Title: Combining Contextuality and Causality - Foundations of Quantum Computational Advantage - Colloquium

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Series: Colloquium

Date: May 02, 2024 - 2:00 PM

URL: https://pirsa.org/24050024

Abstract: This colloquium is presented in collaboration with the Foundations of Quantum Computational Advantage conference.

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Abstract TBA

Zoom link

Combining Contextuality and Causality

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FoQaCiA Workshop/Perimeter Institute Colloquium, May 2nd 2024

Combining contextuality and causality

Reasons:

- Contextuality (with non-locality as a special case) is a key form of **non-classicality** which appears as a necessary ingredient for quantum advantage in several information processing settings.
- In a number of key cases, including measurement-based quantum computation, **adaptivity** (e.g. feedforward of measurement outcomes) plays an essential role. We aim to provide an account of contextuality which can accommodate such information flows.
- Another motivation comes from the breakthrough work on shallow circuits by Bravyi, Gosset and Koenig, which leverages non-locality/contextuality to prove unconditional quantum advantage. This has been generalised by Sivert Aasnaess in his recent thesis, using tools from the sheaf-theoretic approach. A key ingredient here is the information accessible to an output of the system from its "past light cone".
- There is also the basic physics motivation of understanding contextuality in a given causal background.

The quantum view

Quantum mechanics is our most highly confirmed physical theory.
According to its predictions, as confirmed by an enormous body of experimental evidence:

- We **cannot** measure/observe all properties simultaneously
- We **can** observe certain subsets of properties (commuting families of observables) which give us overlapping **observational windows** onto the quantum system
- In effect, what we can observe are partial views of the system – assignments to a subset of the variables
- In general, there is **no** assignment of values to all the variables consistent with all these windows
- In a nutshell: contextuality arises where we have a family of overlapping pieces of data which is **locally consistent**, but **globally inconsistent**.

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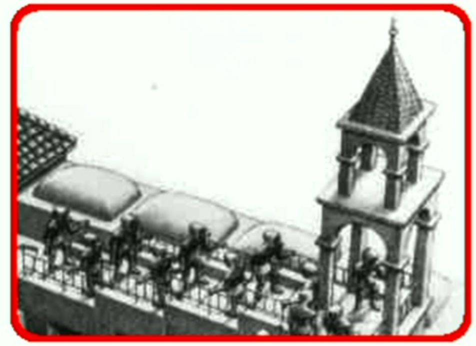
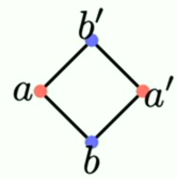
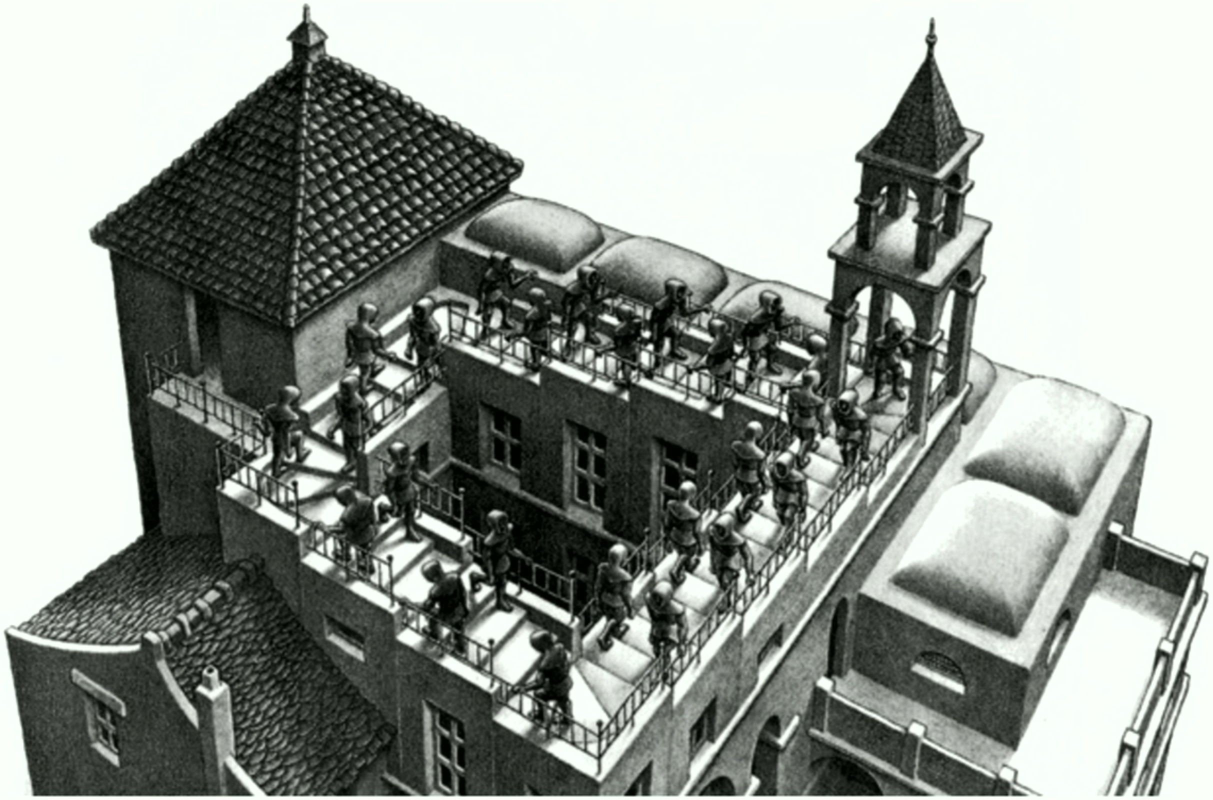
Contextuality Analogy: Local Consistency

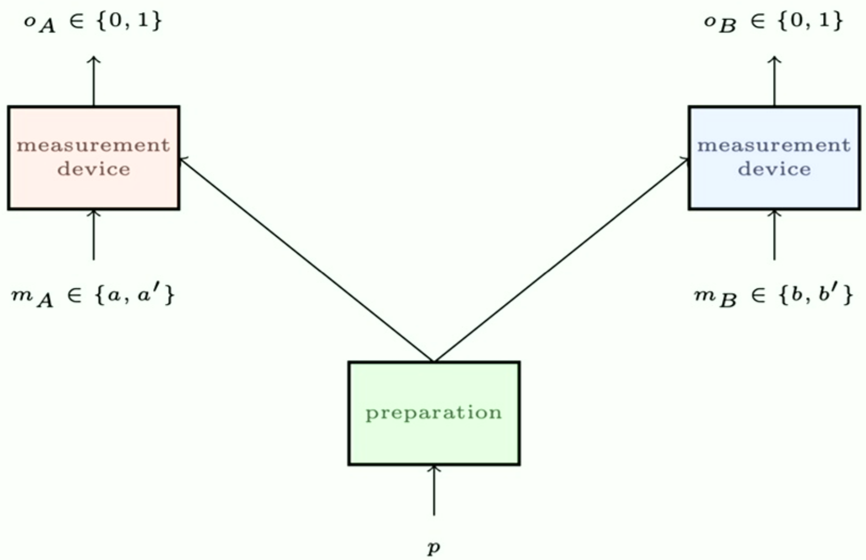




Contextuality Analogy: Global Inconsistency

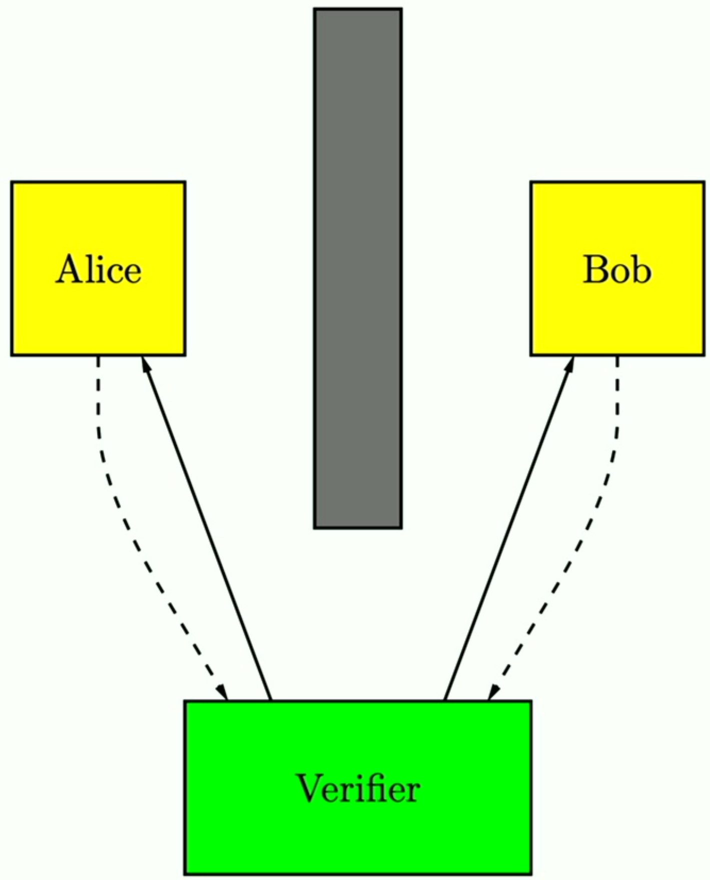


Testing non-local correlations



Nobel Prize in Physics 2022 (Clauser, Aspect and Zeilinger)

Alice-Bob games



The XOR Game

Alice and Bob play a cooperative game against Verifier (or Nature!):

- Verifier chooses an input $x \in \{0,1\}$ for Alice, and similarly an input y for Bob. We assume the uniform distribution for Nature's choices.
- Alice and Bob each have to choose an output, $a \in \{0,1\}$ for Alice, $b \in \{0,1\}$ for Bob, depending on their input. They are **not allowed to communicate during the game**.
- The winning condition: $a \oplus b = x \wedge y$.

A table of conditional probabilities $p(a,b|x,y)$ defines a **probabilistic strategy** for this game. The **success probability** for this strategy is:

$$1/4[p(a=b|x=0,y=0)+p(a=b|x=0,y=1)+p(a=b|x=1,y=0)$$

$$+p(a\neq b|x=1,y=1)]$$

A Strategy for the Alice-Bob game

Example: The Bell Model

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
0	0	1/2	0	0	1/2
0	1	3/8	1/8	1/8	3/8
1	0	3/8	1/8	1/8	3/8
1	1	1/8	3/8	3/8	1/8

The entry in row 2 column 3 says:

If the Verifier sends Alice a_1 *and Bob* b_2*, then with probability* $1/8$*, Alice outputs a* 0 *and Bob outputs a* 1*.*

This gives a winning probability of $\frac{3.25}{4} \approx 0.81$.

A Strategy for the Alice-Bob game

Example: The Bell Model

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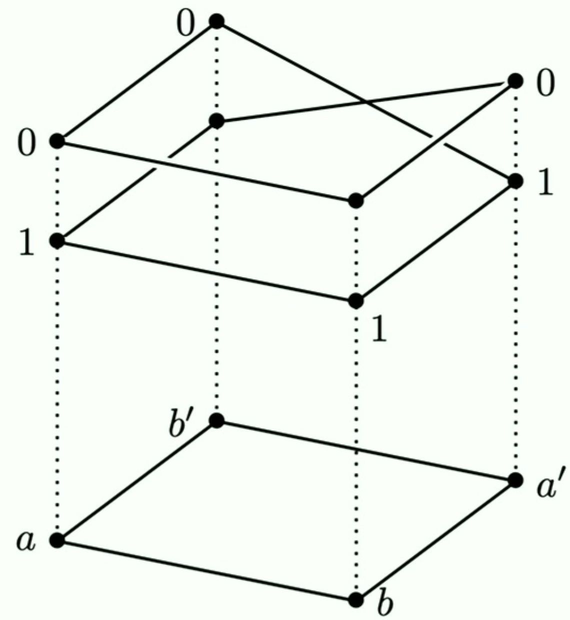
The optimal classical probability is 0.75!

The proof of this uses (and is essentially the same as) the use of **Bell inequalities**.

The Bell table exceeds this bound. Since it is **quantum realizable** using an entangled pair of qubits, it shows that quantum resources yield a **quantum advantage** in an information-processing task.

The Bell table and the "Möbius strip"

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a_1, b_1)	$1/2$	0	0	$1/2$
(a_1, b_2)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2, b_1)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2, b_2)	$1/8$	$3/8$	$3/8$	$1/8$



For real?

This can all be turned into precise (and elegant) mathematics:

- The possible events occurring in a given context vary functorially in the sets of measurements – the “fibres” over the “base” in the bundle diagram
- The data is given by distributions over these events – again described functorially
- This fits perfectly into the language of sheaf theory – exactly the mathematics of the passage from local to global – and **obstructions** to this passage
- These obstructions can be formulated in terms of **cohomology**, which measures the “logical twisting” of the fibres
- The logical twisting relates to paradoxes such as the **Liar paradox**
- Bell inequalities can be characterized in terms of **logical consistency conditions** on events – Boole’s “conditions of possible experience”.
- There is a **resource theory** of contextuality based on this, and notions of **simulation** between contextual systems
- Key measures of contextuality such as the **contextual fraction**
- connections with logic and computation; database theory, constraint satisfaction
- generalized **Vorob’ev theorem**

References

Some papers (all available on the arXiv):

- *The sheaf-theoretic structure of non-locality and contextuality*, SA and Adam Brandenburger, (2011)
- *Logical Bell inequalities*, SA and Lucien Hardy (2012)
- *Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity*, SA (2014)
- *Contextuality, cohomology and paradox*, SA, Rui Soares Barbosa, Kohei Kishida, Ray Lal and Shane Mansfield (2015)
- *The contextual fraction as a measure of contextuality*, SA, Rui Soares Barbosa and Shane Mansfield (2017)
- *A comonadic view of simulation and quantum resources*, SA, Rui Soares Barbosa, Martti Karvonen and Shane Mansfield (2017)
- *The logic of contextuality*, SA and Rui Soares Barbosa (2021)
- *Neither contextuality nor non-locality admits catalysts*, Martti Karvonen (2021)

Desiderata

Our aim is to develop a causal refinement of this setting for contextuality.

The hope is that this leads to a smooth extension of the theory, to which all the current aspects:

contextual fraction, logical Bell inequalities, resource theory, simulations, cohomological criteria, connections with logic and computation, etc. etc.

also lift smoothly.

Grades of causal involvement:

- global ordering on measurements (cf. Mansfield and Gogioso-Pinzani approaches)
- dependence on measurement outcomes, allowing e.g. for feed-forward in MBQC, adaptive computation
- recognizing the different roles played by Nature and Experimenter in their interactions

Dual faces of causality

Causality may be:

- imposed by Nature – a **causal background**
- imposed by the experimenter, e.g. to achieve computational effects (adaptive computation).

We will illustrate these two sources of causality in two basic examples.

Example I: causal background a la G-P

Standard Bell-CHSH bipartite scenario: Alice and Bob, with sets of local measurements M_A and M_B, and outcomes O_A and O_B.

We assume that Alice's events **causally precede** those of Bob.

Thus Bob's backward light-cone includes the events where Alice chooses a measurement and observes an outcome.

Whereas in a standard, "flat" scenario, we would have deterministic outcomes given by functions

$$s_A : M_A \longrightarrow O_A, \quad s_B : M_B \longrightarrow O_B,$$

with these causal constraints, we have functions

$$s_A : M_A \longrightarrow O_A, \quad s_B : M_A \times M_B \longrightarrow O_B$$

That is, the responses by Nature to Bob's measurement may depend on the previous measurement made by Alice.

Example I ctd

If we have measurements $x_1, x_2 \in M_A$, $y \in M_B$, then we can have $\{(x_1, 0), (y, 0)\}$ and $\{(x_2, 0), (y, 1)\}$ as valid histories in a single deterministic model.

Of the usual no-signalling/compatibility equations

$$\begin{array}{lrcl} (1) & e_{\{x_i,y\}}|_{\{x_i\}} & = & e_{\{x_i\}} \\ (2) & e_{\{x_i,y\}}|_{\{y\}} & = & e_{\{y\}} \end{array}$$

only (1) remains: $e_{\{y\}}$ is not even defined, since $\{y\}$ is not a "causally secured" context.

Example II: Anders-Browne (PRL 2009)

This shows how we can use a form of Experimenter-imposed causality to promote two sub-universal computational models (Pauli measurements and mod-2 linear classical processing) to universal MBQC.

Uses GHZ state as a resource state: GHZ $= \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$.

	$+++$	$++-$	$+-+$	$+--$	$-++$	$-+-$	$--+$	$---$
XYY	0	1	1	0	1	0	0	1
YXY	0	1	1	0	1	0	0	1
YYX	0	1	1	0	1	0	0	1
XXX	1	0	0	1	0	1	1	0

In terms of parities (product of $+1/-1$ outputs):

$$\begin{array}{ccccc} X_1 & Y_2 & Y_3 & = & -1 \\ Y_1 & X_2 & Y_3 & = & -1 \\ Y_1 & Y_2 & X_3 & = & -1 \\ X_1 & X_2 & X_3 & = & +1 \end{array}$$

Using Experimenter causal flow to implement OR

Taking X as 0, Y as 1, we consider the measurements for Alice and Bob as inputs to an OR-gate.

We then use the following simple mapping (XOR on the bit representations) from the Alice-Bob measurements to Charlie's measurement to get the OR-function, which we can read off from the XOR of the outcome bits:

$$\begin{array}{lcl} 0,1 & \mapsto & 1 \\ 1,0 & \mapsto & 1 \\ 1,1 & \mapsto & 0 \\ 0,0 & \mapsto & 0 \end{array} \qquad \begin{array}{lcl} X,Y & \mapsto & Y \\ Y,X & \mapsto & Y \\ Y,Y & \mapsto & X \\ X,X & \mapsto & X \end{array}$$

Note that:

- this is purely causality employed by the Experimenter; from Nature's point of view, it is the standard GHZ construction
- the above is a simplified "one-shot" description; really there is a (classically computed) feed-forward of measurement settings needed to represent circuits with embedded OR-gates

Game semantics of causality

We shall conceptualise the dual nature of causality as a two-person game, played between Experimenter and Nature:

- The Experimenter's moves are the choices of measurements to be performed.
- Nature's moves are the outcomes.
- **Strategies** for the two players capture the causal dependencies.

By formalising this, we can develop a theory of causal contextuality which recovers:

- the usual, "flat" contextuality
- the G-P theory of non-locality in a causal background
- MBQC with adaptive computation
- classical causal networks

as special cases, and more.

Note Our formalisation will use ideas from Computer Science: Kahn-Plotkin concrete domains and their representations.

Contextuality scenarios

A (flat) contextuality scenario is $(X, O, \mathcal{C})$, where:

- X is a set of **measurements**.
- $O = \{O_x\}_{x \in X}$ is the set of possible **outcomes** for each measurement.
- $\mathcal{C}$ is a **cover**, i.e. a family $\{C_i\}_{i \in I}$ of subsets $C_i \subseteq X$ such that $\bigcup_{i \in I} C_i = X$.

An **event** has the form (x, o), where $x \in X$ and $o \in O_x$. It corresponds to the measurement x being performed, with outcome o.
Given a set of events s,

$$\mathrm{dom}(s) := \pi_1 S = \{x \mid \exists o. \, (x, o) \in s\}.$$

We say that s is **consistent** if

1. for some $C \in \mathcal{C}$, $\mathrm{dom}(s) \subseteq C$;
2. $(x, y), (x, y') \in s$ implies $y = y'$.

In this case, s defines a function from the measurements in its domain to outcomes.
A consistent set of events is a **section**.

Causal contextuality scenarios

A causal contextuality scenario is $(X, O, \mathcal{C}, \vdash)$, where the additional ingredient is an **enabling relation**, which expresses causal constraints.

The intended interpretation of $s \vdash x$, where s is a section and $x \in X$, is that it is possible to perform x after the events in s have occurred.

Note that this constraint refers to the **measurement outcomes** as well as the measurements which have been performed. This allows adaptive behaviours to be described.

Histories

Given such a causal contextuality scenario M, we can use it to generate a set of **histories**, i.e. of sets of events which can happen in a causally consistent fashion. We associate each measurement x with a unique event occurrence, so histories are required to be consistent.

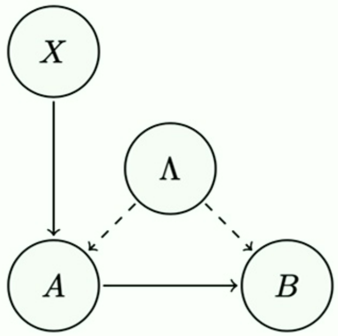
We define the **accessibility relation** $s \triangleright x$ between sections s and measurements x: $s \triangleright x$ iff $x \notin \mathrm{dom}(s)$, $\mathrm{dom}(s) \cup \{x\} \subseteq C$ for some $C \in \mathcal{C}$, and for some $S \subseteq s$, $S \vdash x$.

Now we can define $\mathcal{H}(M)$, the set of histories over M, inductively by

$$\begin{array}{lcl} H_0 & := & \{\varnothing\} \\ H_{k+1} & := & H_k \ \cup \ \{s \cup \{(x,o)\} \mid s \in H_k, s \triangleright x, o \in O_x\}. \end{array}$$

If X is finite, for some k we will have $H_k = H_{k+1}$, and we take $\mathcal{H}(M) = H_k$ for the least such k.

Example: instrumental scenario



Outcomes: $\{1, 2\}$
Measurement settings

- for Alice: $\{x_1, x_2\}$
- for Bob: $\{y_1, y_2\}$

Enablings:

$$\varnothing \vdash x_i, \qquad (x_i, j) \vdash y_j$$

Thus Alice's measurement outcome determines Bob's measurement setting, without any information as to what Alice's measurement setting was.
The variant where there **is** such information flow can also be represented.

Strategies

We can regard a causal contextuality scenario $M = (X, O, \mathcal{C}, \vdash)$ as specifying a **game** between Experimenter and Nature:

- Events (x, o) correspond to the Experimenter choosing a measurement x, and Nature responding with outcome o.
- The histories correspond to the **plays** or runs of the game.

Given this interpretation, we define a **strategy for Nature** over the game M as a set of histories $\sigma \subseteq \mathcal{H}(M)$ satisfying the following conditions:

- σ is downwards closed: if $s, t \in \mathcal{H}(M)$ and $s \subseteq t \in \sigma$, then $s \in \sigma$.
- σ is deterministic and total: $\varnothing \in \sigma$, and if $s \in \sigma$ and $s \rhd x$, then there is a unique $o \in O_x$ such that $s \cup \{(x, o)\} \in \sigma$.

Thus in any position s reachable under σ, it has a unique response to any measurement which can be chosen by the Experimenter.

The sheaf of strategies

Given a causal contextuality scenario $M = (X, O, \mathcal{C}, \vdash)$, we can define a presheaf

$$\Gamma : \mathcal{P}(X)^{\mathsf{op}} \longrightarrow \mathbf{Set}$$

For each $U \subseteq X$, $\Gamma(U)$ is the set of strategies for M_U, the restriction of the scenario to measurements in U.

Given $U \subseteq V$, the restriction map $\Gamma(U \subseteq V) : \Gamma(V) \longrightarrow \Gamma(U)$ is given by $\sigma \mapsto \sigma|_U := \sigma \cap \mathcal{H}(M_U)$.

Proposition

Γ *is a presheaf, and satisfies the sheaf condition for "causally secured" covers.*

Running the sheaf theory script

We can now follow the same structure as in Abramsky-Brandenburger, replacing the "flat" event sheaf of local sections by the sheaf of strategies.

We recall the distribution monad $\mathcal{D}_R$, where R is a semiring; when R is the non-negative reals, we recover discrete probability distributions. We have the presheaf $\mathcal{D}_R\Gamma$, obtained by composing the endofunctor part of the monad with Γ.

An **empirical model** over the scenario $(M, \mathcal{C})$ is a family $\{e_i\}$, where $e_i \in \mathcal{D}_R\Gamma(C_i)$, subject to the usual compatibility conditions: for all i, j, $e_i|_{C_i \cap C_j} = e_j|_{C_i \cap C_j}$. Thus e_i assigns a probability to each extensional strategy over M_{C_i}.

The model is **causally non-contextual** if there is a distribution $d \in \mathcal{D}_R\Gamma(X)$ such that, for all i, $d|_{C_i} = e_i$.

We can show that this recovers

- Standard "flat" contextuality when the enabling is trivial (all measurements initially enabled)
- The Gogioso-Pinzani theory of contextuality for causal Bell scenarios

Experimenter strategies and adaptive computation

But this is only part of the picture!

The strategies considered so far have been strategies for Nature, which choose an outcome for each measurement which can be chosen by the Experimenter.
Using the duality inherent in game theory, there is also a notion of **strategy for Experimenter**.

We define a **strategy for Experimenter** over M to be a set $\tau \subseteq \mathcal{H}(M)$ which is co-total: if s is a non-maximal history in τ, then there is x such that $s \cup \{(x, o)\} \in \tau$ for all $o \in O_x$.

Thus at each stage, the Experimenter chooses the next measurement to be performed. It must then accept any possible response from Nature. The future choices of the Experimenter can then depend on Nature's responses, allowing for adaptive protocols.

We can use Experimenter strategies to capture adaptive MBQC.

The Big Picture

We shall refer to strategies for Nature as N-strategies, and to strategies for Experimenter as E-strategies.

If we are given an N-strategy σ and an E-strategy τ, we can play them off against each other:

$$\langle \sigma \mid \tau \rangle := \sigma \cap \tau.$$

If τ is deterministic, at each stage τ chooses a unique measurement, and σ a unique outcome for that measurement, so this will be the down-set of a unique maximal history s. In general, it determines a set of histories.

A general causal empirical model will specify a distribution on N-strategies ("mixed N-strategy") and a distribution on E-strategies for each context.
These distributions can be pushed forward through the evaluation map to yield distributions on histories.

This provides a basis for exploring a wide range of phenomena.

Anders–Browne revisited

We now show how the Anders–Browne construction of an OR gate can be formalised using an Experimenter strategy.

First, we have the description of the standard GHZ construction. This is given by a flat measurement scenario with $X = \{A_i, B_j, C_k \mid i, j, k \in \{0, 1\}\}$, and $O_x = \{0, 1\}$ for all $x \in X$.

The maximal compatible sets of measurements are all sets of the form $\{A_i, B_j, C_k\}$ with $i, j, k \in \{0, 1\}$, i.e. a choice of one measurement per each site or agent. We regard each measurement as initially enabled. The N-strategies for this scenario form the usual sections assigning an outcome to each choice of measurement for each site, and the GHZ model assigns distributions on these strategies as in the table shown previously.

To get the Anders–Browne construction, we consider the E-strategy which initially allows any A or B measurement to be performed, and after a history $\{(A_i, o_1), (B_j, o_2)\}$ chooses the C-measurement $C_{i \oplus j}$. Playing this against the GHZ model results in a strategy that computes the OR function with probability 1.

Anders-Browne ctd

The full power of adaptivity is required when using this as a building block to implement a more involved logical circuit.

Suppose that the output of the OR gate above is to be fed as the first input of a second OR gate, built over a GHZ scenario with measurements labelled $\{A'_i, B'_j, C'_k \mid i, j, k \in \{0, 1\}\}$.

The E-strategy implements the first OR gate as above, with any B' measurement also enabled, being a free input. After that, the A'-measurement can be determined: after a history containing $\{(A_i, o_1), (B_j, o_2), (C_{i \oplus j}, o_3)\}$, the E-strategy chooses the A'-measurement $A'_{o_1 \oplus o_2 \oplus o_3}$.
The second OR gate is then implemented like the first.

Note that the choice of A'-measurement depends not only on previous measurement choices, but on outcomes provided by Nature.

Where we are

- Abramsky, Samson, Rui Soares Barbosa, and Amy Searle. "Combining contextuality and causality: a game semantics approach." *Philosophical Transactions of the Royal Society A* 382.2268 (2024): 20230002.

The paper is an initial proof of concept.

We establish the formal framework, and show that it subsumes:

- Standard "flat" contextuality scenarios
- The (quite extensively developed) Gogioso-Pinzani framework for Bell scenarios with causal background
- Adaptivity in MBQC setting, e.g. the Anders-Browne construction.

Is causality eliminable?

Can we in fact **eliminate** causal structure from contextuality analysis?

Some evidence in favour:

- We have recently shown that there is a **flattening construction** which takes Gogioso-Pinzani scenarios (Bell scenarios with a partial order on the sites/agents) to standard flat contextuality scenarios.
- Moreover, the flat scenario is contextual (in the usual sense) if and only if the original G-P scenario was (in the causal sense).
- The idea is treat each measurement event in the G-P scenario with a given history of past measurements in its backwards light cone as a distinct measurement in the flat scenario.
- Formally, showing this works amounts to some manipulation of dependent products (as in HoTT).
- Thus for G-P scenarios, **in principle** causality is eliminable.
  However, there may be succinctness and complexity costs in doing so.

Eliminating causality II

For the case of Experimenter strategies over a flat contextuality scenario (i.e. no causality imposed by Nature), we conjecture that these are captured in an equivalent form by the **Measurement Protocol comonad** on the category of standard flat contextuality scenarios described in

- SA, Rui Soares Barbosa, Martti Karvonen, and Shane Mansfield. "A comonadic view of simulation and quantum resources." In 2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), pp. 1–12. IEEE, 2019.
- Barbosa, Rui Soares, Martti Karvonen, and Shane Mansfield. "Closing bell boxing black box simulations in the resource theory of contextuality." In Samson Abramsky on Logic and Structure in Computer Science and Beyond, pp. 475–529. Cham: Springer International Publishing, 2023.
- Karvonen, Martti. "Neither contextuality nor nonlocality admits catalysts." Physical Review Letters 127, no. 16 (2021): 160402.

Question What about Experimenter strategies with non-trivial causal background?

Temporal Correlations

- Amy Searle, Rui Soares Barbosa, and Samson Abramsky, "Mapping Temporal Correlations to Contextuality Correlations", poster submitted to QPL 2024, draft paper.
- Amy Searle, "Detecting nonclassicality in causal correlations: obstructions to global descriptions", draft thesis, to be submitted May 2024.

In-depth study of linearly ordered temporal scenarios.
Key ideas:

- The amount of signalling from the past is bounded by a **memory constraint**, which is a parameter of the scenario. This is formalized in terms of “lookback” of the model.
- A key role is played by a **flattening construction**, which turns a k-lookback model into a flat contextuality scenatio, preserving all non-classicality properties.
- This allows standard contextuality results, notably **Vorob’ev’s Theorem**, to be applied to temporal scenarios.

Shallow circuits

Work in progress by SA, Carmen Constantin and Martti Karvonen

- Path-breaking work by Bravyi, Gossett, Koenig, and Tomamichel on **unconditional quantum advantage in shallow circuits**.
- This leverages some classic contextual constructions, e.g. GHZ or Peres-Mermin, to show separation for circuits of bounded depth and fan-in.
- Sivert Aasnaess showed in his recent thesis that one can generalize this to a large class of (essentially all) non-local games.
- His work implicitly used many of the structural tools we have been developing for contextuality.
- His work is quite dense and not easy to follow!
- Our aim, based partly on his work and partly on the concrete constructions of Bravyi et al., is to make this more explicit and systematic, to show the various pieces being used and how they fit together, and hopefully produce an account which is clearer and more perspicuous, and easier to apply and generalize to other settings.
- Good progress, particularly on the side of the classical bounds.