Title: Entanglement-based probes of topological phases of matter

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Abstract: I will discuss recent progress in understanding entanglement-based probes of 2D topological phases of matter. These probes are supposed to extract universal topological information from a many-body ground state. Specifically, I will discuss (1) the topological entanglement entropy, which is supposed to give information about the number of anyon excitations, and (2) the modular commutator, which is supposed to tell us the chiral central charge.

Entanglement-based probes of topological phases of matter

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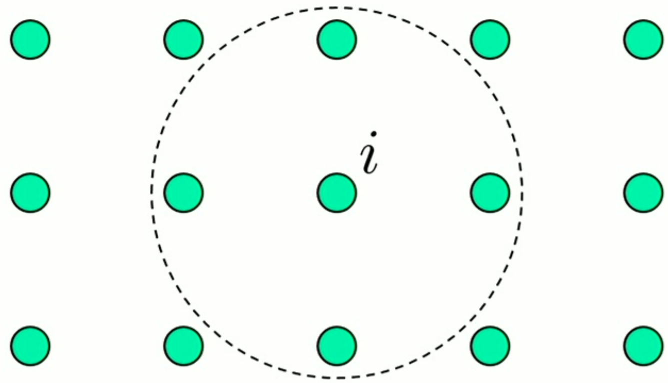
TEE: (Kim, Levin, Lin, Ranard, Shi, arXiv:2302.00689)
(ML, in prep)

Modular commutator: (Gass, ML, arXiv:2405.15892)

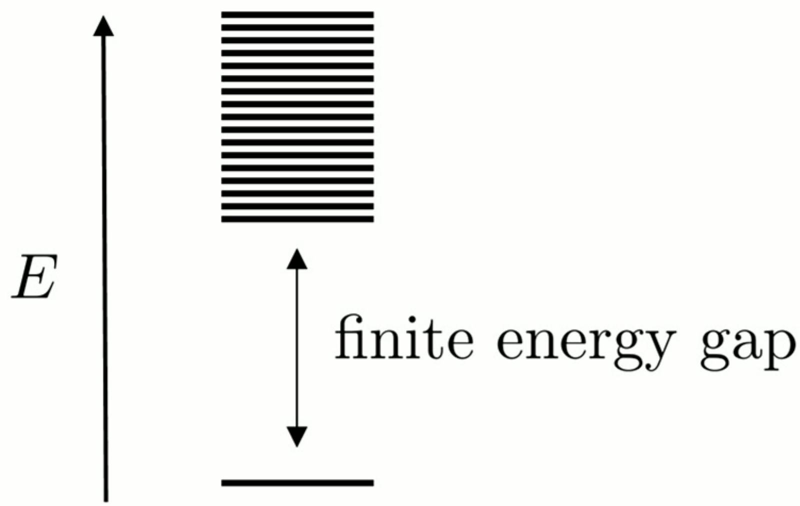
(2D) lattice models



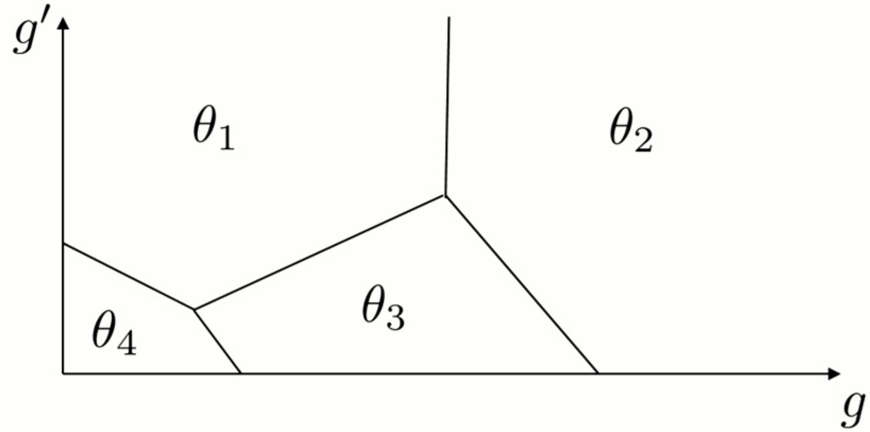
Hilbert space: $\mathcal{V} = V \otimes V \otimes V \otimes \cdots$

Hamiltonian: $H = \sum_i H_i$

Focus on *gapped* lattice models:



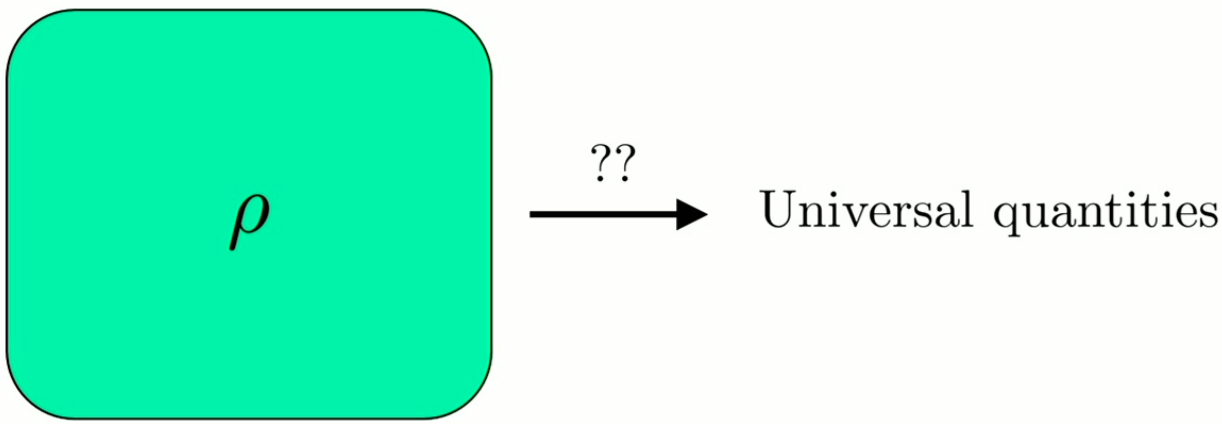
Universal quantities



"universal quantity" = quantity that is constant throughout gapped *phase*

Examples: anyon data, chiral central charge c_-

Expect universal quantities are encoded in ground state:



But how to extract them?

This talk: two entanglement based probes

1. Topological entanglement entropy

   $\Longrightarrow$ number of anyon excitations

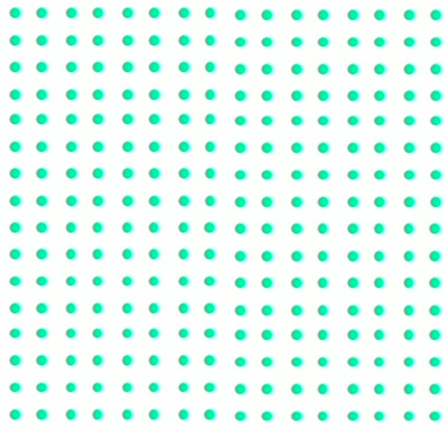
2. Modular commutator

   $\Longrightarrow$ chiral central charge c_-

(Kitaev, Preskill, hep-th/0510092)
(Levin, Wen, cond-mat/0510613)

Topological entanglement entropy

Let $\rho =$ GS density operator



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Topological entanglement entropy

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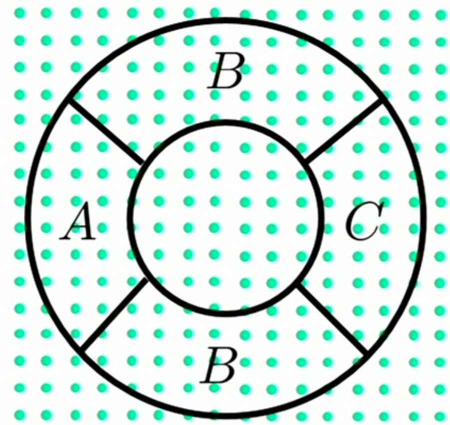
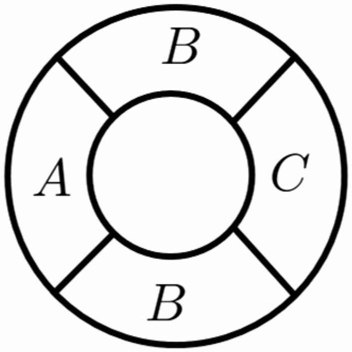


Define:

$$\gamma = \frac{1}{2}[S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC})]$$

$$\rho_R = \mathrm{Tr}_{R^c}(\rho), \qquad S(\rho_R) = -\mathrm{Tr}(\rho_R \log \rho_R)$$

Equivalently:



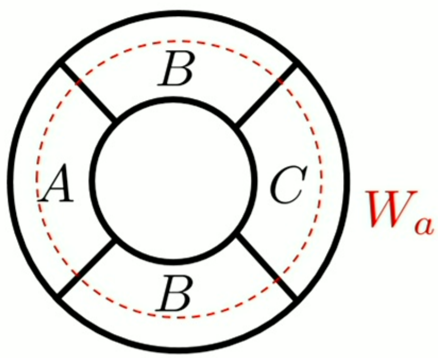
$$\gamma = \frac{1}{2}\underbrace{[S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC})]}_{I(A:C|B)_\rho}$$

"conditional mutual information"

Measures "global" correlations in ABC

Intuition/motivation

States with anyon excitations have these "global" correlations:



$$\implies \text{expect} \begin{cases} \gamma > 0 \text{ if there are anyons} \\ \gamma = 0 \text{ otherwise} \end{cases}$$

Original claim about TEE

In the limit of large regions,

$$\gamma = \log \mathcal{D}$$

where

$$\mathcal{D} = \sqrt{\sum_a d_a^2}$$

d_a: quantum dimensions

$\sum_a$: sum over anyon types

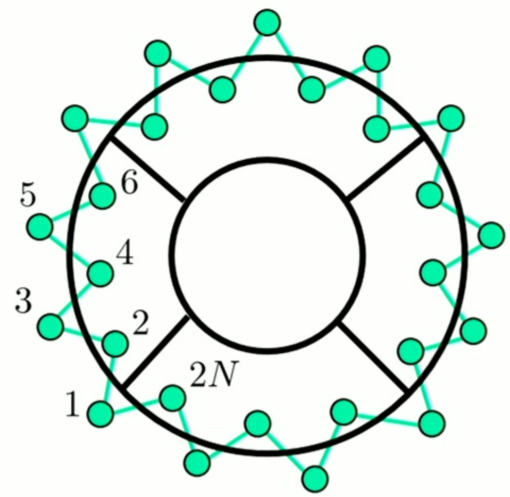
(Haah, Zou, arXiv:1604.06101)

Bravyi's counterexample

Consider ground state of

$$H = -\sum_{i=1}^{2N} Z_{i-1} X_i Z_{i+1}$$

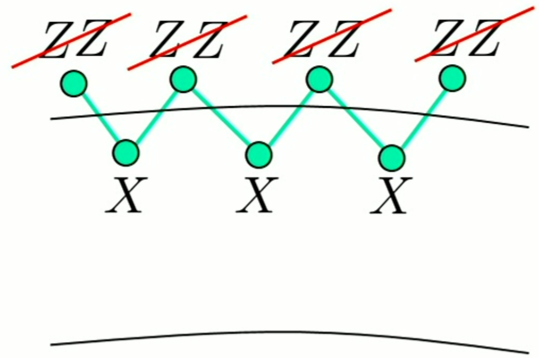
i.e. cluster state $|\psi_{cs}\rangle$



- Connected to product state by depth-2 circuit
- But $\gamma = \frac{1}{2}\log 2 > 0$

Why does $\gamma = \frac{1}{2}\log 2$?

$$Z_{i-1}X_iZ_{i+1}|\psi_{cs}\rangle = |\psi_{cs}\rangle$$



$$\implies X_2X_4\cdots X_{2N}|\psi_{cs}\rangle = |\psi_{cs}\rangle$$

$\implies$ $|\psi_{cs}\rangle$ has a "global" correlation in ABC

More quantitatively:

$$\rho_{ABC} = \tfrac{1}{2^{N-1}}(1 + X_2 X_4 \cdots X_{2N})$$

$$\implies S(\rho_{ABC}) = (N-1)\log 2$$

$$S(\rho_{AB}) = N_{AB}\log 2$$

$$S(\rho_{BC}) = N_{BC}\log 2$$

$$S(\rho_{B}) = N_{B}\log 2$$

$$\implies \gamma = \frac{1}{2}\log 2$$

Q: What is the precise relationship between TEE and anyon data?

A: In the limit of large regions,

$$\gamma \geq \log \mathcal{D}$$

(Kim, Levin, Lin, Ranard, Shi, arXiv:2302.00689)
(ML, in prep)

Q: What is the precise relationship between TEE and anyon data?

A: In the limit of large regions,

$$\gamma \geq \log \mathcal{D}$$

$\implies$ γ gives *upper bound* to anyon content

(Kim, Levin, Lin, Ranard, Shi, arXiv:2302.00689)
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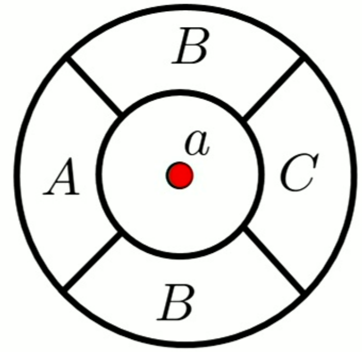
Proof of inequality: Abelian case

Consider a state ρ with N Abelian anyons $a \in \mathcal{A}$

Want to show:

$$I(A:C|B)_\rho \geq \log N$$

Anyon assumptions:



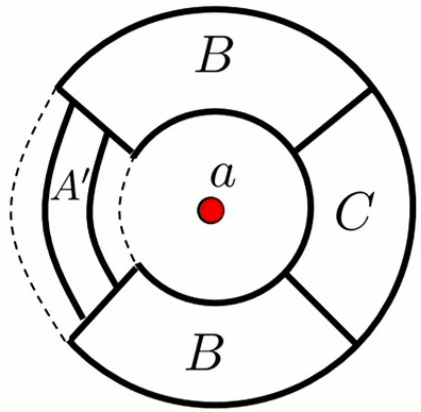
There exists a family of density operators $\{\rho^{(a)}, a \in \mathcal{A}\}$ with $\rho^{(1)} = \rho$, satisfying the following properties, for any ABC centered at origin:

1. **Global distinguishability:** $\rho^{(a)}_{ABC}$, $\rho^{(b)}_{ABC}$ orthogonal for $a \neq b$

2. **Local indistinguishability:** $\rho^{(a)}_{AB} = \rho^{(b)}_{AB}$ and $\rho^{(a)}_{BC} = \rho^{(b)}_{BC}$ for all a, b

3. **Fusion**: Let A' be a thinner version of A. There exist unitary operators $\{W_s, s \in \mathcal{A}\}$ supported in A with:

$$(W_s \rho^{(a)} W_s^\dagger)_{A'BC} = \rho_{A'BC}^{(s \times a)}$$



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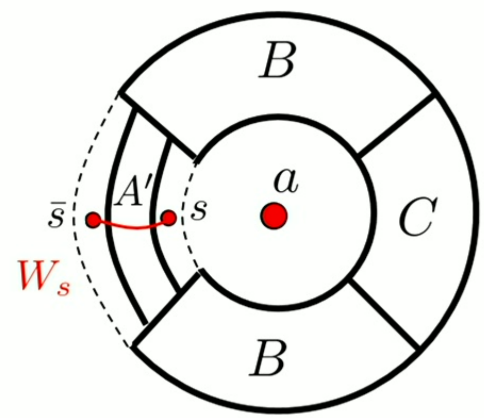
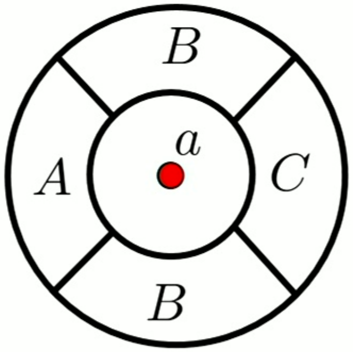


Anyon assumptions:



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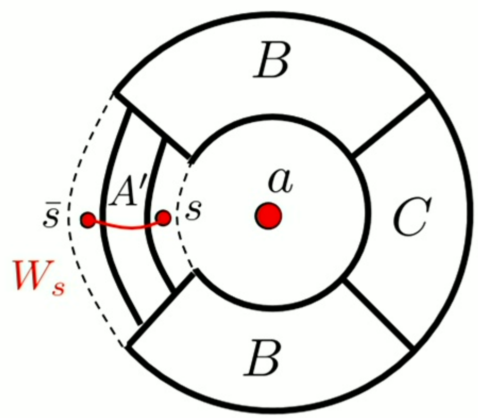


Outline of proof:

1. Consider mixed state $\lambda = \sum_a p_a \rho^{(a)}$.

2. Assumptions 1,2 $\implies$ $I(A:C|B)_\lambda = \sum_a p_a I(A:C|B)_{\rho^{(a)}} + \sum_a p_a \log p_a$.

3. Strong subadditivity $\implies$ $\boxed{\sum_a p_a I(A:C|B)_{\rho^{(a)}} \geq -\sum_a p_a \log p_a.}$

4. Assumption 3 + SSA $\implies$ $\boxed{I(A:C|B)_{\rho^{(a)}} \geq I(A':C|B)_{\rho^{(s\times a)}}.}$

5. Claim follows: $I(A:C|B)_{\rho^{(1)}} \geq \frac{1}{N} \sum_s I(A':C|B)_{\rho^{(s)}} \geq \log N$.

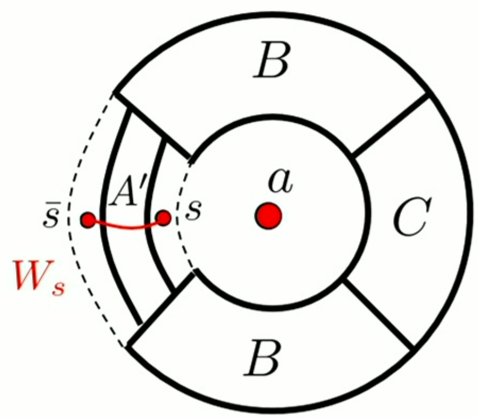
Comments

- Proof only uses SSA and basic anyon properties.

- Easy to generalize: higher dimensions, boundaries, mixed states, etc.

3. **Fusion**: There exist unitary operators $\{W_s, s \in \mathcal{A}\}$ supported in A with:

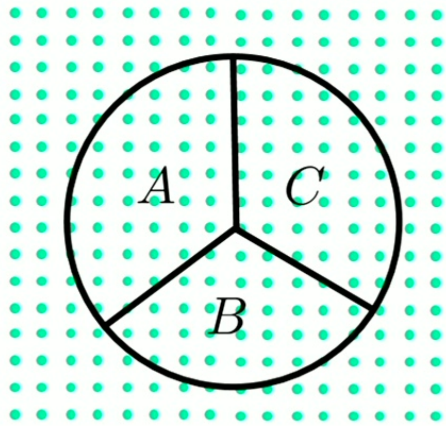
$$(W_s \rho^{(a)} W_s^\dagger)_{A'BC} = \sum_b p_{s \times a \to b} \cdot \rho^{(b)}_{A'BC}$$



(Kim, Shi, Kato, Albert, arXiv:2110.06932)

Modular commutator

Let $\rho =$ GS density operator



Define:

$$J(A,B,C)_\rho = i\mathrm{Tr}(\rho_{ABC}[K_{AB}, K_{BC}]),$$

$K_R \equiv -\log \rho_R$ "modular Hamiltonian"

(Kim, Shi, Kato, Albert, arXiv:2110.06932)

Proposal

In the limit of large A, B, C,

$$J(A,B,C)_\rho = \frac{\pi}{3} c_-$$

where

$$c_- = \frac{(\text{thermal Hall conductance } \kappa_H)}{\pi^2 k_B^2 T / 3h}$$

$c_- = 1$

Evidence

- Heuristic argument for entanglement bootstrap states

  (Kim, Shi, Kato, Albert, arXiv:2110.06932)

- Proof for non-interacting fermion GS

  (Fan, Zhang, Gu, arXiv:2211.04510)

- CFT based arguments

  (Zou, Shi, Sorce, Lim, Kim, arXiv:2206.00027)
  (Fan, arXiv:arXiv:2206.02823)

- Numerical results for Laughlin-like state

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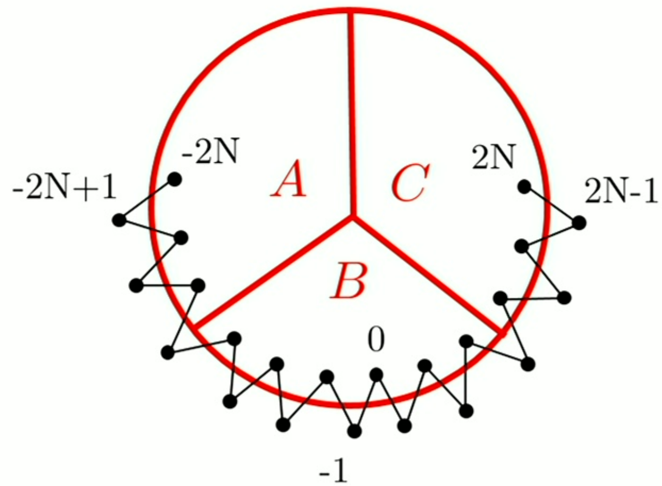
But does it always work?

(Gass, ML, arXiv:2405.15892)

1D counterexample

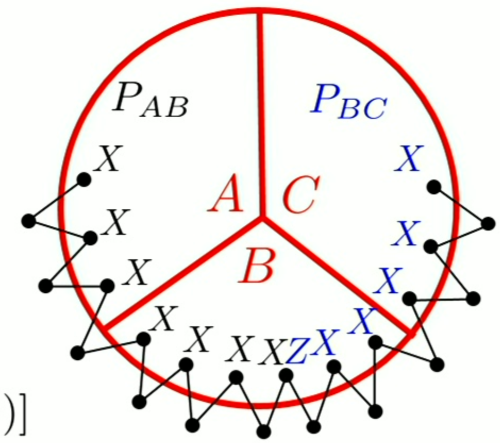
$$H = -\sum_{i=-2N}^{2N} h_i$$



$$h_i = Z_{i-1} X_i Z_{i+1}, \quad (i \neq 0, 1)$$
$$h_{-2N} = X_{-2N} Z_{-2N+1},$$
$$h_{2N} = Z_{2N-1} X_{2N},$$

$$h_0 = \frac{1}{\sqrt{3}}(Z_{-1}X_0 + Z_0 Z_1 + Z_{-1} Y_0 Z_1),$$
$$h_1 = Z_{-1} X_0 X_1 Z_2$$

$$\rho = \prod_i \left(\frac{1+h_i}{2}\right)$$



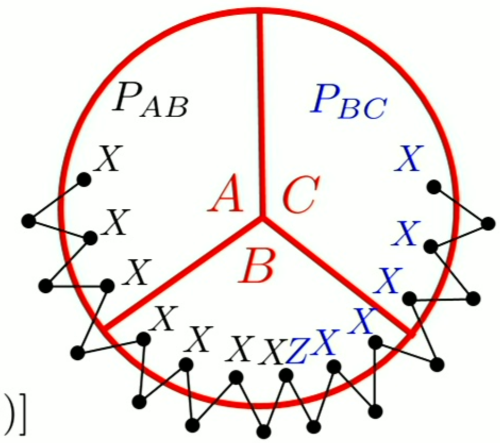
$$\implies \rho_{ABC} = \mathrm{Tr}_{\mathrm{odd}}\rho$$
$$= \frac{1}{2^{2N+1}}[1 + \frac{1}{\sqrt{3}}(P_{AB} + P_{BC} + iP_{AB}P_{BC})]$$

$$P_{AB} = X_{-2N}X_{-2N+2}\cdots X_{-2}X_0, \qquad P_{BC} = Z_0X_2\cdots X_{2N-2}X_{2N}$$

$$\implies \begin{array}{l} \rho_{AB} \propto 1 + P_{AB}/\sqrt{3} \\ \rho_{BC} \propto 1 + P_{BC}/\sqrt{3} \end{array} \qquad \implies \begin{array}{l} K_{AB} = c_{AB}1 - \lambda P_{AB} \\ K_{BC} = c_{BC}1 - \lambda P_{BC} \end{array}$$

$$\lambda = \log\left(\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right)$$

$$\rho = \prod_i \left(\frac{1 + h_i}{2} \right)$$



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$$P_{AB} = X_{-2N} X_{-2N+2} \cdots X_{-2} X_0, \qquad P_{BC} = Z_0 X_2 \cdots X_{2N-2} X_{2N}$$

$$\implies \begin{matrix} \rho_{AB} \propto 1 + P_{AB}/\sqrt{3} \\ \rho_{BC} \propto 1 + P_{BC}/\sqrt{3} \end{matrix} \qquad \implies \begin{matrix} K_{AB} = c_{AB} 1 - \lambda P_{AB} \\ K_{BC} = c_{BC} 1 - \lambda P_{BC} \end{matrix}$$

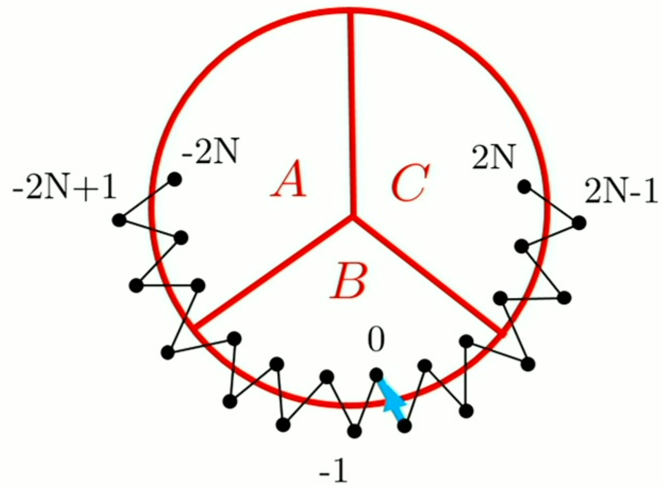
$$\lambda = \log \left(\frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}} \right)$$

$$\implies J(A, B, C)_\rho = 2\lambda^2/\sqrt{3}$$

Entangling circuit

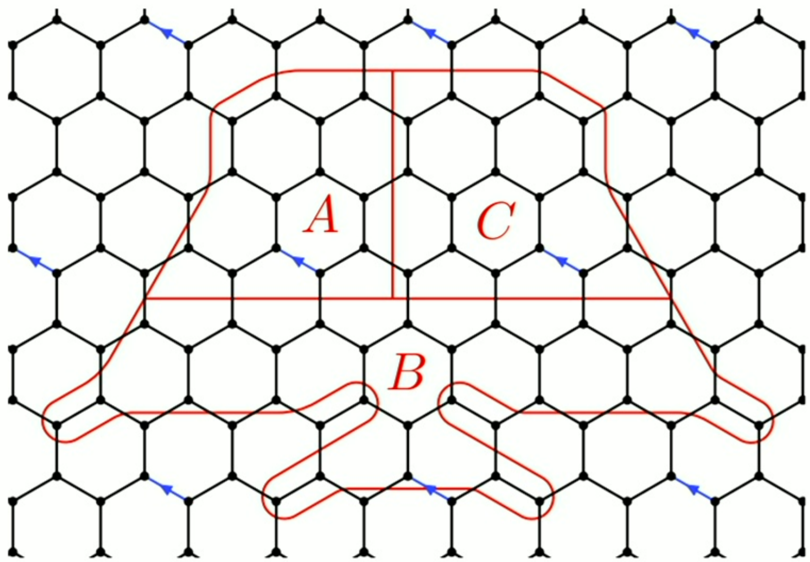
i — j $\equiv CZ_{i,j}$

i → j $\equiv \underbrace{V_{i,j}}_{CNOT_{i,j}\mathbf{R}_j}$



$$|\psi\rangle = \prod_{i\neq 0} CZ_{i,i+1} \cdot V_{1,0} |++\cdots+\rangle$$

2D counterexample



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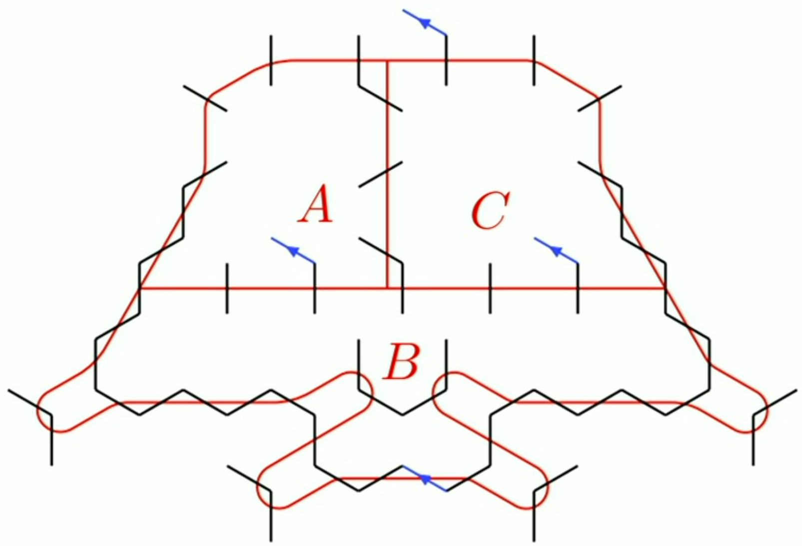

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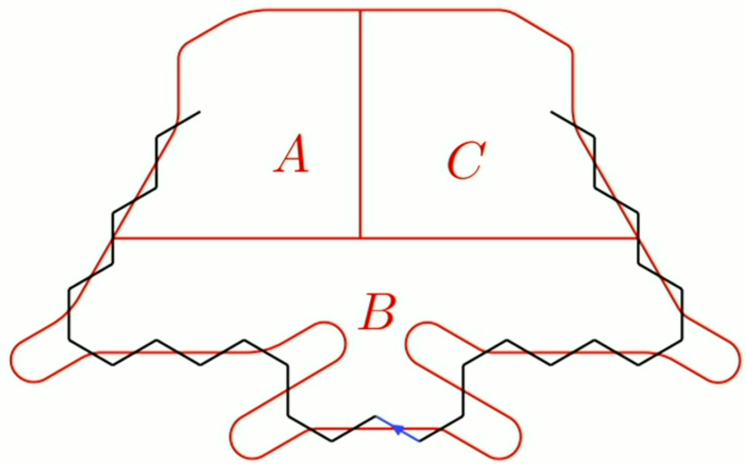

$$\Longrightarrow \quad J(A,B,C)_\rho = 2\lambda^2/\sqrt{3}$$

Comments/questions

- Counterexamples are *fine-tuned* (similar to spurious TEE)

- Could modular commutator work for “generic” ground states?

- What is the proper bulk definition of c_-?