Title: Defining stable steady-state phases of open systems

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Abstract: The steady states of dynamical processes can exhibit stable nontrivial phases, which can also serve as fault-tolerant classical or quantum memories. For Markovian quantum (classical) dynamics, these steady states are extremal eigenvectors of the non-Hermitian operators that generate the dynamics, i.e., quantum channels (Markov chains). However, since these operators are non-Hermitian, their spectra are an unreliable guide to dynamical relaxation timescales or to stability against perturbations. We propose an alternative dynamical criterion for a steady state to be in a stable phase, which we name uniformity: informally, our criterion amounts to requiring that, under sufficiently small local perturbations of the dynamics, the unperturbed and perturbed steady states are related to one another by a finite-time dissipative evolution. We show that this criterion implies many of the properties one would want from any reasonable definition of a phase. We prove that uniformity is satisfied in a canonical classical cellular automaton, and provide numerical evidence that the gap determines the relaxation rate between nearby steady states in the same phase, a situation we conjecture holds generically whenever uniformity is satisfied. We further conjecture some sufficient conditions for a channel to exhibit uniformity and therefore stability.

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defining steady-state phases of open systems

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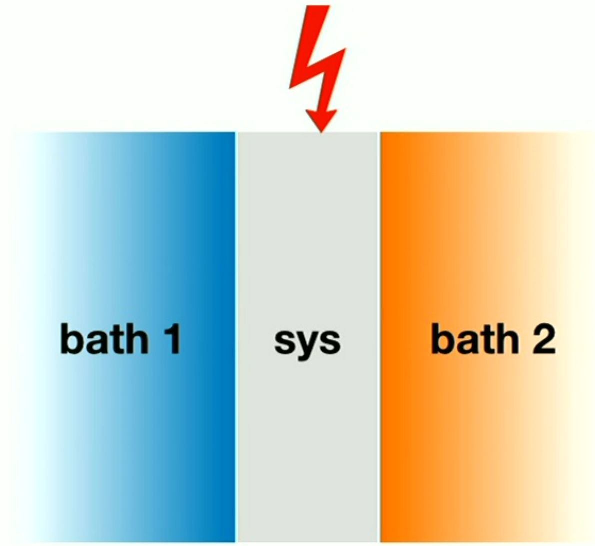
rakovszky, sg, von keyserlingk, arXiv:2308.15495

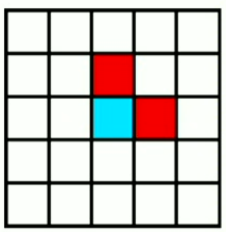
nonequilibrium steady states

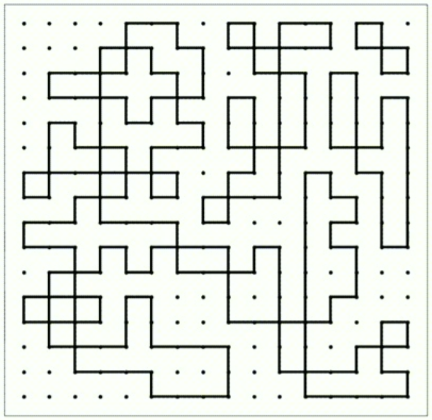
- System coupled to very large environment
- Expected to reach a steady state (possibly non-unique) if you wait long enough
- What are the properties of this state?
- Order of limits for steady state
  - First take bath size to infinity
  - Take system size and time to infinity as $L^z/t = O(1)$ for some z
- Does this order of limits give rise to distinct **phases**?

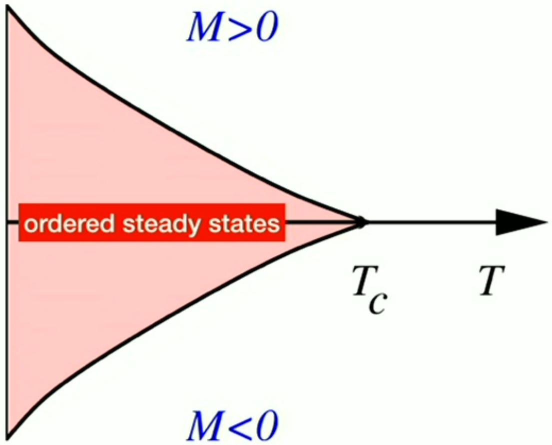


stable steady-state phases exist…

- In the classical context: fault-tolerant cellular automata (Toom, Gacs, …)







- Some have distinctively quantum orders (e.g., error correcting codes)

two basic questions

- Hard question: proving that a given steady-state phase is stable to arbitrary local perturbations
  - Concrete results in math literature for certain dynamical systems
  - Several concrete examples with nonperturbative instabilities

- Easier question (**this talk**): What is the “landscape” of open systems like? What does it *mean* to be in a phase?

outline

- Phases of gapped zero-temperature systems: quasi-adiabatic continuation and its implications
- Case study of a nonequilibrium system: biased random walk
  - Essential role of non-Hermiticity
  - Spectra and pseudospectra
  - What the spectrum controls
- “Uniformity” criterion for open-system phases
  - Consequences for correlation functions, response to local perturbations, …
  - Example: Stavskaya automaton

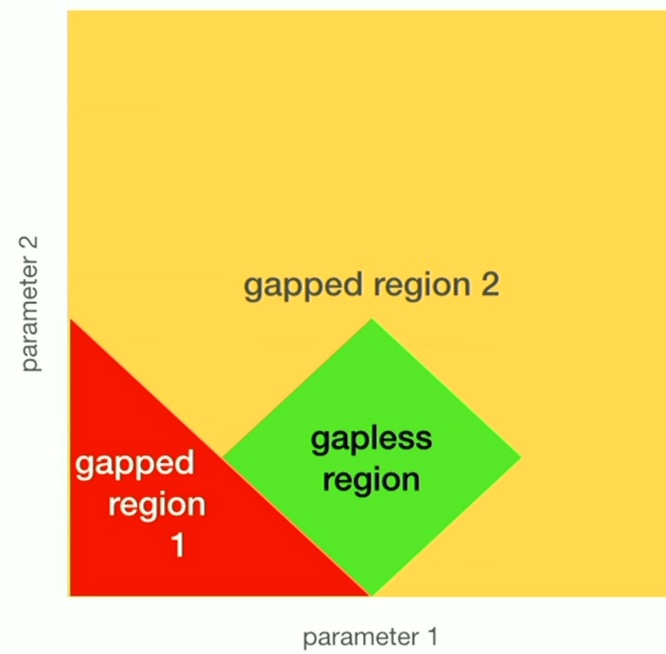
gapped zero-temperature systems

gapped phases of ground states

- Gapped regions: finitely many ground states, degenerate and separated by O(1) gap from all other states as size $L \to \infty$
- Standard properties of ground states in gapped regions:
  - area-law entanglement
  - analytic evolution of expectation values and few-point correlators
  - correlation functions decay exponentially to their asymptotic values
  - if one point in a gapped phase has long-range order, so does every other point
  - "local perturbations perturb locally" (LPPL): perturbing a gapped ground state at point x has a weak effect at distant point y,

$$\hat{H} = \hat{H}_0 + V(x)\hat{O}(x), \quad \frac{\delta\langle \hat{O}'(y)\rangle}{\delta V(x)} \sim e^{-|x-y|/\xi}$$

- What have these got to do with one another?

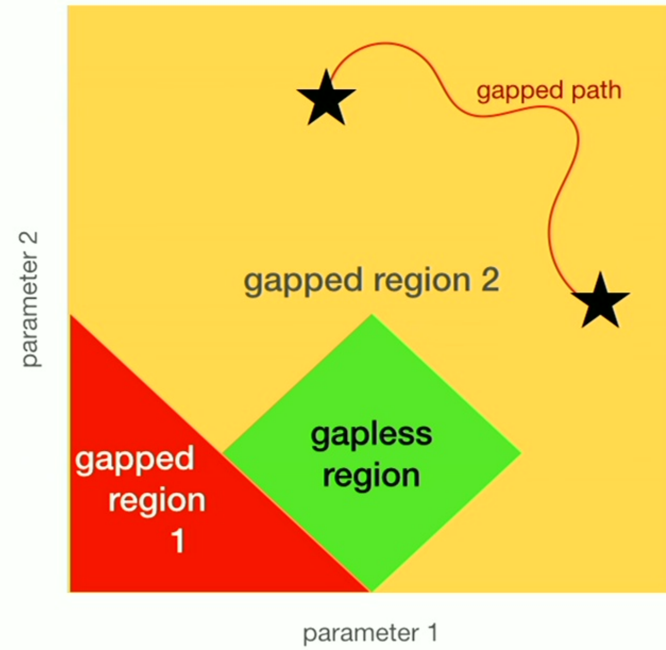


gapped paths and finite-depth circuits

- Move along a gapped path (i.e., gap always stays open)
- By **adiabatic theorem**, if we move slow enough (relative to the gap) we remain close to* the instantaneous ground state
- Gap remains open throughout: there is a finite-time evolution that connects two ground states along the path
- By **Lieb-Robinson theorem**, this finite-time evolution does not change correlations at asymptotically large distances: there is a light cone
- Can Trotterize the finite-time evolution, getting that

  $$|\psi_a\rangle \approx \hat{U}\,|\psi_b\rangle$$

  where $\hat{U}$ is a finite-depth (local) unitary circuit (FDLU)
- Wavefunctions fall into equivalence classes under FDLU equivalence (FDLU is an equivalence relation)



FDLU and long-range order

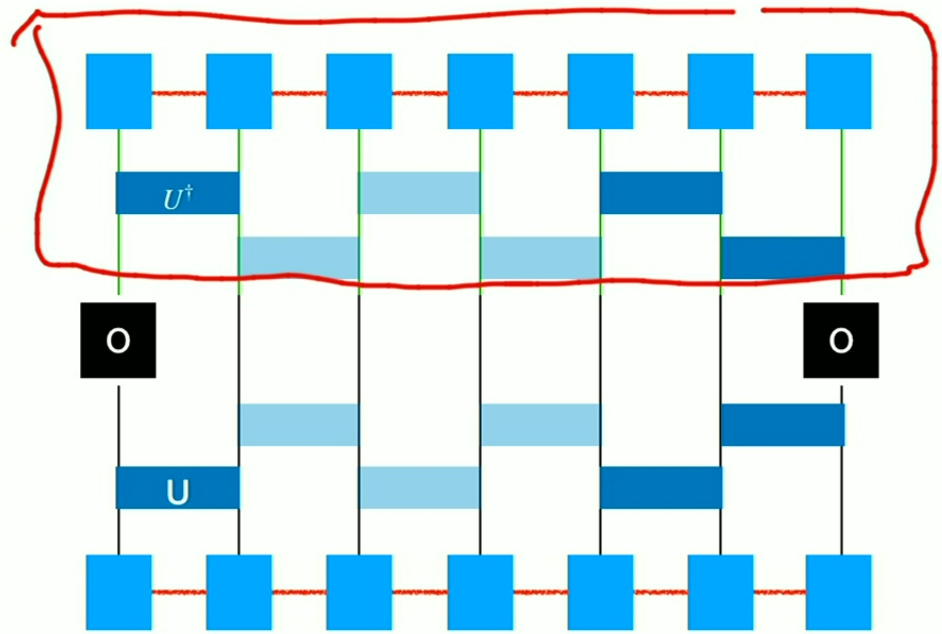
$\langle\psi_a|O(x)O(0)|\psi_a\rangle \approx \langle\psi_b|U^\dagger O(x)O(0)U|\psi_b\rangle = \langle\psi_b|\tilde{O}(x)\tilde{O}(0)|\psi_a\rangle$

where $\tilde{O}(x) = U^\dagger O(x)U$

Local operator with LRO in $|\psi_a\rangle$ means *some* fattened local operator with LRO in $|\psi_b\rangle$

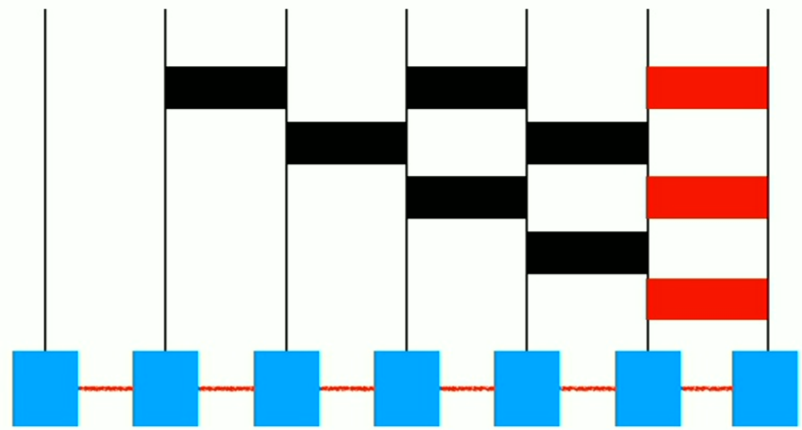
Long-range order is either present (or not) throughout a gapped phase

Generally $\mathrm{Tr}(O(x)U^\dagger O(x)U) = O(1)$ so the same correlator picks up LRO throughout a phase (but this can fail at some points)



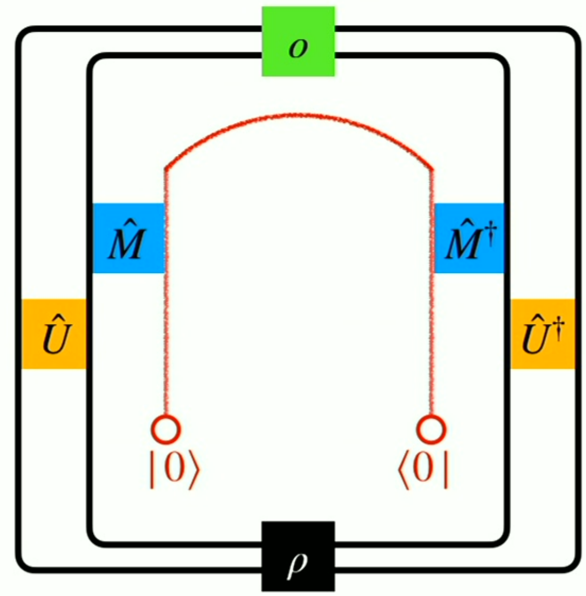
“local perturbations perturb locally”

- Start with a gapped Hamiltonian H, perturb it with a weak local perturbation $\lambda O(x)$ near x
- Assume gap remains open for all $H(\lambda') = H + \lambda' O(x), \quad 0 \leq \lambda' \leq \lambda$
- This means ground states of $H(0)$ and $H(\lambda)$ are related by FDLU consisting of:
  - Gates from $H(0)$ acting everywhere
  - Perturbations acting at x
- These only perturb the ground state within the FDLU light cone
- Farther away, expectation values are unaffected
- LPPL can also be adapted to finite temperature



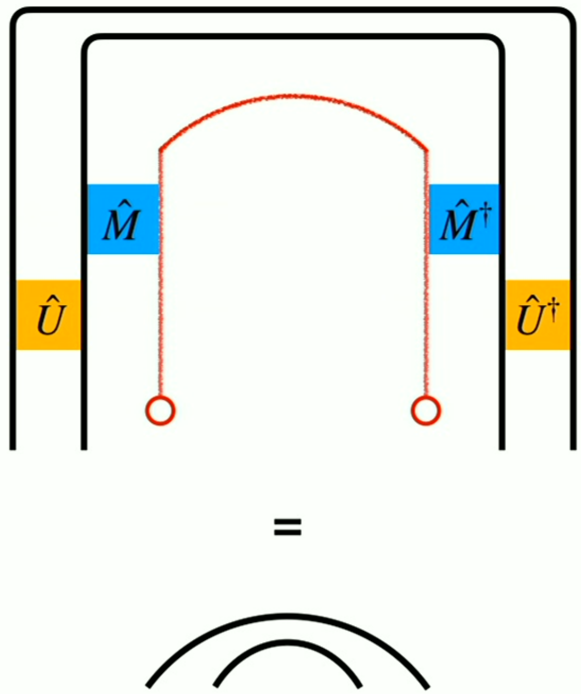
quantum channels

- System interacts with environment in reference state
- After interaction, environment qubit is lost/traced over
- "Superoperator": takes (system) density matrices to density matrices $\mathscr{E}(\rho) = \rho'$
- Schrödinger (bottom-to-top) and Heisenberg (top-to-bottom) versions of a channel have different properties
  - Schrödinger evolution is trace-preserving
  - Heisenberg evolution is unital (maps identity to identity)
- Can write states as "kets" (vectors in the Hilbert space) and observables as "bras" (vectors in the dual space)
- Expectation values are"matrix elements" $(o \,|\, \mathscr{E} \,|\, \rho)$



unital property in the heisenberg picture

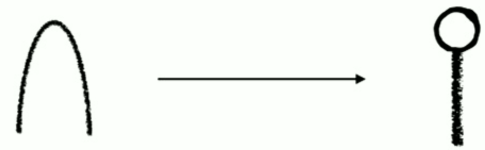
- In vector notation, $(\mathbb{I}\,|\,\mathscr{E} = (\mathbb{I}\,|$
- In Schrödinger picture, corresponds to trace preservation
- In Heisenberg picture, corresponds to unital property: "it doesn't matter when you don't perturb the system"
- Note that the corresponding property is not true acting to the right, $\mathscr{E}\,|\,\mathbb{I}) \neq |\,\mathbb{I})$ in general



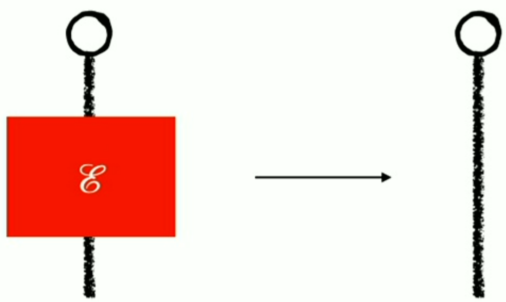
fat-line notation

- Fat lines carry density matrices,
  a channel is a superoperator on density matrices

- Observables are vectors acting from the top, identity

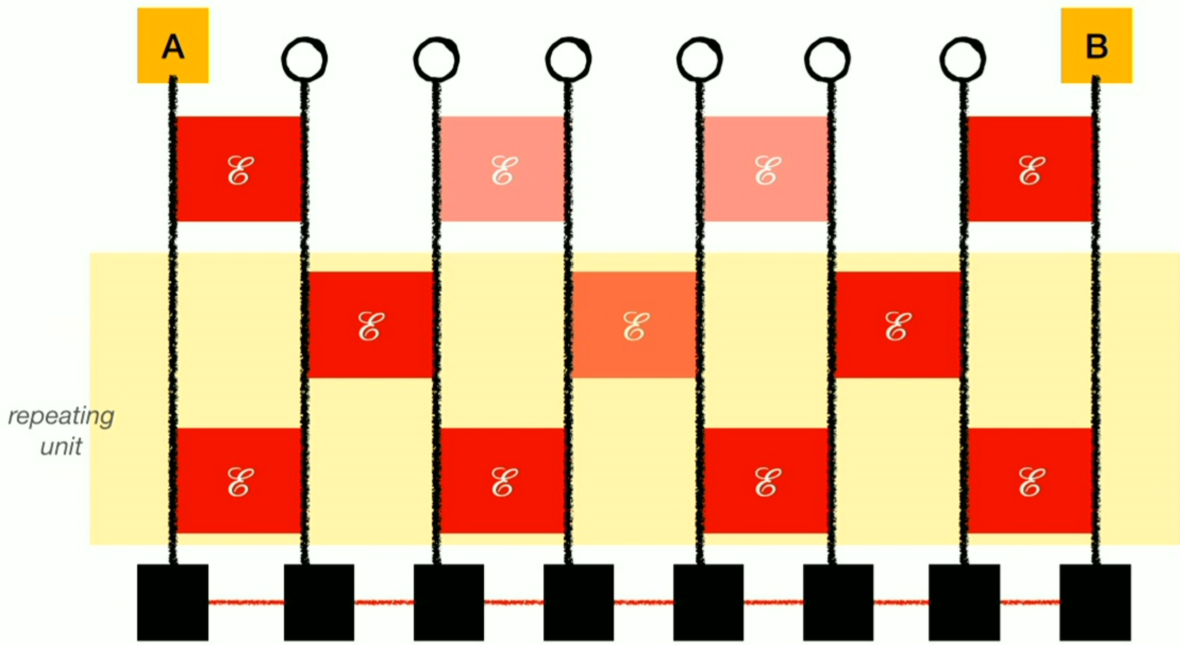


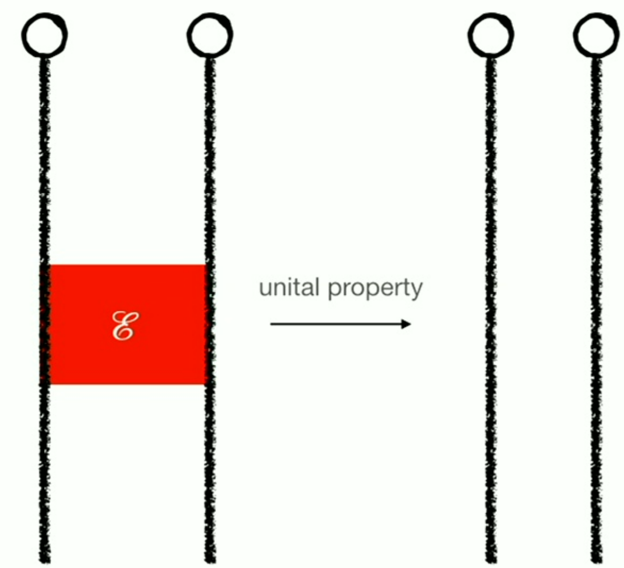
- Unital property:



circuits composed of quantum channels

- Lieb-Robinson arguments carry over more or less directly from unitary systems



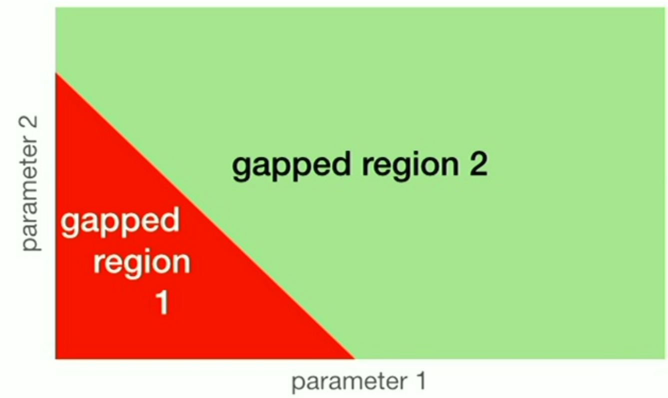


spectra of quantum channels

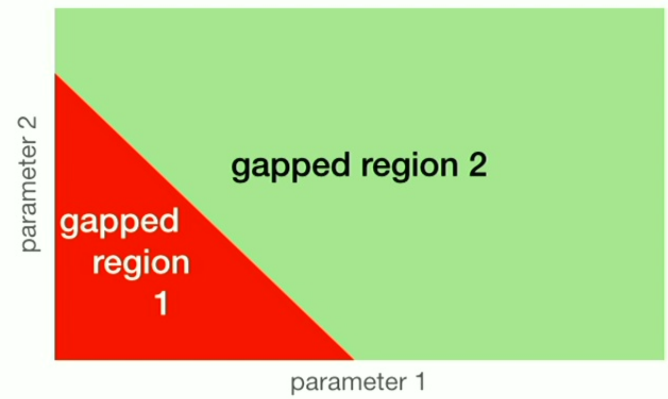
- We are interested in steady states, $\mathscr{E}(\rho) = \rho$
- In general, any initial state evolved to long times can be written as

  $$\mathscr{E}^t(\rho) = \rho_{\text{s.s.}} + \sum_n c_n \lambda_n^t \sigma_n$$

  where σ_n are traceless Hermitian matrices
- Apparent characteristic timescale $t_* \sim 1/|\log \lambda_1|$
- Spectrally gapped regions: t_* stays finite in large-system limit
- **Do these behave like zero-temperature gapped phases?**
- Instead of working with channel, could work with continuous time version, Lindbladian, s.t. $\partial_t \rho = \mathscr{L}(\rho)$



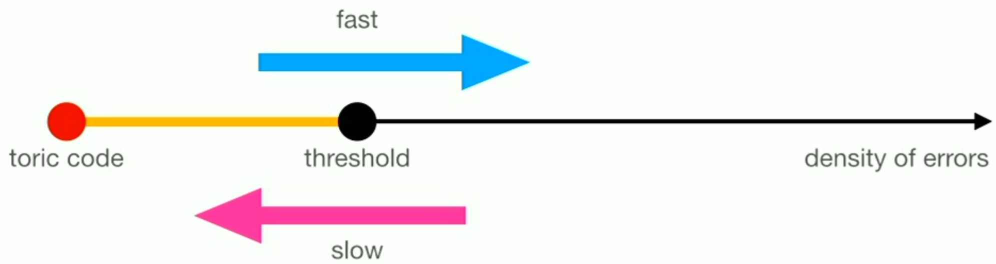
ground-state phases vs. steady-state phases



- Extremal eigenvector of Hermitian matrix $\hat{H}$
- If gap stays open there is a finite-time evolution that remains in the instantaneous ground state
- Lieb-Robinson: finite-time evolution cannot form long-range correlations
- Inverse of finite-time evolution is finite-time evolution, so cannot *destroy* long-range correlations either

- Extremal eigenvector of non-Hermitian matrix $\mathscr{L}$
- Not obvious that a system would remain in the steady state along a gapped path
- Lieb-Robinson: finite-time evolution cannot form long-range correlations
- Inverse of a channel is not a channel, so finite-time evolution can destroy long-range correlations

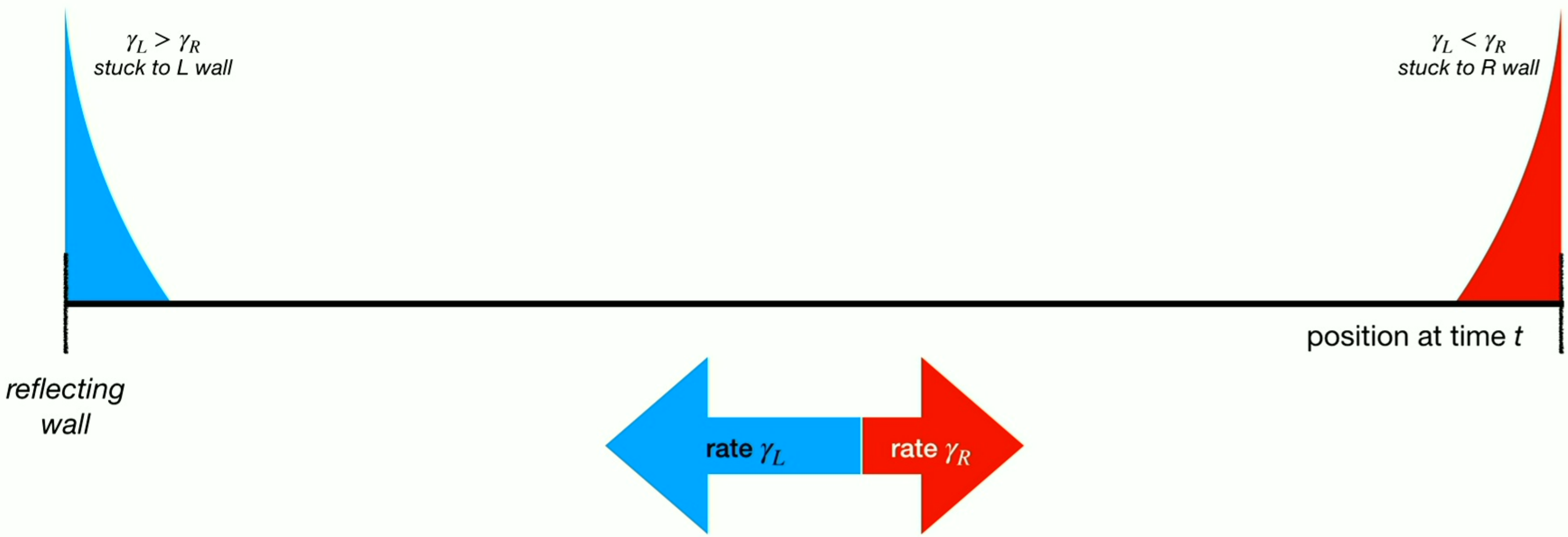
two-way equivalence



- Proposal (Coser + Perez-Garcia '19): if $\rho_2 = \mathscr{E}(\rho_1)$ and $\rho_1 = \mathscr{R}(\rho_2)$ under FDLC, then we say the two states are in the same phase
- At the threshold the recovery map ceases to be an FDLC (Sang + Hsieh '24)
- The below-threshold toric code is not the steady state of any obvious parent channel: what kinds of mixed states have parent channels?
- "Standard" active error correction requires nonlocal classical processing, unclear how to work this into our concept of steady-state phases

biased random walk

a simple nonequilibrium phase transition in 1d



- Master equation (in bulk): $\dot{p}_i = \gamma_R p_{i-1} + \gamma_L p_{i+1} - (\gamma_R + \gamma_L) p_i$
- Master equation (at origin): $\dot{p}_0 = -\gamma_L p_0 + \gamma_L p_1$
- For these b.c.'s, the spectrum is **gapped** whenever the rates are asymmetric

totally asymmetric limit

- Turn off rightward hopping, work in discrete time, arrive at a Markov chain:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad Mv = \lambda v$$

$$\lambda = 1,\ v = (1,0,0,0,0)$$

$$\lambda = 0,\ v = (-1,1,0,0,0)$$

- This matrix (regardless of size) has only two eigenvalues (1,0) and **only two eigenvectors** (i.e., it contains a giant Jordan block)
- Gapped unique ground state! But clearly a relaxation time that diverges with system size
- Relaxation of generic initial states is not controlled by spectrum but by Jordan block structure

away from the limit

- Matrix is generally diagonalizable but has nearly parallel eigenvectors (all localized near left end)

- Spectrum remains gapped

- Initial states near right end have coefficients $\sim e^{L}$ in eigenbasis, can be written as

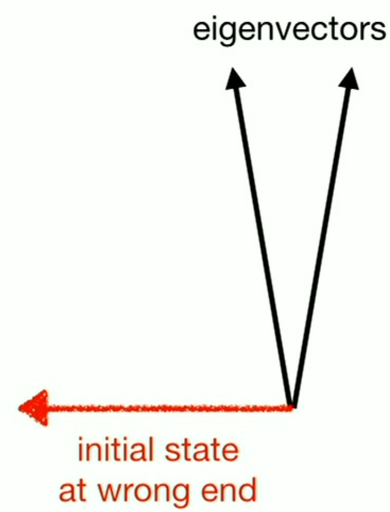
  $$w = \sum_i \lambda_i e^{cL} v_i$$

- Under time evolution these go to

  $$w(t) = \sum_i \lambda_i e^{cL - \lambda_i t} v_i$$

- Non-steady-state coeffs become small when $L \sim t$

- $L \to \infty, t \to \infty$ limits do not commute: spectrum only controls relaxation if you take the long-time limit first!



pseudospectrum

- Eigenvalues are poles of resolvent $R_M(z) = Q(z\mathbb{I} - M)^{-1}Q$ where Q is a projector that removes the steady state

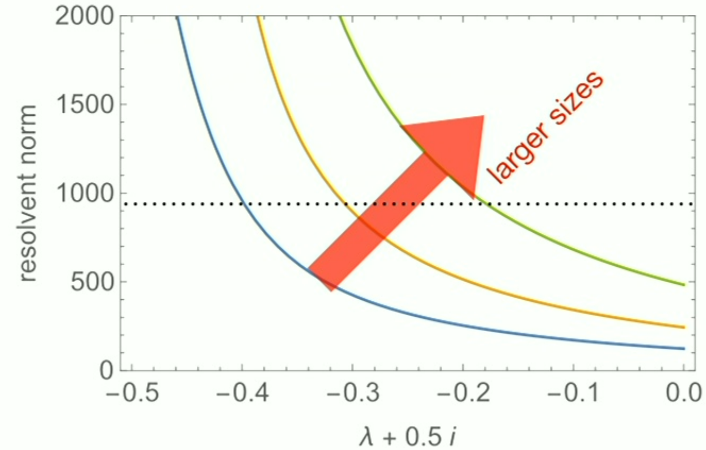
- Define ϵ-pseudospectrum in terms of "large resolvent,"

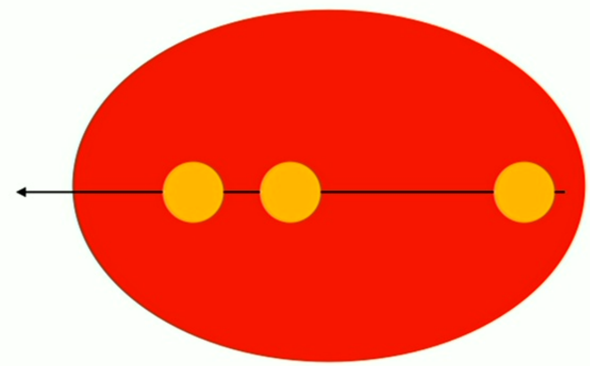
  $S_\epsilon = \{z \in \mathbb{C} \mid \|R_M(z)\|_2 \geq 1/\epsilon\}$

- Pseudospectrum contains info about relaxation of generic initial states (Trefethen…)

- In the totally asymmetric random walk, $\|R(z)\| \sim 1/(z-1)^L$

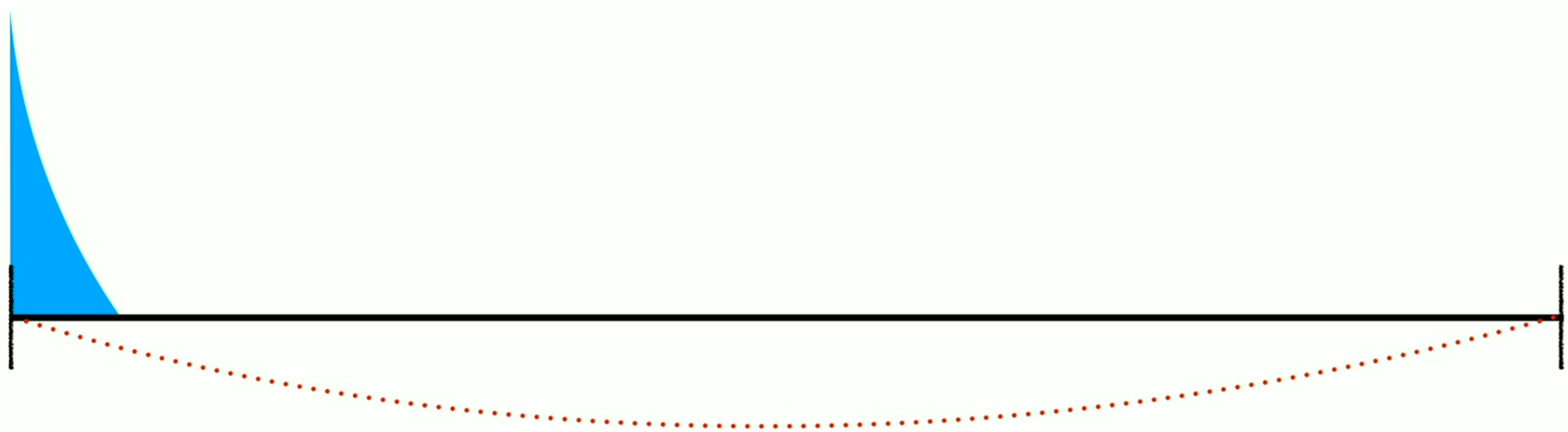
- Taking $L \to \infty$ first, the entire interval [0,1] lies in the pseudospectrum for any ϵ so it is gapless





boundary-condition dependence

- Resolvent also controls perturbation theory: regions of large resolvent are susceptible
- Instability: connect opposite ends of the system, creating a ring
- On a ring the steady state is current carrying and there is no density pile-up at the left end
- Lesson 1: connecting opposite ends of the Jordan block creates an instability
- Lesson 2: LPPL fails on the ring — a local perturbation can collapse gap, destroy steady state



other cellular automata

- All deterministic cellular automata are Markov chains with eigenvalues that live either on the unit circle or at zero: all either gapped or degenerate
- Many have long relaxation times and nontrivial dynamics: again, due to Jordan block structure
- Jordan blocks in *configuration* space
- A natural conjecture:
    - Automata are stable when perturbations that connect the two ends of the block are illegal
    - Why might they be illegal? E.g., because of locality constraints

steady state phases of quantum channels

general picture

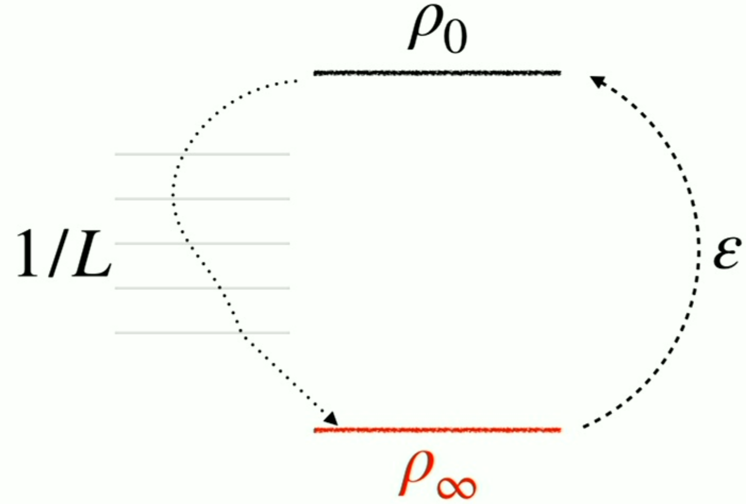
- Nontrivial gapped phase needs:
  - Gap setting *some* characteristic O(1) relaxation timescale
  - Initial states that take O(L) time to relax
- Informal proposal to put these together:
  - Long relaxation timescale $\Rightarrow$ large emergent Jordan blocks, gapless pseudospectrum $\Rightarrow$ instability to fully general perturbations
  - **Local perturbations relax on timescale set by gap**
  - Perturbing dynamics from a steady state ~ starting with a locally perturbed steady state of the new dynamics
  - Short channels relating steady states $\Rightarrow$ steady states of unperturbed and perturbed channels are very similar

instability to general perturbations

- A nontrivial steady state is one that you cannot reach from a trivial one at finite depth
- Take a channel $\mathscr{E}_0$ with a nontrivial steady state ρ_∞
- Add a perturbation acting as

  $$P_\varepsilon \rho = (1-\varepsilon)\rho + \varepsilon \rho_0$$

  where ρ_0 is your favorite product density matrix
- For any $\varepsilon > 0$ in the thermodynamic limit, the steady state of $\mathscr{E}_0 P_\varepsilon$ is trivial, so $\mathscr{E}_0$ is unstable
- But clearly this perturbation is highly nonlocal



uniformity condition

This is a proposal for what it means for an open set of parameters D to be in a stable phase:

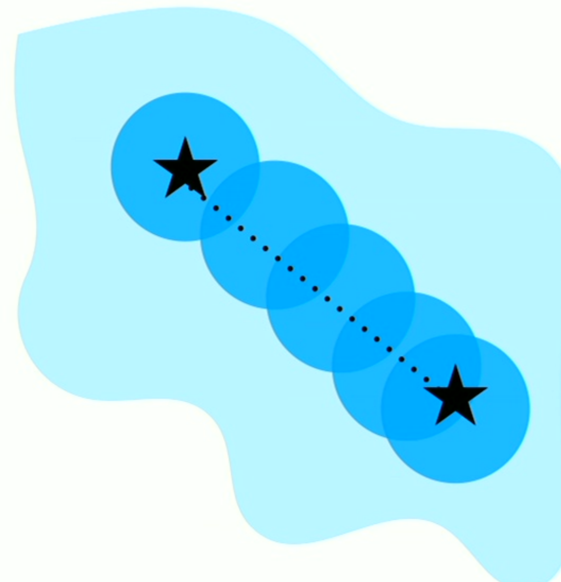
The (δ, τ)-uniformity condition means $\exists(\delta > 0, \tau > 0)$ such that D can be finitely covered with balls of size δ with the property:

Given a channel $\mathscr{E}$ with steady state ρ in a ball B, every other channel $\mathscr{E}' \in B$ has a steady state ρ' such that

$|(o|\{\mathscr{E}^t|\rho') - |\rho)\}| \leq O(e^{-t/\tau})$ for any local operator

- NB τ is size-independent **and uniform** throughout D.
- Primes are counterintuitively placed: we are saying "every steady state is downstream of steady states elsewhere in B"
- A phase is the closure of uniform regions for all (δ, τ)

Allows for moving within the phase using finite-depth channels



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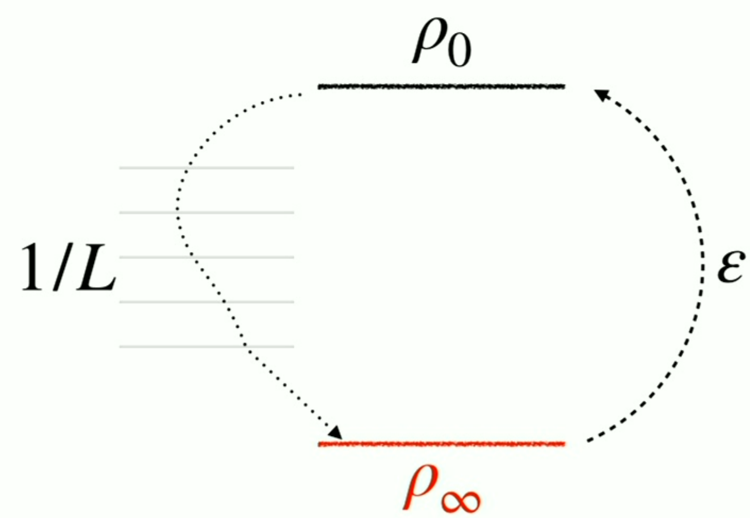

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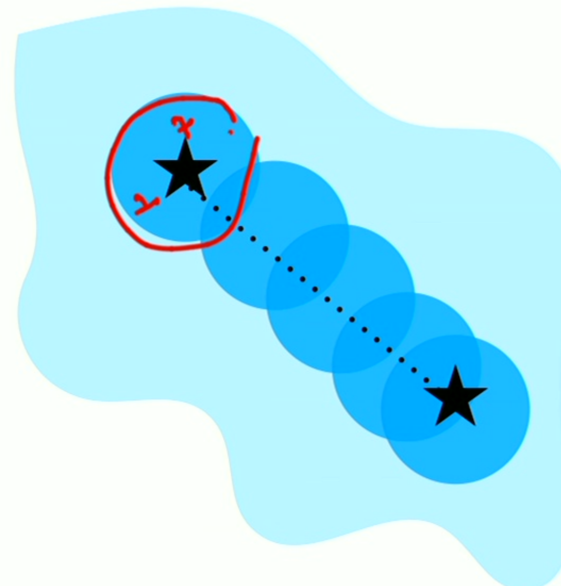
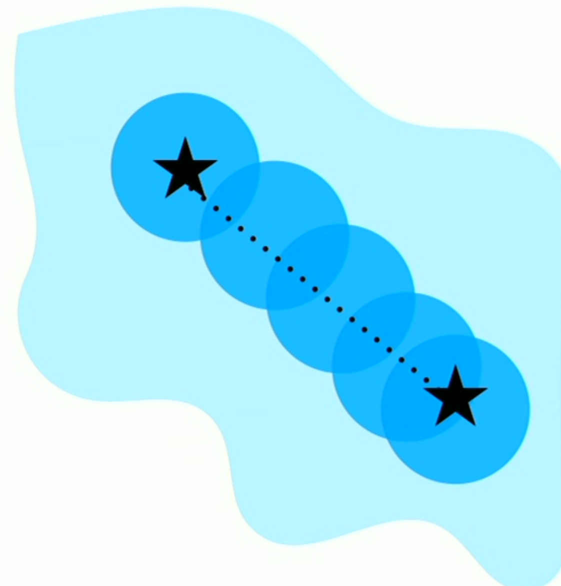


why would this be an equivalence?

- One answer: we defined it that way (but why is this sensible?)
- Defined w.r.t. *local* modifications of the dynamics
- Consider adding a weak perturbation P_ϵ to a channel $\mathscr{E}_0$ with a nontrivial steady state
- Even if P_ϵ destroys the steady state, it must take a long time to do so: for times $\ll 1/\epsilon$, $P_\epsilon \mathscr{E}_0 \sim \mathscr{E}_0$
- If we had a nontrivial-trivial phase transition, steady states near the phase boundary would relax slowly when quenched weakly across the transition *in either direction*



implications of uniformity

Direct consequences of finite depth channel + light cone:

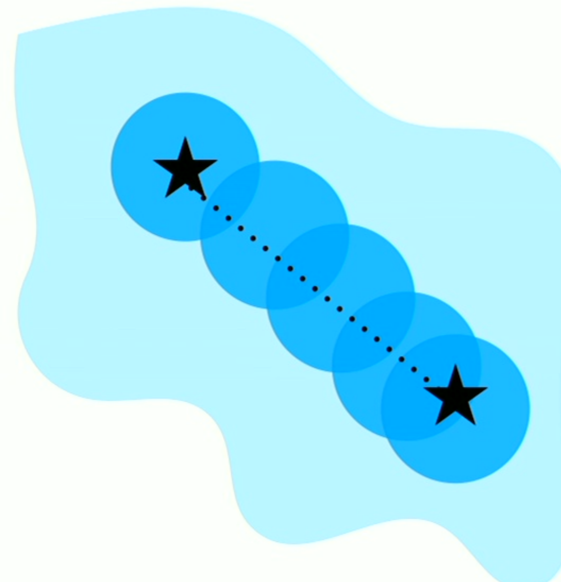
- Analyticity of correlations inside a phase
- Local perturbations perturb locally
- If one point in the phase has long-range order then so does every other point

Spectral implication of uniformity: perturbations have controlled matrix elements between steady state and **local** observable:

$|(o|(R_0 V)^n|\rho_0)| \leq e^{cn}$ for some c

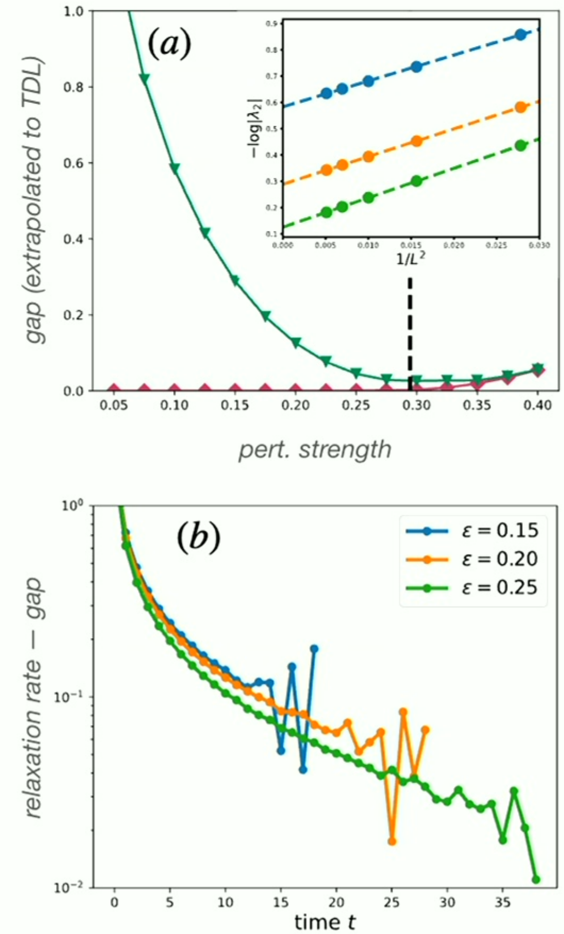
where $R_0 = Q(\mathbb{I} - \mathscr{E}_0)^{-1}Q$, resolvent of unperturbed channel

Deriving these results from uniformity follows standard Hastings logic



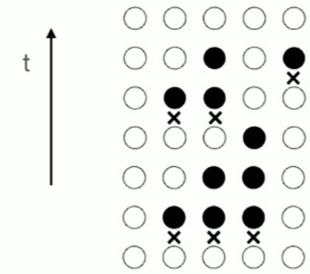
uniformity and the spectral gap

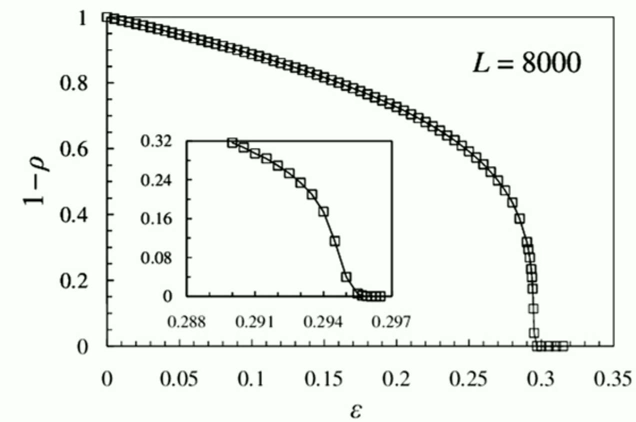
- Two conceptually distinct questions:
  - At what rate do steady states in the same phase reach each other, in the thermodynamic limit?
  - What is the spectral gap?
- We conjecture that these coincide
- Difficult to prove without additional assumptions on the spectrum of the channel
- Numerics supports this identification (in Stavskaya's model, where uniformity is known to hold)



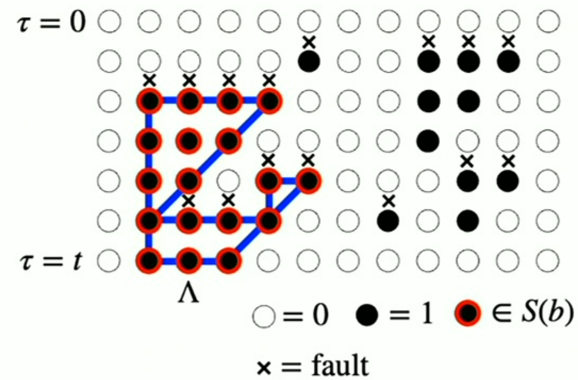
stavskaya's automaton

- One classical bit at each site, one-dimensional geometry
- Two-site update rule: $b_x \to \min(b_x, b_{x+1})$ applied probabilistically
- Absent errors, two steady states: all-0, all-1
- All-1 state is unstable to errors of type $1 \to 0$
- Consider maximally biased errors, $0 \to 1$ at some rate ε mixed in with Stavskaya dynamics: all-1 state is an exact steady state by construction
- Claim: there's a phase transition at $\varepsilon_c > 0$ below which there is another steady state (to which almost all initial states are absorbed)





uniformity from erosion of errors



- In Stavskaya, unperturbed dynamics erodes errors
- When there are multiple local steady states, linear erosion is the best you can do
- Naive conjecture: erosion is a sufficient condition for stability
- This criterion misses nonperturbative instabilities: roughly, the growth of the erroneous region can be

$$\partial_t R = -1 + \epsilon f(R)$$

- When the surface area for growth is large, very large errors can grow instead of shrinking
- Less naive conjecture: nonperturbative instability requires a finite entropy of distinct steady state configurations

summary/outlook

- Physical picture of nontrivial steady states of many-body channels/Lindbladians
  - Cannot be quickly reached from trivial state because of emergent large Jordan blocks
  - Can be quickly reached from other states in the same phase (on timescale set by the gap): this assumption plus Lieb-Robinson implies various familiar properties of a gapped phase
  - Evades non-invertibility of channels by defining a local equivalence relation
- Open questions/issues:
  - Establishing uniformity for higher-dimensional models
  - Counterexamples with stable steady states violating uniformity, e.g., symmetric diffusion
  - Efficiently checkable criteria for uniformity
- How does active error correction fit into this framework?

uniformity from erosion of errors

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