Title: How much entanglement is needed for quantum error correction?

Speakers: Zhi Li

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Abstract: It is commonly believed that logical states of quantum error-correcting codes have to be highly entangled such that codes capable of correcting more errors require more entanglement to encode a qubit. Here we show that this belief may or may not be true depending on a particular code. To this end, we characterize a tradeoff between the code distance d quantifying the number of correctable errors, and geometric entanglement of logical states quantifying their maximal overlap with product states or more general ``topologically trivial" states. The maximum overlap is shown to be exponentially small in d for three families of codes: (1) low-density parity check (LDPC) codes with commuting check operators, (2) stabilizer codes, and (3) codes with a constant encoding rate. Equivalently, the geometric entanglement of any logical state of these codes grows at least linearly with d. On the opposite side, we also show that this distance-entanglement tradeoff does not hold in general. For any constant d and k (number of logical qubits), we show there exists a family of codes such that the geometric entanglement of some logical states approaches zero in the limit of large code length.

How much entanglement is needed for quantum error correction?

Zhi Li
Perimeter Institute

Physics of Quantum Information, May 28th

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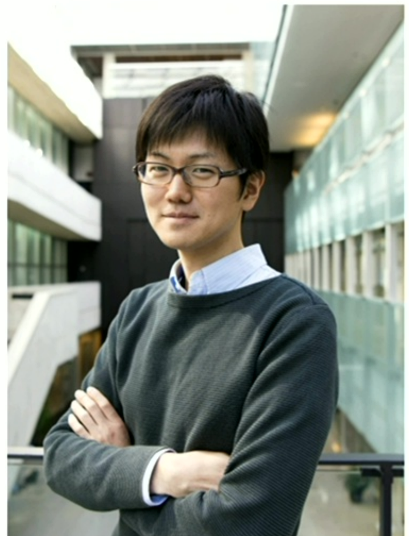
Thanks my collaborators!

Sergey Bravyi
(IBM)



Beni Yoshida
(Perimeter)

Dongjin Lee
(Perimeter)

based on: 2405.01332, 2405.07970

Q: How much entanglement is needed for quantum error correction?

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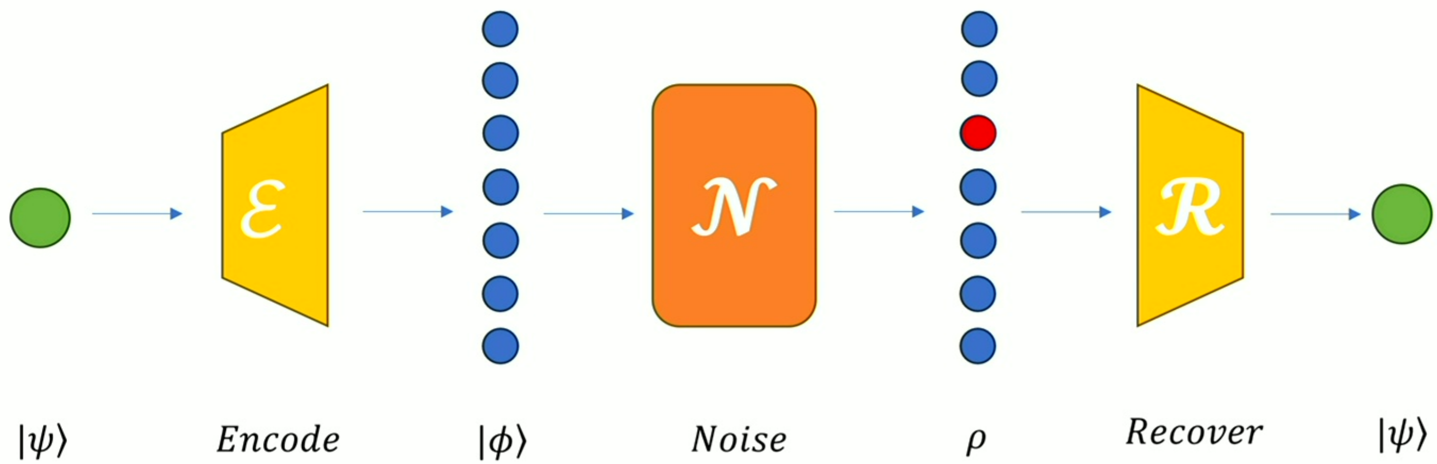
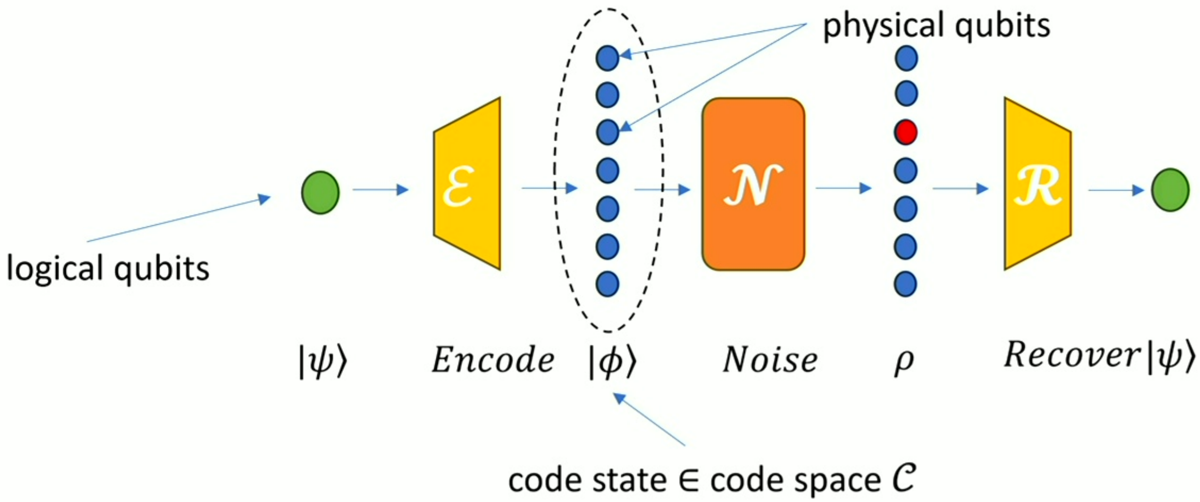
Quantum error correction

protecting (quantum) information by redundancy



QEC: $\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}(|\psi\rangle) = |\psi\rangle$

Quantum error-correcting code

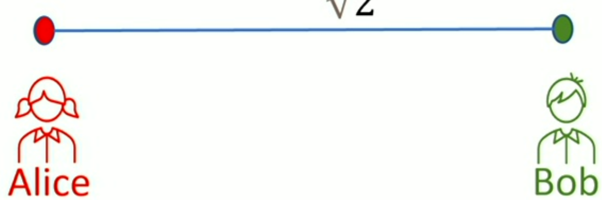


- $n =$ number of physical qubits, $\dim(\mathcal{H}) = 2^n$
- $k =$number of logical qubits, $\dim(\mathcal{C}) = 2^k$
- code distance d: (roughly) maximal number of faulty physics qubits

Q: How much entanglement is needed for quantum error correction?

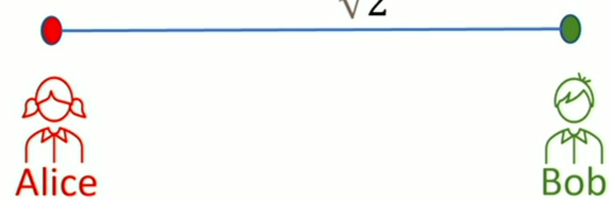
Entanglement

$$|\phi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



$$\rho_A = \rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$|\phi'\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$



$$\rho_A = \rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

complete knowledge of individual subsystems ≠ knowledge of whole system

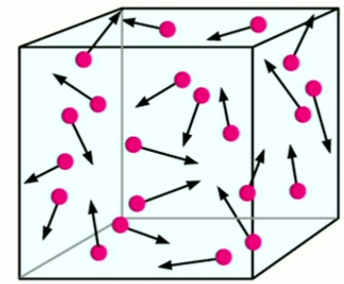
“Classical”

- Classical physics

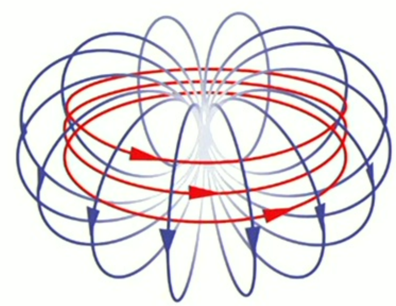
  n particles in 3D: $x_i, y_i, z_i, v_{xi}, v_{yi}, v_{zi} \quad (i = 1, \ldots, n)$

  EM field: $\vec{E}(x, y, z), \vec{B}(x, y, z)$

- Full description = descriptions for individual components



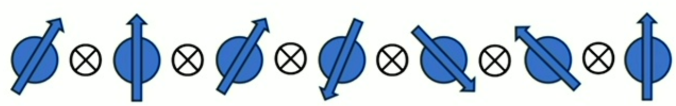
(fig from Mikhailovsky)



(fig from tabitarezaire.com)

- Counterpart in quantum physics: product states

$$|\phi\rangle = \otimes_{i=1}^{n} |\phi_i\rangle$$

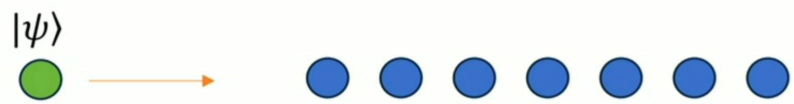


* for experts: “non-classical” things can still be classically simulatable

Q: Why Is entanglement needed for quantum error correction?

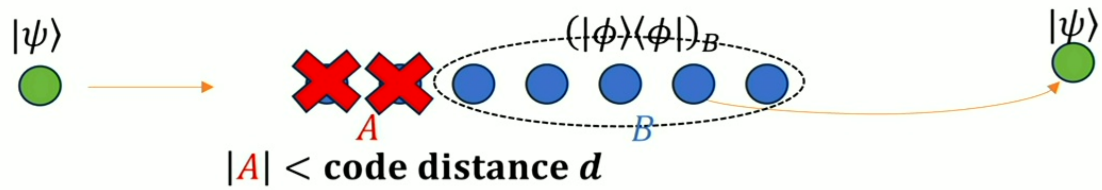
A: Quantum error-correcting codes must be entangled

Quantum error-correcting code



No Cloning: $|\psi\rangle \not\rightarrow |\psi\rangle|\psi\rangle|\psi\rangle|\psi\rangle|\psi\rangle|\psi\rangle|\psi\rangle$

Quantum error-correcting code



No Cloning: full information in B $\leftrightarrow$ no information in A

[*for experts: monogamy of entanglement]

➢ code states are **locally indistinguishable** within **code distance** $\boldsymbol{d}$**:**

local indistinguishability
(KL condition)

$$\forall A \; s.t. |A| < d, \; \forall |\phi\rangle, |\phi'\rangle \in \mathcal{C}$$
$$(|\phi\rangle\langle\phi|)_A = (|\phi'\rangle\langle\phi'|)_A$$

(Knill, Laflamme 1996)

$|\phi\rangle = |0^{\otimes n}\rangle \in \mathcal{C}$

$|\psi\rangle \in \mathcal{C}$

local indistinguishability
(KL condition)

$\forall \text{qubit } q, (|\psi\rangle\langle\psi|)_q = |0\rangle\langle 0|$

$\Downarrow$

$|\psi\rangle = |0^{\otimes n}\rangle$

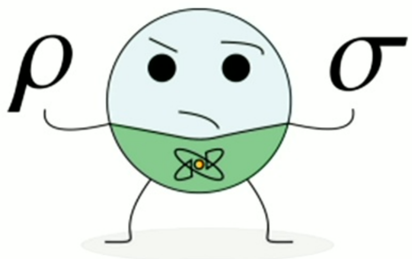
Entanglement is not sufficient…

product state	entangled state	QECC
$\lvert\phi\rangle = \otimes_{i=1}^{n} \lvert\phi_i\rangle$ local determines global	$\lvert\phi\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$ $\lvert\phi'\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$ locally same, globally different (however, $\frac{\lvert\phi\rangle + \lvert\phi'\rangle}{\sqrt{2}} = \lvert 00\rangle$)	all code states, including all superpositions, are locally indistinguishable with each other

Q: How much entanglement is needed for quantum error correction?

A2: (Useful) Quantum error-correcting codes must be highly entangled

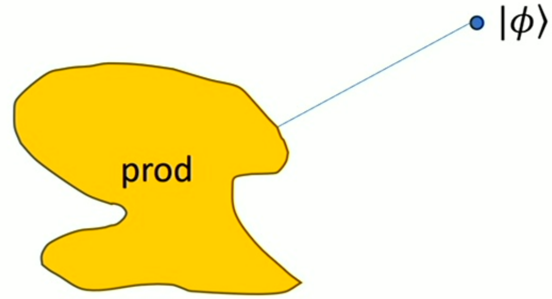
Geometric Entanglement



Fidelity *(fig from: Cerezo)*

$$F(\rho,\sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$$

$$F(|\phi_1\rangle, |\phi_2\rangle) = \cos\theta = |\langle\phi_1|\phi_2\rangle|$$



$$\max_{|P\rangle:prod} |\langle P|\phi\rangle|^2 = exp(-\mathrm{GEM})$$

Wavefunctions

Five-qubit code

(Laflamme et al 1996, Bennett et al 1996)

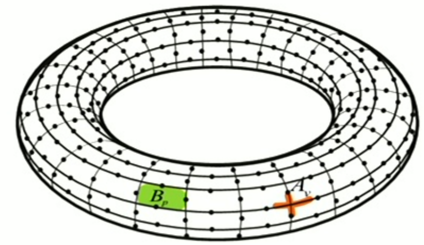
$$|0_L\rangle = \frac{1}{4}\Big[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle$$
$$+|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle$$
$$-|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle$$
$$-|10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big]$$

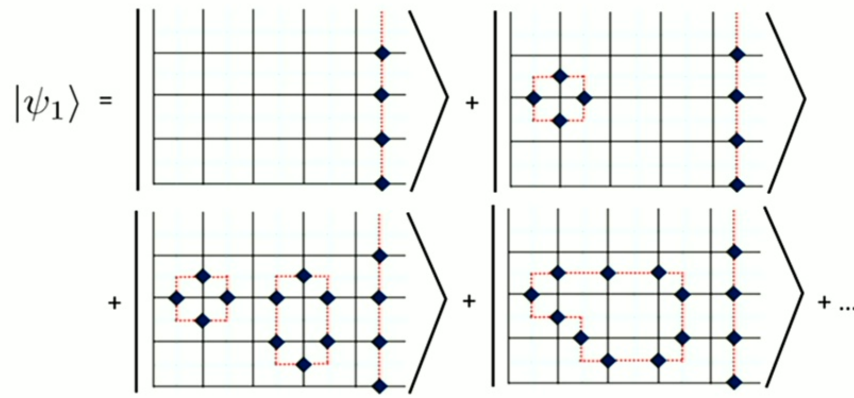
$$|1_L\rangle = \frac{1}{4}\Big[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle$$
$$+|10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle$$
$$-|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle$$
$$-|01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]$$

wavefunctions (fig from Nielsen&Chuang)

Kitaev's toric code

(Kitaev 1997)





toric code wavefunction (fig from: Kufel)

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large GEM → large "expansion number"

$$|\phi\rangle = \sum_{i=1}^{M} a_i \, |P_i\rangle$$

$$\implies 1 = \sum_{i=1}^{M} |a_i|^2 = \sum_{i=1}^{M} |\, \langle P_i|\phi\rangle \,|^2 \leq M \exp\left(-\mathrm{GEM}\right)$$

$$\implies \exp\left(\mathrm{GEM}\right) \leq M$$

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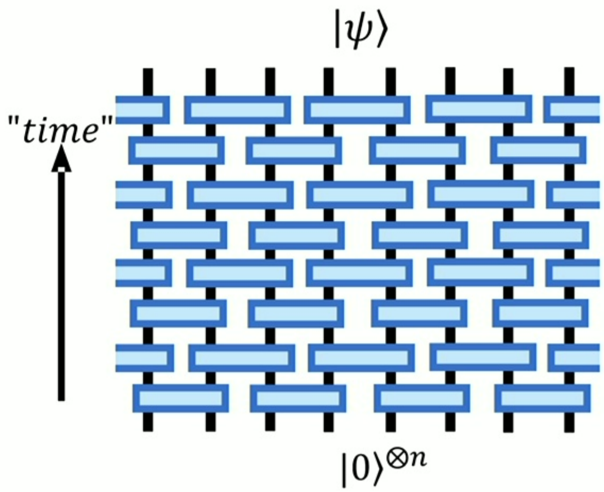
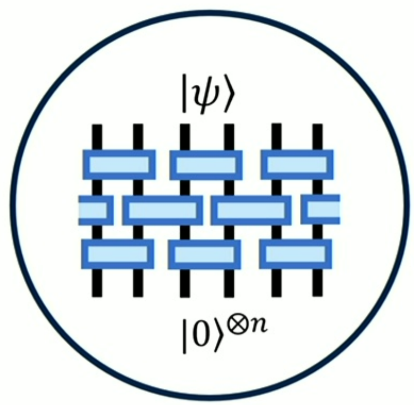
Circuit Complexity

circuit complexity = minimal circuit depth

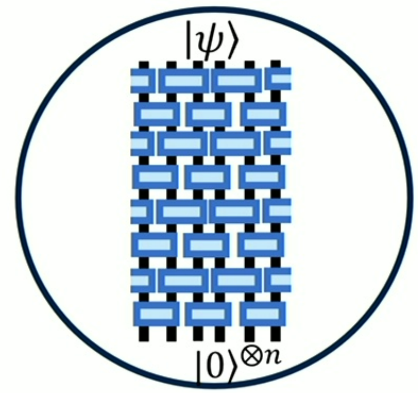


local unitary circuit

geometrically local
"k-local"



Complexity=$O(1)$
Short-range entangled (SRE)

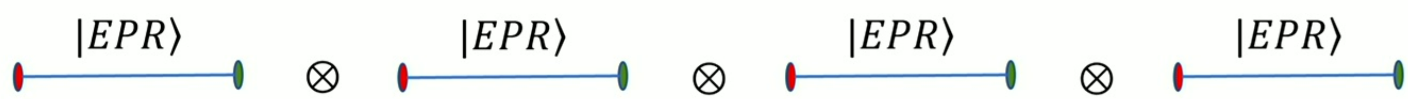


Complexity=$\omega(1)$
Long-range entangled (LRE)

Geometric Entanglement v.s. Circuit Complexity

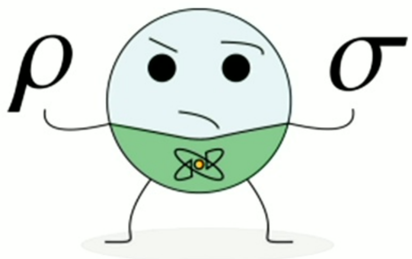
$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|00000\ldots\rangle + |11111\ldots\rangle)$$

GEM=O(1), depth=O(n) or O(log n)



GEM=O(n), depth=1

Geometric Entanglement



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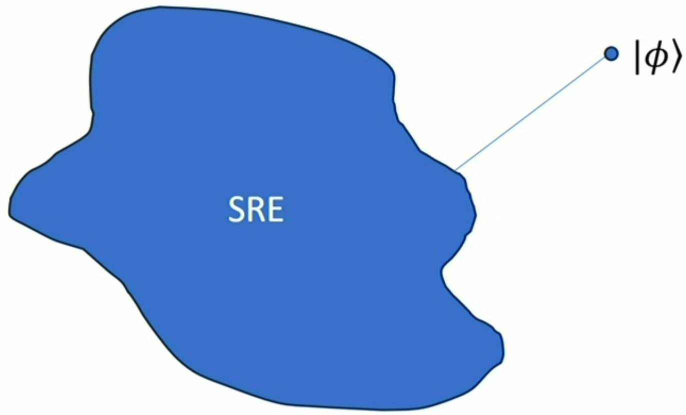
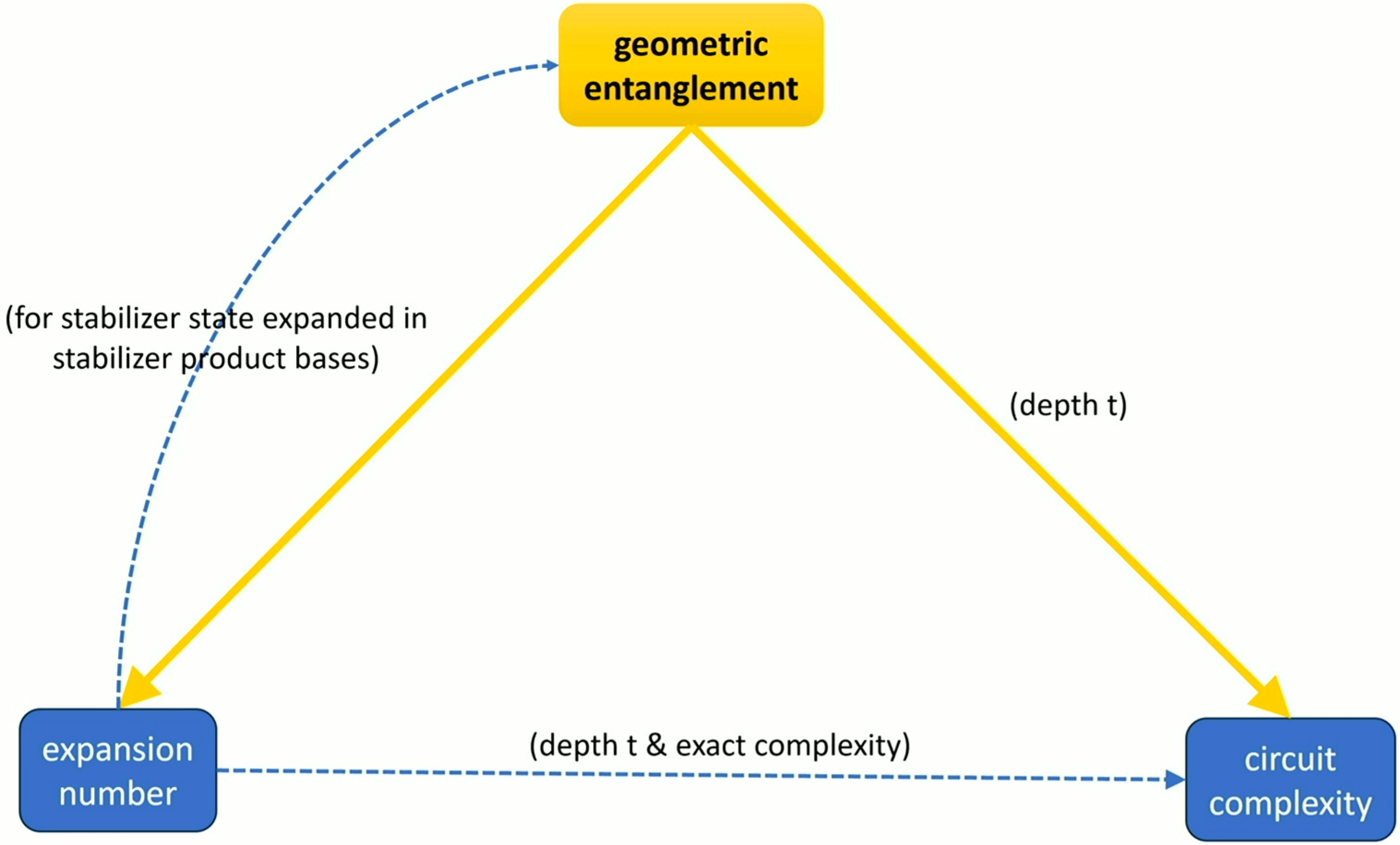


$$\max_{|P\rangle : depth \le t} |\langle P|\phi\rangle|^2 = exp(-GEM_t)$$


geometric entanglement
(for stabilizer state expanded in stabilizer product bases)
(depth t)
expansion number
(depth t & exact complexity)
circuit complexity

Our Main Results

- General QEC: expansion number lower bound

- GEM in general QEC: a non-example

- GEM lower bound
  - High rate QEC
  - Stabilizer QEC
  - LDPC codes

- GEM lower bound based on topological order

Expansion number

Theorem 1 (any QECC)

$\forall\, |\phi\rangle \in$ QECC with code distance d,

expansion number $\geq 2^{d-1}$

- works for any QEC
  - do not assume stabilizer structure
  - do not assume locality
  - do not assume dimensionality

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A non-example

$$|\psi_0\rangle = \sqrt{1-p}|0\rangle^{\otimes n} + \sqrt{p}|1\rangle^{\otimes n}, \quad |\psi_1\rangle = \sqrt{\frac{2}{n(n-1)}} \sum_{|x|=2} |x\rangle$$

$p = \frac{2}{n}$

Hamming weight

- QEC with $d = 2, k = 1$, yet overlap= $1 - O(\frac{1}{n})$!
- Higher d and k possible via code concatenation

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Geometric entanglement for high rate QECC

Theorem 2 (any QECC)

$\forall |\phi\rangle \in$ QECC with code distance d and dimension 2^k, $\forall$ product state $|P\rangle$,

$$|\langle P|\phi\rangle|^2 \leq \prod_{i=0}^{d-2}\left[1 - H^{-1}\left(\frac{k}{n}\right)\right]$$

Recall

- k =number of logical qubits
- $2^k = dim$(code space)
- code rate=$\frac{k}{n}$

$H = -x\log(x) - (1-x)\log(1-x)$
H^{-1} = inverse function

Corollary (high rate QECC)

$\forall |\phi\rangle \in$ QECC with const rate ($\frac{k}{n} = O(1)$), $\forall$ product state $|P\rangle$,

$$|\langle P|\phi\rangle|^2 \leq e^{-O(d)}$$

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Stabilizer code examples

Five-qubit code *(Laflamme et al 1996, Bennett et al 1996)*

$$
\begin{aligned}
|0_L\rangle = \frac{1}{4}\Big[&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\
&+|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\
&- |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\
&- |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big] \\
|1_L\rangle = \frac{1}{4}\Big[&|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\
&+|10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\
&- |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle \\
&- |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]
\end{aligned}
$$

wavefunctions (fig from Nielsen&Chuang)

g_1	$XZZXI$
g_2	$IXZZX$
g_3	$XIXZZ$
g_4	$ZXIXZ$

Stabilizer operators: $[g_i, g_j] = 0$

Stabilizer Code space: $\{|\phi\rangle|\ g_i|\phi\rangle = |\phi\rangle, \forall i\}$

Geometric entanglement for stabilizer QEC

Theorem 3 (stabilizer QECC)

$\forall |\phi\rangle \in$ stabilizer QECC with code distance d, $\forall$ product state $|P\rangle$,

$$|\langle P|\phi\rangle|^2 \leq \frac{1}{2^{d-1}}$$

- more protection wanted, more entanglement needed

- for stabilizer state expanded in stabilizer product bases,

$$\max_{|P\rangle \in \mathrm{stab}} |\langle P|\phi\rangle|^2 = \big(\text{(stab) expansion number}\big)^{-1}$$

- Thm. 1 $\nRightarrow$ Thm. 2:

$$\max_{|P\rangle \in \mathrm{stab}} |\langle P|\phi\rangle|^2 < \max_{|P\rangle} |\langle P|\phi\rangle|^2$$

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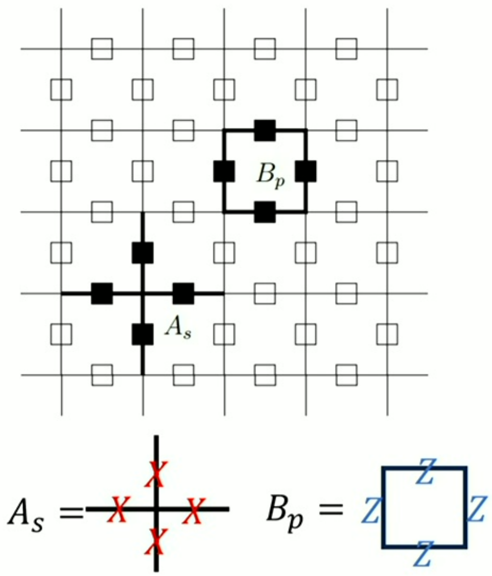
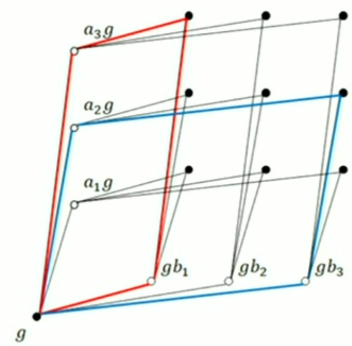
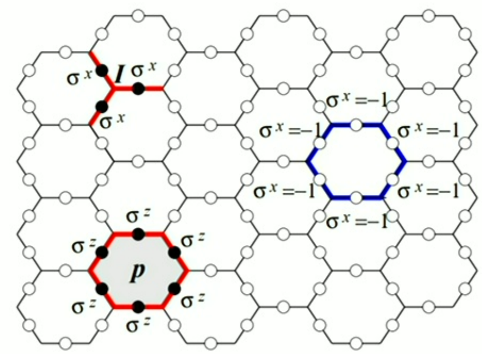
LDPC: low density parity check







(stabilizer) check operators

(non-stabilizer) check operators

Geometric entanglement for LDPC code

Theorem 4 (LDPC code)

$\forall|\phi\rangle \in$ LDPC QECC with code distance d, $\forall$ product state $|P\rangle$,

$$|\langle P|\phi\rangle|^2 \leq e^{-O(d)}$$

- coefficient in $O(d)$ depends on sparsity
- does not require large k
- also works for non-stabilizer codes

Theorem 4' (high rate LDPC code)

$\forall|\phi\rangle \in$ LDPC QECC with const rate ($\frac{k}{n} = O(1)$), $\forall$ product state $|P\rangle$,

$$|\langle P|\phi\rangle|^2 \leq e^{-O(n)}$$

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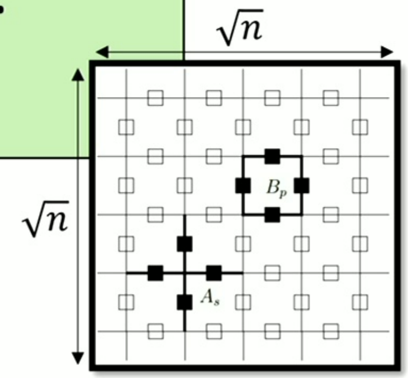
Geometric entanglement for Toric Code

Theorem 5: ∀ toric code state $|TC\rangle$, ∀ finite-depth unitary circuit U (depth t), ∀ product state $|P\rangle$:

$$|\langle P|U|TC\rangle|^2 \leq e^{-\Theta(\frac{n}{t^2})}$$



- n = area
- U = id: $|\langle P|TC\rangle|^2 \leq e^{-\Theta(n)}$
- U = id: Thm. 3 or 4 → $|\langle P|TC\rangle|^2 \leq e^{-\Theta(\sqrt{n})}$
- $U \neq$ id: "stability" under finite-depth unitary

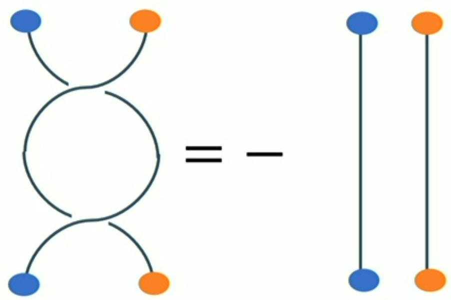
Corollary: to prepare ∀ $|TC\rangle$ from $|0^n\rangle$, circuit depth

$$t \geq \Theta(\sqrt{n})$$

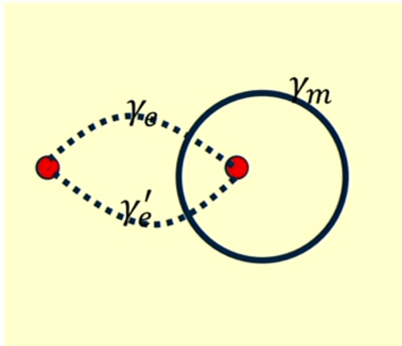
(Bravyi, Hastings, Verstraete 2006)

Corollary: expansion number of ∀ $|TC\rangle$ is at least $e^{\Theta(n)}$

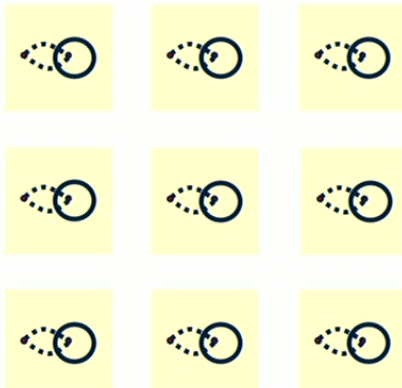
Anyons in Toric Code



braiding statistics



→ long range entanglement



→ GEM $\geq O(n)$

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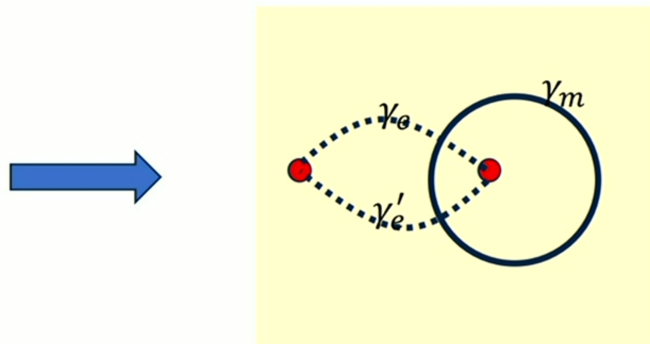
Product overlap 2D stabilizer QECC

Theorem 5': $\forall$ 2D stabilizer code $|\Psi\rangle$, $\forall$ finite-depth unitary circuit U (depth t), $\forall$ product state $|P\rangle$:

$$|\langle P|U|\Psi\rangle|^2 = e^{-\Omega(\frac{d^2}{t^2})}$$

- Idea

2D stabilizer codes



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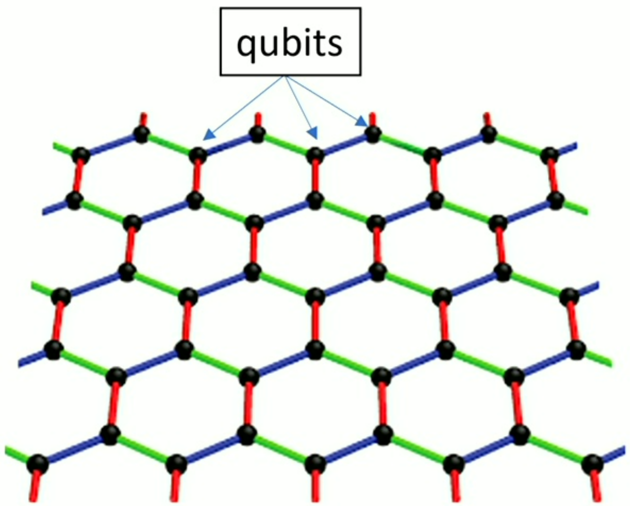
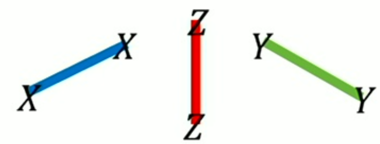
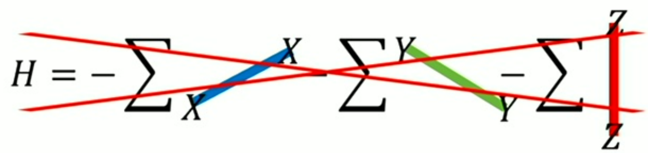
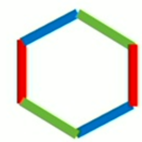
Kitaev Honeycomb model *(Kitaev 2005)*

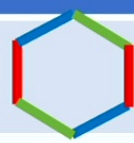
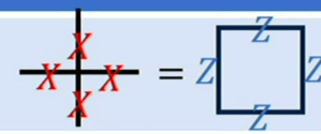
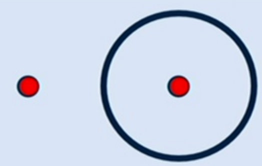


link operator: 



symmetry generator: 

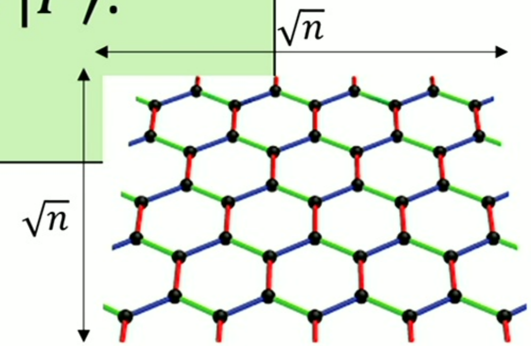
$H = -\sum$ 

	Honeycomb model	Toric code model
"code space"	 $= 1$ ("symmetric sector")	 $= 1$
code distance	2	$\sqrt{n}$
dimension	$e^{O(n)}$	4 (on torus)
particle	Majorana fermions	$\mathbb{Z}_2$ anyons
detection		

Product overlap for Kitaev Honeycomb model

Theorem 4: $\forall$ symmetric state $|Hc\rangle$, $\forall$ finite-depth unitary circuit U (depth ω), $\forall$ product state $|P\rangle$:

$$|\langle P|U|TC\rangle|^2 = e^{-\Omega(\frac{n}{\omega^2})}$$



- $n =$ area
- U: "stability" under finite-depth unitary
- $d = 2$! Theorem 2 is helpless

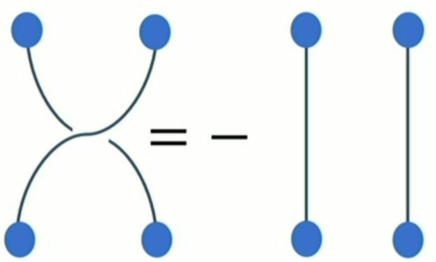
Corollary: to prepare $\forall\ |Hc\rangle$ from $|0^n\rangle$, circuit depth

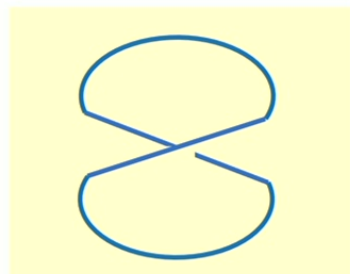
$$\omega \geq \Theta(\sqrt{n})$$

Corollary: expansion number of $\forall\ |Hc\rangle$ is at least $e^{\Omega(n)}$

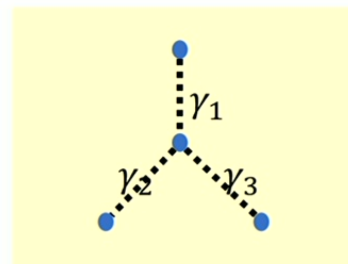
Fermions in Honeycomb model

exchange statistics





$e^{i\alpha} = -1$



$\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_1\gamma_2\gamma_3 = -1$

(Levin Wen 2003)

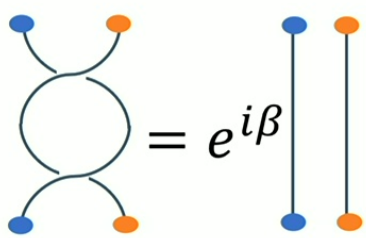
Conclusion

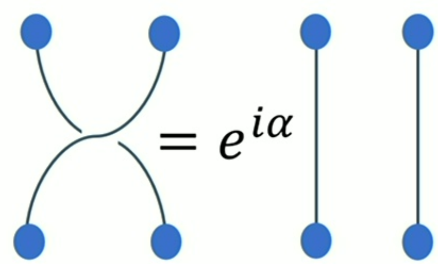
- ➢Quantum information: (useful) QECC are highly entangled
  - ➢LDPC, stabilizer, constant rate
  - ➢GEM$\geq O(d)$
  - ➢Circuit complexity $\geq O(\log d)$
  - ➢Expansion number $\geq e^{O(d)}$

- ➢More protection wanted, more entanglement needed

Conclusion

- Physics:
  - $GEM \geq O(n)$
  - circuit depth lower bound $O(\sqrt{n})$
  - expansion number lower bound $\exp(O(n))$
- Emergent phenomena (anyons/fermions) require large entanglement





Thanks!