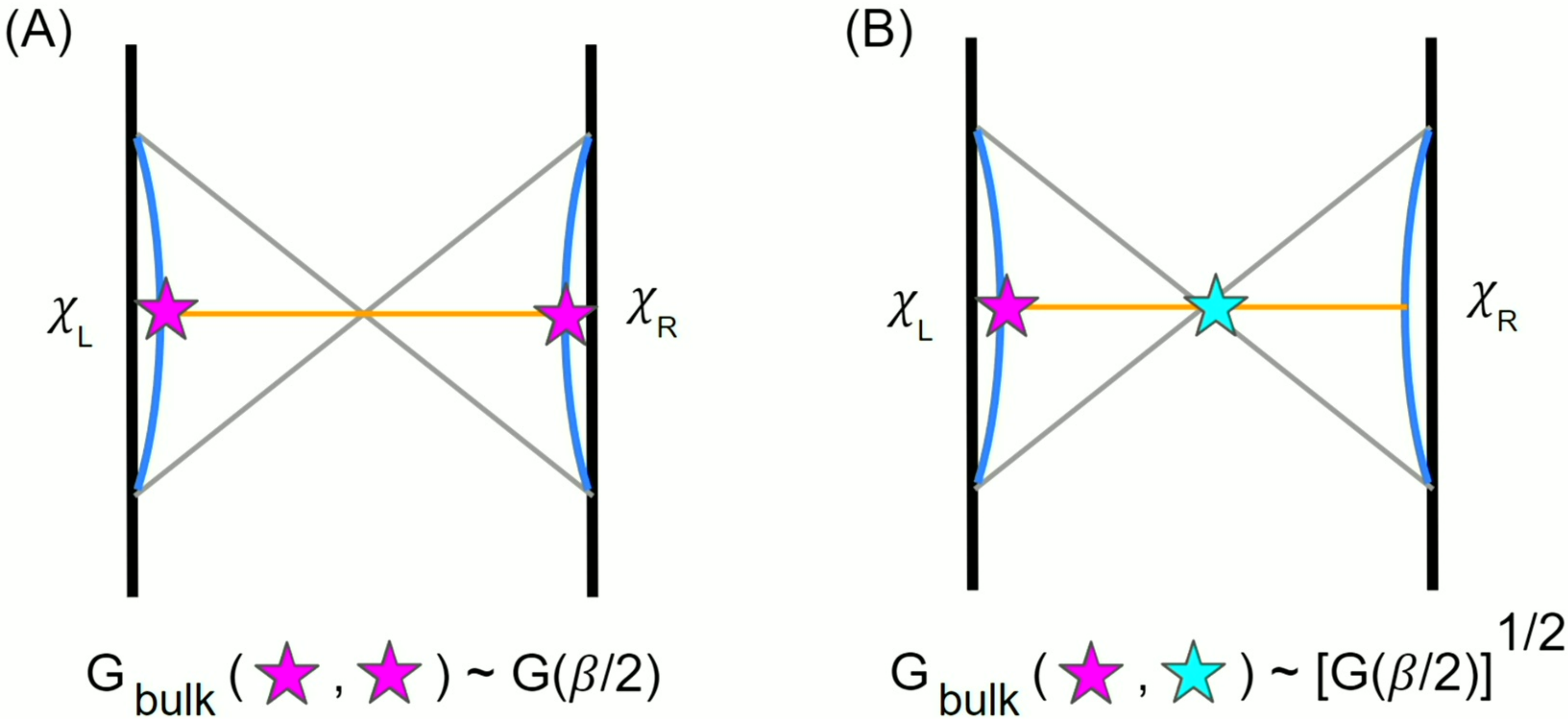
- Dilaton value at ●
- Bulk entanglement on ▬
- Minimize this combination over the location of ●



We also give a new perspective on this formula via dimensional reduction of a magnetic black hole [Bentsen-Nguyen-S]

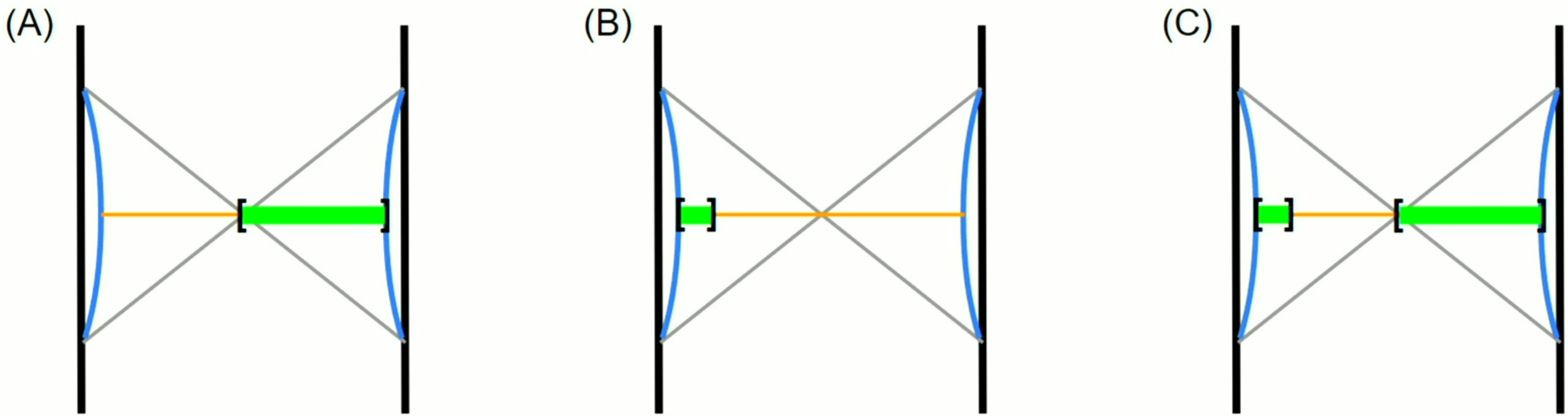
Correlations

$G(\tau)$ = thermal Green function



Mutual information

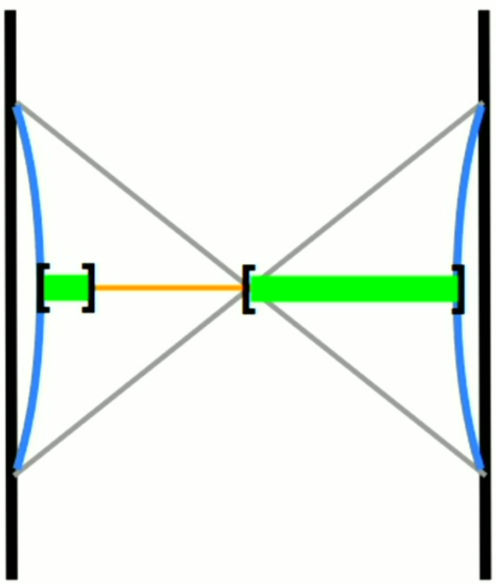
Entropy = Dilaton + Bulk entanglement



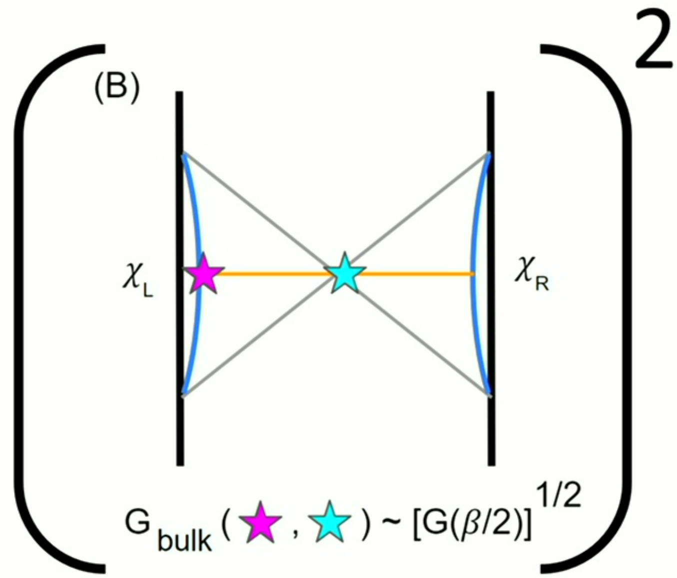
$$I(K_L : N_R)\text{: (A) } S(R)\text{, (B) } S(K_L)\text{, (C) } S(K_L \cup R).$$

Limit of a long wormhole

(C)



single interval entropies +
bulk mutual information



Final result and comments

- For general beta, the result is $I(K_L: R) \sim K/\beta^{2\Delta}$; if we now take the delicate limit $\beta \to O(N)$, we get $I \sim K/N^{2\Delta}$
- Hence, a subset of $\epsilon N^{2\Delta}$ fermions has no more than ϵ information about the code space
- We did a purely SYK calculation giving the same result
- For SYK, $2\Delta \leq \frac{1}{2}$ so the distance is never linear in N; for "low-rank SYK" [Kim-Cao-Altman] the dimension is tunable and the distance can be arbitrarily close to linear, e.g. $2\Delta \sim .99$ while maintaining a non-vanishing rate

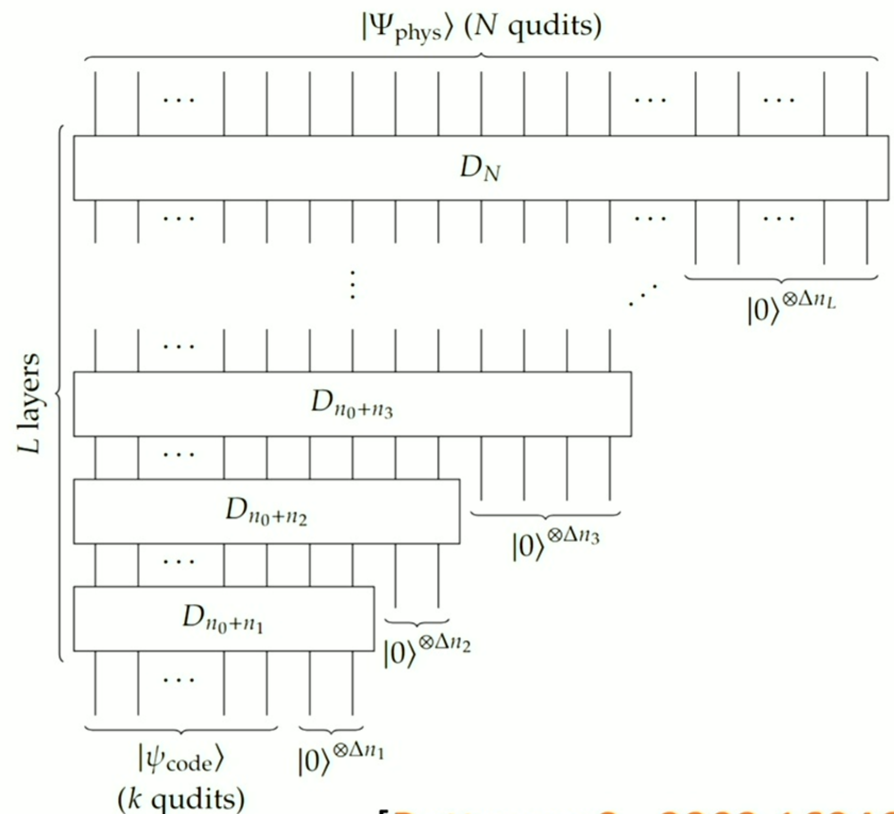
Summary and outlook

- Summary: approximate ground spaces as codes, rate is ground state entropy, distance obtained from a two-sided mutual information
- Holographic perspective: large (emergent) physical distance → large code distance
- If we had a stabilizer code with these properties, it would be a fantastic; for SYK, it's not clear if these codes are useful
- Many questions remain …

Decoding and encoding?

$H = \sum_a h_a$; h_a = all terms including fermion a

Different h_a nearly commute and h_a jumps in expectation value when fermion a is applied (error), but there are large fluctuations; decoding is plausibly hard from black hole perspective; also related to bulk reconstruction with strong quantum fluctuations



[Bettaque-S - 2303.16946]

NLTS-ish?

- Also interesting to compare to recent "good QLDPC" codes [Panteleev-Kalachev, …]; one thing those codes have [Anshu-Breuckmann-Nirkhe] is a robust kind of entanglement related to the absence of "trivial low-energy states" (NLTS) [Freedman-Hastings]
- A system has NLTS if all low energy states below a certain energy density are significantly entangled (no constant depth circuit)
- A morally similar criterion: replace constant depth circuit with time evolution for a constant time (not clearly the same for mean-field models)
- By invoking the Maldacena-Qi eternal wormhole, we show that our SYK codes DO NOT have this NLTS-like property; we can prepare states of arbitrarily low but fixed (with N) energy density [Bentsen-Nguyen-S]; this also dovetails nicely with holographic complexity conjectures

N=2 SUSY case

We now have complex fermions with a conserved U(1) and supersymmetry with a complex supercharge

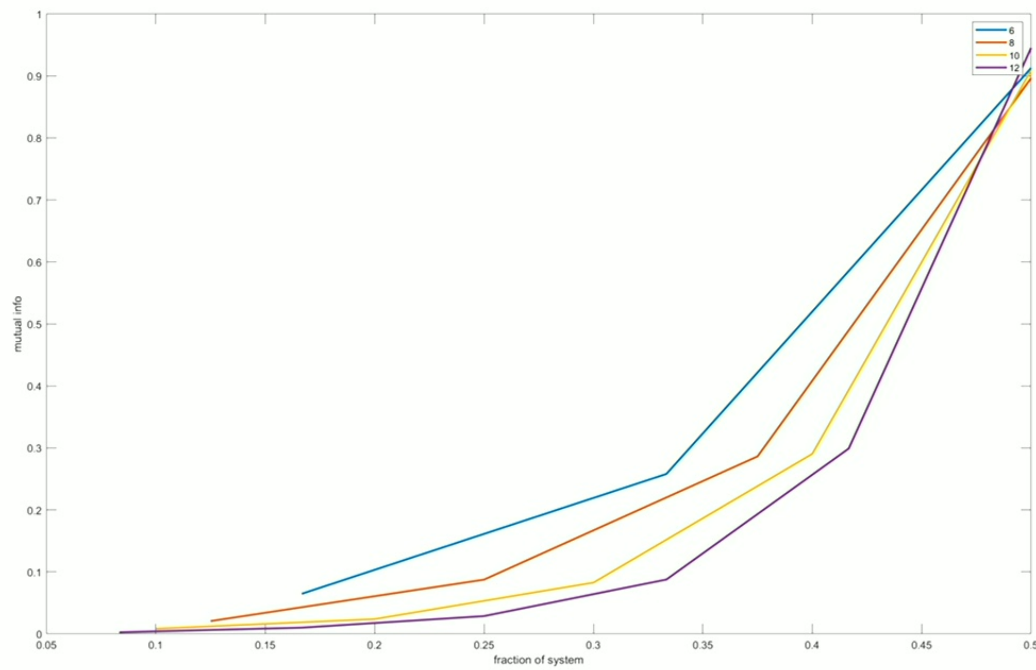
$$Q = \sum_{a,b,c} i\, C_{abc}\psi_a\psi_b\psi_c$$

$$H = \{Q, Q^{+}\}$$

Ground state degeneracy is now exact

[Fu-Gaiotto-Maldacena-Sachdev]

Our methods above should generalize and one can study this system in exact diagonalization:



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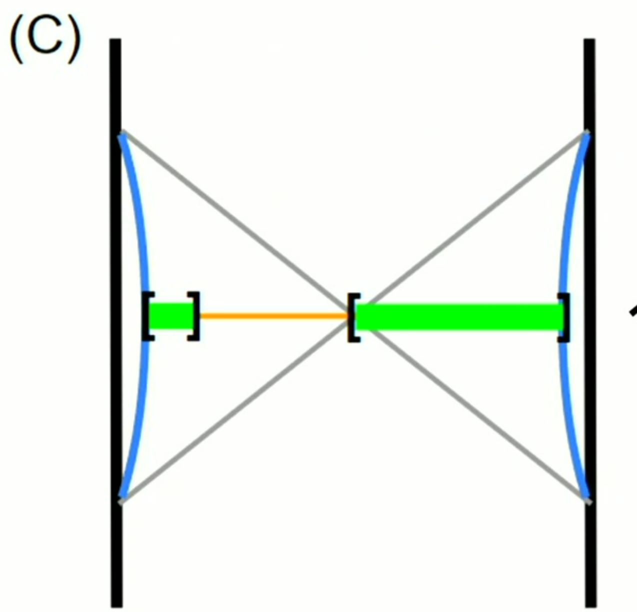

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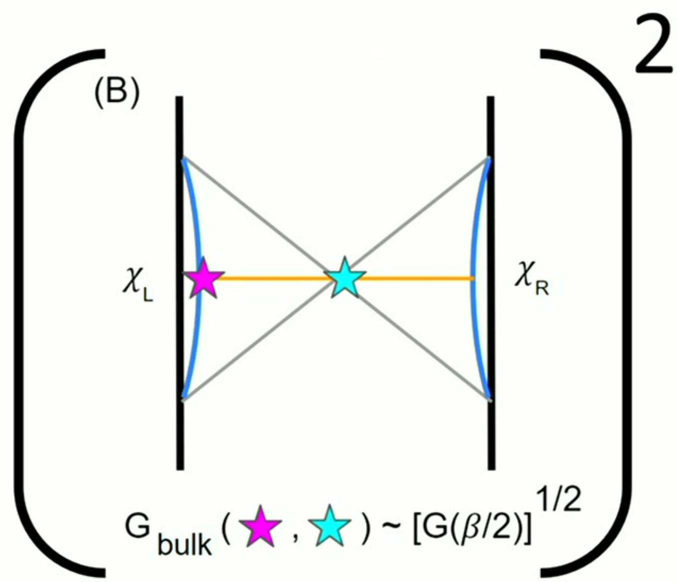

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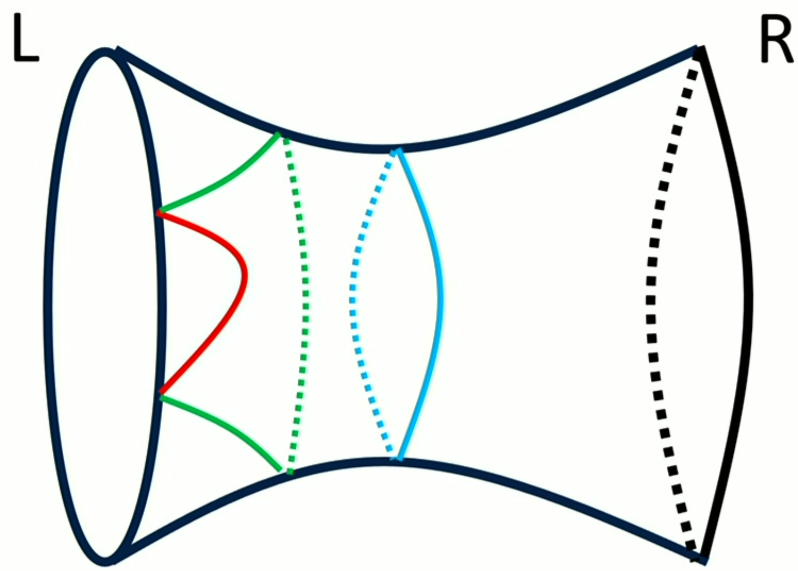

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