Title: Stability of mixed-state quantum phases via finite Markov length

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Abstract: For quantum phases of Hamiltonian ground states, the energy gap plays a central role in ensuring the stability of the phase as long as the gap remains finite. In this talk we introduce Markov length, the length scale at which the quantum conditional mutual information (CMI) decays exponentially, as an equally essential quantity characterizing mixed-state phases and transitions. For a state evolving under a local Lindbladian, we argue that if its Markov length remains finite along the evolution, then it remains in the same phase, meaning there exists another quasi-local Lindbladian evolution that can reverse the former one. We apply this diagnostic to toric code subject to decoherence and show that the Markov length is finite everywhere except at its decodability transition, at which it diverges. This implies that the mixed state phase transition coincides with the decodability transition and also suggests a quasi-local decoding channel.

Stability of mixed-state quantum phases via finite Markov length

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Based on:

SS, Tim Hsieh. 2404.07251

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Outline

1. Introduction and background

- Motivations
- Definition of mixed-state phase equivalence: two-way connectability

2. Markov length ξ

- "ξ's role in mixed-state phase is similar to energy gap's role in ground-state phase"
- Relation to quasi-local decoders for topological codes

3. Example: dephased toric code

- Phase diagram from Markov length

Why many-body mixed-states?

Physics:

Can quantum phase survive finite temperature & decoherence?

Finite temperature state

$$\exp(-\beta H_{\mathrm{local}})$$

Open system dynamics

$$\exp(\mathcal{L}_{\mathrm{local}} t)[\rho_0]$$

Quantum information:

Quantum error correction: How to reliably store and process quantum information in a noisy environment?

Why many-body mixed states, cont.

Let $\{\, |\boldsymbol{s}\rangle \,\}$ be a basis of a quantum spin system

level of "coherence"

classical	mixed states	purely quantum
$\rho = \sum_{\mathbf{s}} p_{\mathbf{s}} \lvert\mathbf{s}\rangle\langle\mathbf{s}\rvert$	$\rho = \sum_{\mathbf{s}\mathbf{s}'} q_{\mathbf{s}\mathbf{s}'} \lvert\mathbf{s}\rangle\langle\mathbf{s}'\rvert$	$\lvert\psi\rangle = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} \lvert\mathbf{s}\rangle$
Stat-mech models $\Downarrow$ Classical phases	Mixed-state phase: Unification and generalization quantum & classical phases?	Ground states $\Downarrow$ Quantum phases

Many-body physics

LU circuit & ground state phase equiv.

Definition of ground state phase equivalence:

$$\Delta(H_s) > 0 \ \ \forall s \in (0,1)$$

Adiabatic path: $|\psi(H_0)\rangle$ $|\psi(H_1)\rangle$

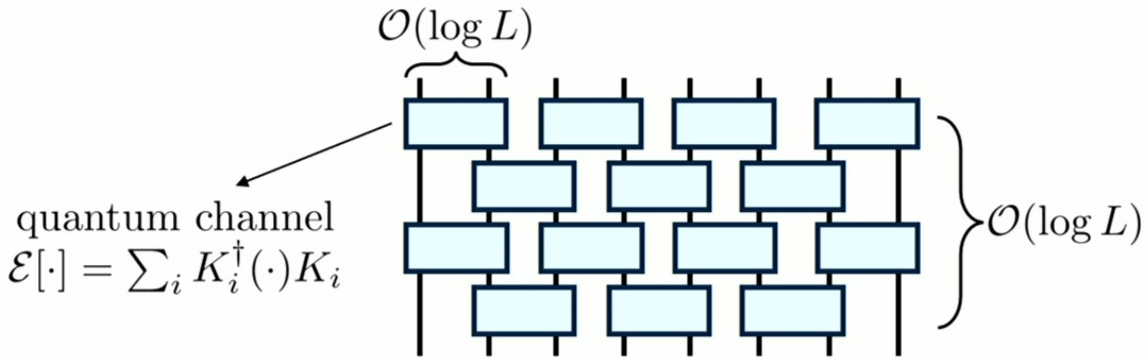
Hastings, Wen. 2019

Circuit definition: $|\psi_0\rangle = U\,|\psi_1\rangle \quad U \in \text{local unitary circuits}$

Benefits of the circuit definition:

- Phase equivalence is a property of *states only*
- Common properties of states within phase (Lieb-Robinson bound)

LC circuit & mixed state phase equiv.



1. Non-invertible in general
2. Can only destroy but not generate long-range feature

Def: Mixed-state phase equivalence

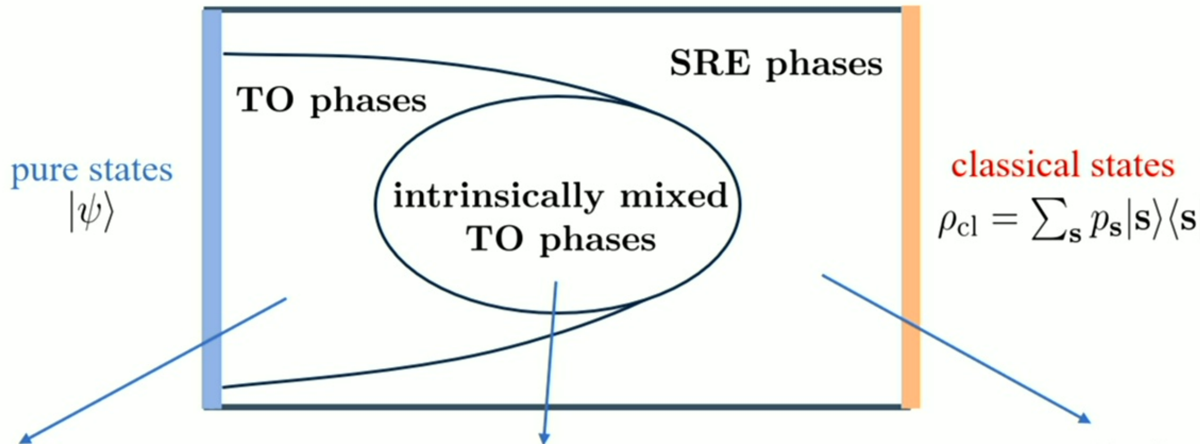
Two states ρ_1 and ρ_2 are in the same phase if there exists two LCs $\mathcal{C}_1, \mathcal{C}_2$ such that:

$$\mathcal{C}_1[\rho_1] \approx \rho_2 \qquad \mathcal{C}_2[\rho_2] \approx \rho_1$$

Coser, Pérez-García 2019

Are mixed-state phases stable small perturbation?

Zoo of mixed-state phases



$\rho_{\rm TO} \sim |\psi_{\rm TO}\rangle$

e.g.
Under-decohered TO
Low-T toric code in 4D

Dennis, Kitaev, Landahl, Preskill. 2001
Lu, Hsieh, Grover. 2020
Fan, Bao, Altman, Vishwanath. 2023
SS, Zou, Hsieh. 2023

$\rho_{\rm imTO}$ is LRE
$\rho_{\rm imTO} \not\sim |\psi_{\rm TO}\rangle$

e.g.
ZX-dephased toric code
decohering anyon theories

Wang, Wu, Wang. 2023
Sohal, Prem. 2024
Ellison, Cheng. 2024

$\rho = \mathcal{C}[\rho_{\rm cl}]$
$\approx \sum_i p_i \,|{\rm SRE_i}\rangle\,\langle{\rm SRE_i}|$

e.g.
Finite temp toric code in 2D
Over-decohered TO

Hastings. 2011
Chen, Tarun. 2023
Chen, Tarun. 2024

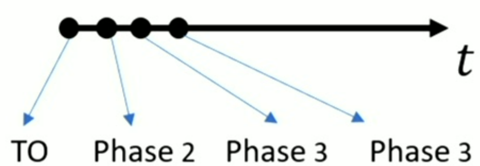
Are mixed-state phases stable small perturbation?

LRE ρ_0 & local Lindbladian $\mathcal{L}$

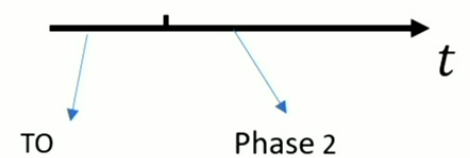
$$\rho_t = \mathcal{G}_t[\rho_0] = e^{t\mathcal{L}}[\rho_0]$$

ρ_t's phase of matter as a function of time?

Conjecture 1



Conjecture 2

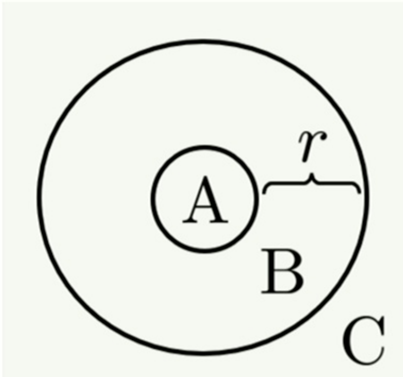


Why it is hard?

- Need to either find the reversal LC circuit, or prove its non-existence

Markov length: A computable quantity of ρ_t that tells the phase diagram

Quantum Conditional Mutual Information (CMI) and Markov length



Conditional mutual information (CMI)

$$I_\rho(A:C|B) = I_\rho(A:BC) - I_\rho(A:B)$$

correlation between A & A^c

correlation between A & B

Small CMI → Correlation between A and A^c is mostly captured by a buffer B surrounding A

Definition: A state ρ has a finite Markov length ξ if for any A ∪ B ∪ C tripartition, the CMI satisfies:

$$I(A:C|B) \lesssim \exp\left(-\frac{\mathrm{dist(A,C)}}{\xi}\right)$$

E.g. Gapped ground states, Gibbs state of commuting projector Hamiltonian…

Main result

$\mathcal{L}_{t\in(0,1)}$ being a local Lindbladian $\quad \rho_0 \xrightarrow{\mathcal{L}_t} \rho_1 \quad$ s.t. $\xi_t = \mathcal{O}(1) \quad \forall t$

Implies

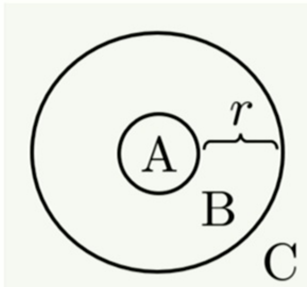
$\exists\tilde{\mathcal{L}}_{t\in(0,1)}$ being a local Lindbladian $\quad \rho_0 \xleftarrow{\tilde{\mathcal{L}}_t} \rho_1$

Implies

All states along the path are of the same mixed-state phase

In mixed-state phases, ξ^{-1} plays a similar role as the energy gap Δ in ground state phases!

Lemma: Local recovery problem



Q: Suppose $\mathcal{E}_A$ ('error') acts on A, can we find a quantum channel on AB that recovers $\mathcal{E}_A$'s influence on ρ?

$$\exists ? \quad \tilde{\mathcal{E}}_{AB} \quad \text{s.t.} \quad \tilde{\mathcal{E}}_{AB} \circ \mathcal{E}_A[\rho] \approx \rho$$

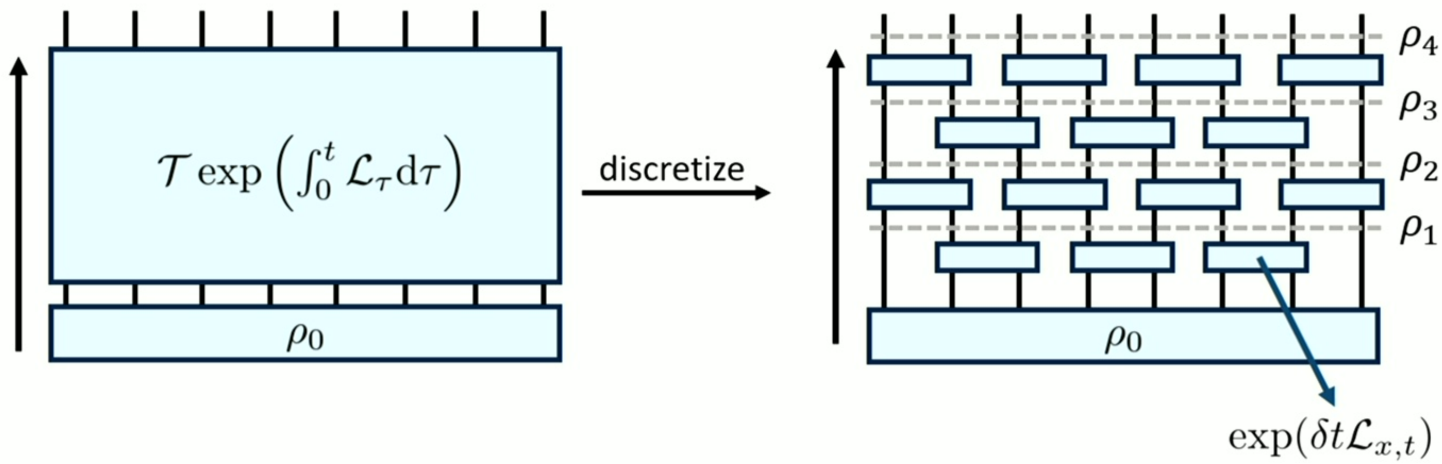
By using the Petz's map as the recovery map $\tilde{\mathcal{E}}_{AB} = \mathcal{P}(\mathcal{E}_A, \rho_{AB})$, the recovery error is bounded as:

$$\left| \tilde{\mathcal{E}}_{AB} \circ \mathcal{E}_A[\rho] - \rho \right|_1^2 \leq I_\rho(A:C|B)$$
$$\sim \exp(-r/\xi)$$

Based on Junge et al. 2018

$$\mathcal{P}(\mathcal{E}, \rho)[\cdot] = \int_{-\infty}^{\infty} \frac{\pi}{2(\cosh(\pi\tau)+1)} \rho^{\frac{1-i\tau}{2}} \mathcal{E}^\dagger \left[\mathcal{E}[\rho]^{\frac{-1+i\tau}{2}} (\cdot) \mathcal{E}[\rho]^{\frac{-1-i\tau}{2}} \right] \rho^{\frac{1+i\tau}{2}} \mathrm{d}\tau$$

How to construct the reversal circuit



Assumption:

state ρ_ℓ has Markov length ξ_ℓ, for $\ell \in \{1,2,\ldots,t/\delta t\}$

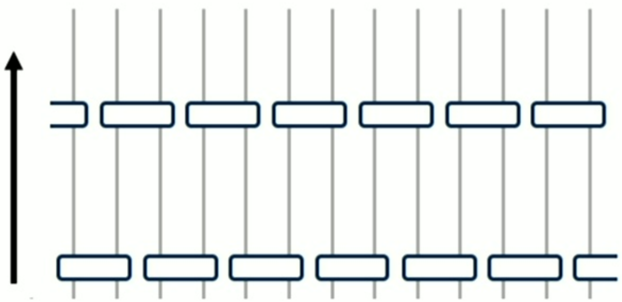
Big picture:

Reverse the dynamics gate by gate, such that

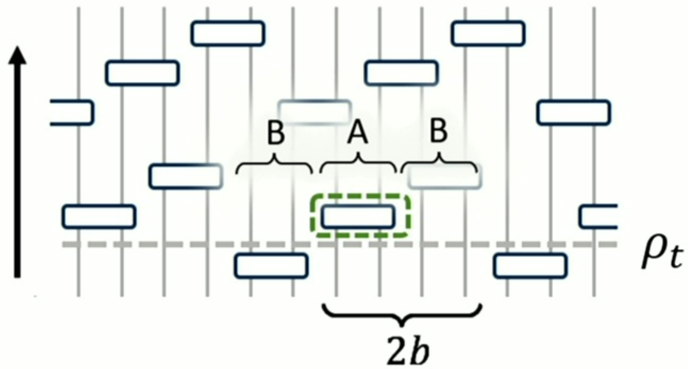
- Reversal dynamics is an LC circuit
- Total approximation error is under control

Constructing reversal circuit

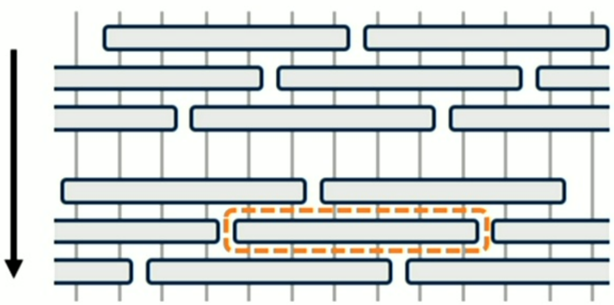
$\mathcal{G}$



$\mathcal{G}$ (reorganized)



$\widetilde{\mathcal{G}}$



(Solution to a local recovery problem)

$$(\tilde{\mathcal{E}}_{x,t})_{AB} = \mathcal{P}((\mathcal{E}_{x,t})_A, (\rho_t)_{AB})$$

To achieve:

$$|\tilde{\mathcal{G}} \circ \mathcal{G}[\rho_0] - \rho_0|_1 \leq \epsilon$$

It suffices to require:

$$b \geq \xi \cdot \log\left(\frac{\text{poly}(L)}{\epsilon \cdot \delta t}\right)$$

$\Rightarrow$ Reversal circuit is LC

$\tilde{\mathcal{G}}$ as a quasi-local decoder

Let $|\psi\rangle \in V$, a topologically degenerate ground state subspace.
Suppose $\mathcal{L}_{t\in(0,1)}$ is a noise process.

If $\quad |\psi\rangle \xrightarrow{\mathcal{L}} \rho_\psi \quad \xi_t = \mathcal{O}(1) \ \ \forall t$

Then $\exists \tilde{\mathcal{L}}_\psi$: $\quad |\psi\rangle \xleftarrow{\tilde{\mathcal{L}}_\psi} \rho_\psi$

Due to local indistinguishability, $\tilde{\mathcal{L}}_\psi = \tilde{\mathcal{L}}$ is independent of $|\psi\rangle \in V$

Thus:

$$\tilde{\mathcal{G}} \circ \mathcal{G}\left[|\psi\rangle\langle\psi|\right] = |\psi\rangle\langle\psi| \quad \forall\, |\psi\rangle \in V$$

where $\tilde{\mathcal{G}} = \mathcal{T} e^{\int_0^1 \tilde{\mathcal{L}} \mathrm{d}t}$ and $\mathcal{G} = \mathcal{T} e^{\int_0^1 \mathcal{L} \mathrm{d}t}$

$\tilde{\mathcal{G}}$ is a (quasi-)local decoder for the noise channel $\mathcal{G}$.

Q: What happens when the Markov length diverges?

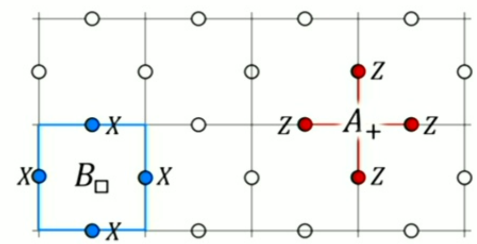
A: Dissipation-driven quantum phase transition

Example: dephased toric code

Kitaev's toric code model

$|\text{t.c.}\rangle$ is the ground state of

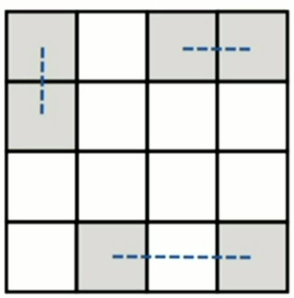
$$H_{\text{t.c.}} = -\sum_{\square \in P} B_{\square} - \sum_{+ \in V} A_{+}$$



Dephased T.C. state

$$\rho_0 = |\text{t.c.}\rangle\langle\text{t.c.}| \qquad \mathcal{L}[\rho] = \sum_i \tfrac{1}{2}(Z_i \rho Z_i - \rho)$$

$$\rho_{p_t} = e^{\mathcal{L}t}[\rho_0] = \mathcal{N}_{p_t}^{\otimes L}[\rho_0] \qquad p_t = \tfrac{1}{2}(1 - e^{-t})$$



Aspects of dephased toric code

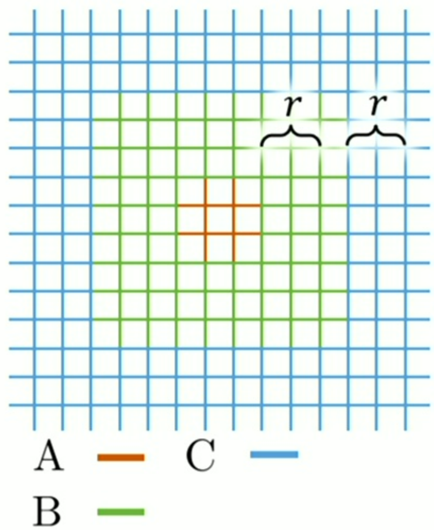
ρ_p can be mapped to a random bond Ising model (RBIM), having a Z_2 symmetry-breaking transition at $p_c \approx 0.11$

1. ρ_p losses stored quantum info at p_c [Dennis, Kitaev, Landahl, Preskill 2001]
2. ρ_p's topological entanglement negativity vanishes at p_c [Fan, Bao, Altman, Vishwanath 2023]
3. ρ_p becomes is SRE after p_c [Tarun's talk]
4. $\rho_p \sim |\mathrm{t.c.}\rangle$ (i.e. LC bi-connected) when $p < 0.04$ [SS, Zou, Hsieh]

Our focus…

- Behavior of ρ_t 's CMI & Markov length
  - Argument based on RBIM mapping
  - Numerical results

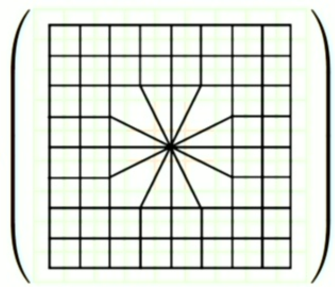
CMI as point defect energy cost



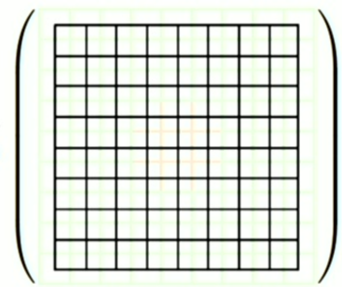
$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$$

$$S_{\rho_p}(AB) - S_{\rho_p}(B)$$

$= \overline{F_{\mathrm{RBIM},p}}$



$- \overline{F_{\mathrm{RBIM},p}}$



=(free energy cost of a point defect in the center of a $2r \times 2r$ RBIM)

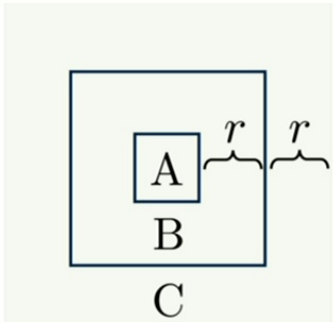
$\equiv F_{\mathrm{def},p}(2r)$

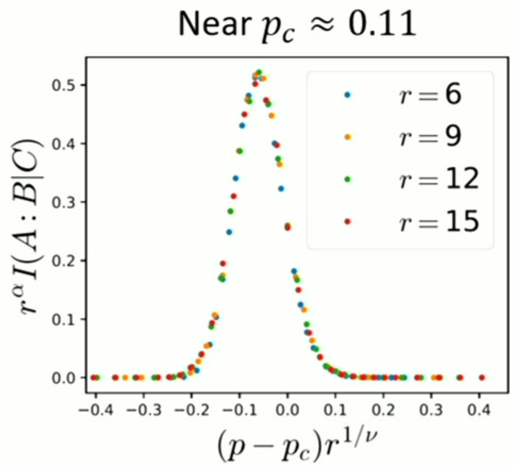
similarly $S_{\rho_p}(ABC) - S_{\rho_p}(BC) = F_{\mathrm{def},p}(4r)$

$$I(A:C|B) = \overline{F_{\mathrm{def}}(4r)} - \overline{F_{\mathrm{def}}(2r)}$$

$$= \begin{cases} e^{-r/\xi_p} & p \neq p_c \\ r^{-\alpha} & p = p_c \end{cases}$$

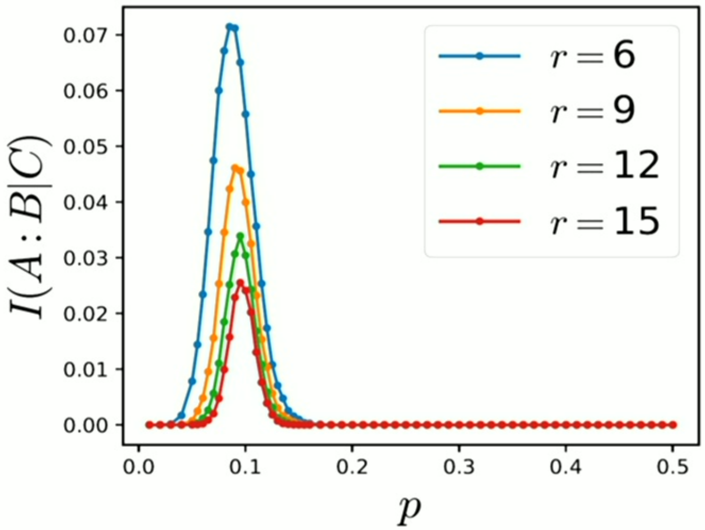
Numerical results

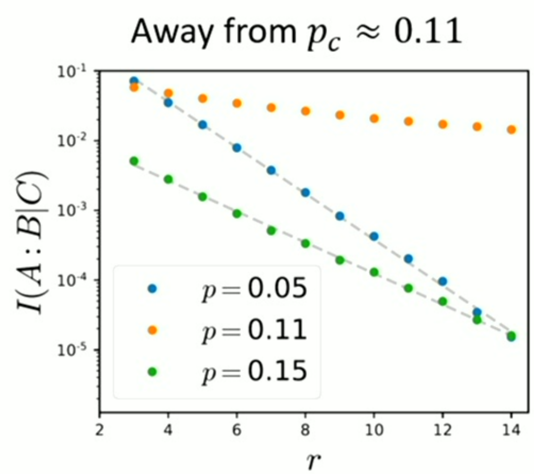




$$I(A:C|B) = r^{-\alpha}\Phi\left((p-p_c)r^{1/\nu}\right)$$

$$\text{Best collapse: } (p_c, \nu, \alpha) = (0.11, 1.8, 1.1)$$





Dephased TC's phase diagram

ρ_p's Markov length is finite everywhere except at p_c

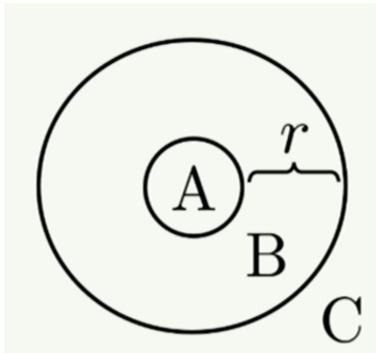
$\xi = \mathcal{O}(1)$ $\xi = \infty$ $\xi = \mathcal{O}(1)$

$p = 0$ p_c $p = 0.5$

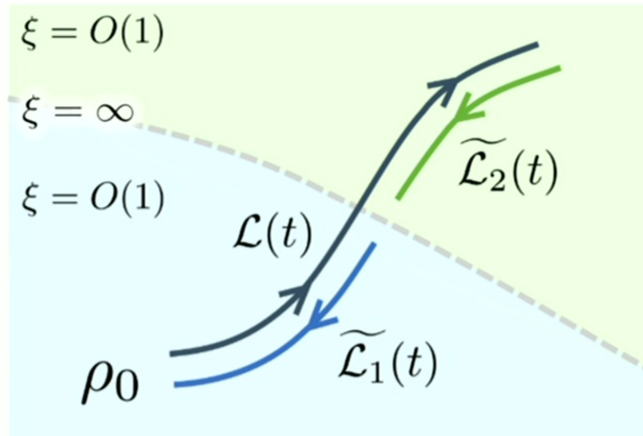
- toric code phase
- logical info preserved
- $\exists$ quasi-local decoder

SRE phase

Summary



$$I(A : C|B) \sim e^{-r/\xi}$$



Along a local Lindbladian evolution, if the Markov length…

- …stays finite ⇒ phase equivalence along the path, preservation of logical info
- …diverges ⇒ dissipation-induced phase transition

Q: What happens when the Markov length diverges?

A: Dissipation-driven quantum phase transition

Dephased TC's phase diagram

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$\xi = \mathcal{O}(1)$ $\xi = \infty$ $\xi = \mathcal{O}(1)$

$p = 0$ p_c $p = 0.5$

toric code phase
logical info preserved
$\exists$ quasi-local decoder

SRE phase