Title: Mathematical Physics Lecture

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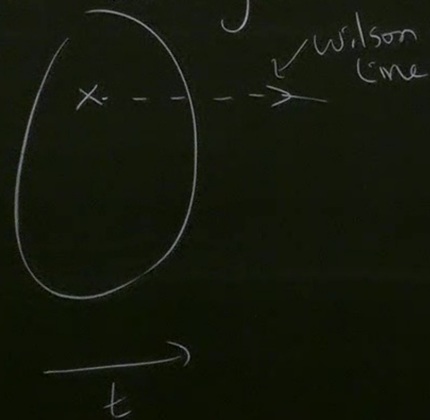
Wilson line = Particle whose Hilbert space is a rep. of gauge group

M is a symplectic manifold

G, gauge group, G acts on M

Expect, geometric quant of M will give such a particle.



Phase space $\sim$ Gauge theory phase space $\times M$

G, gauge group, G acts on M

Eg. $G = U(1)$

We need M with a $U(1)$ action

Eg. $M = \mathbb{R}^2$

$U(1)$ acts by Hamiltonian $p^2 + q^2$

Hilbert space $= \sum$ eigenstates of harmonic oscillator

Correspond to Bohr-Sommerfeld orbits

$\omega = dp\,dq = r\,dr\,d\theta$

$r =$ an integer.

$\mathcal{H} = \bigoplus_{n} \mathbb{C}_{n^2}$ $\quad \mathbb{C}_{n^2} =$ irrep of $U(1)$ with charge n^2.

Or: $M = S^1 \times S^1$ $\quad \omega = n\,dp\,dq$

$U(1)$ acts by p; Hilbert space is n dimensional,

$\bigoplus_{k=0}^{n-1} \mathbb{C}_k$

$\underline{SU(2)}$

$M = S^2$

$\omega = \varepsilon^{ijk} x_i \, dx_j \, dx_k \cdot n$

normalized so area n

$SU(2)$ action $\ni$ Hamiltonian fns whose Poisson bracket = algebra of Pauli matrices

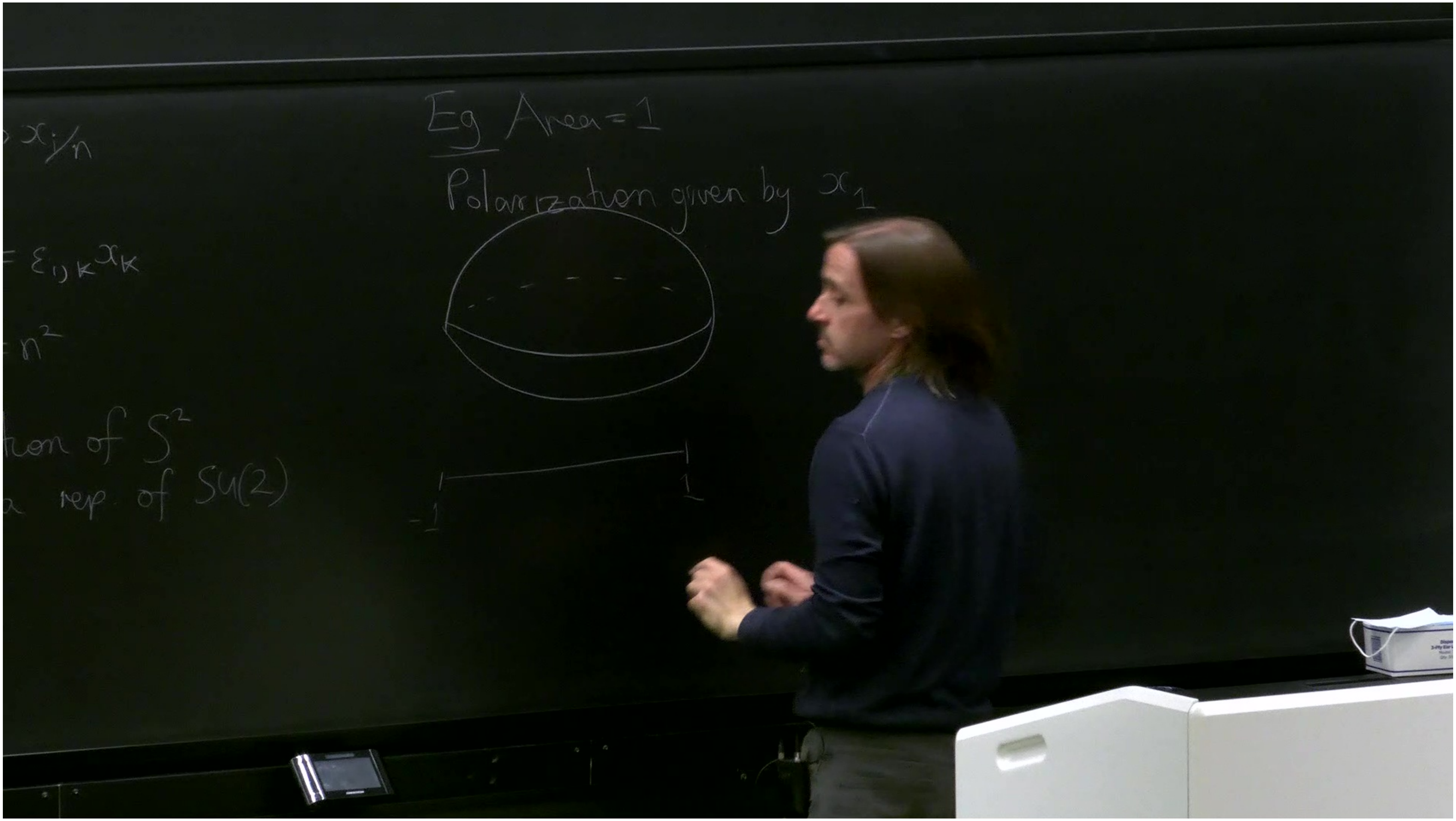
$\sigma_i \rightsquigarrow x_i/n$

Note,

$\{x_i, x_j\} = \varepsilon_{ijk} x_k$

$\sum x_i^2 = n^2$

Quantization of S^2 will give a rep of $SU(2)$


Eg Area = 1
Polarization given by x1
-1
1
of S2
rep. of SU(2)

$x_1 = \pm 1$ degenerate

Lagrangians, automatically satisfy BS condition

as $\int_{\text{Lagrangian}} \alpha = 0$ if $\omega = d\alpha$

Because Area $= 1$, no room for other states

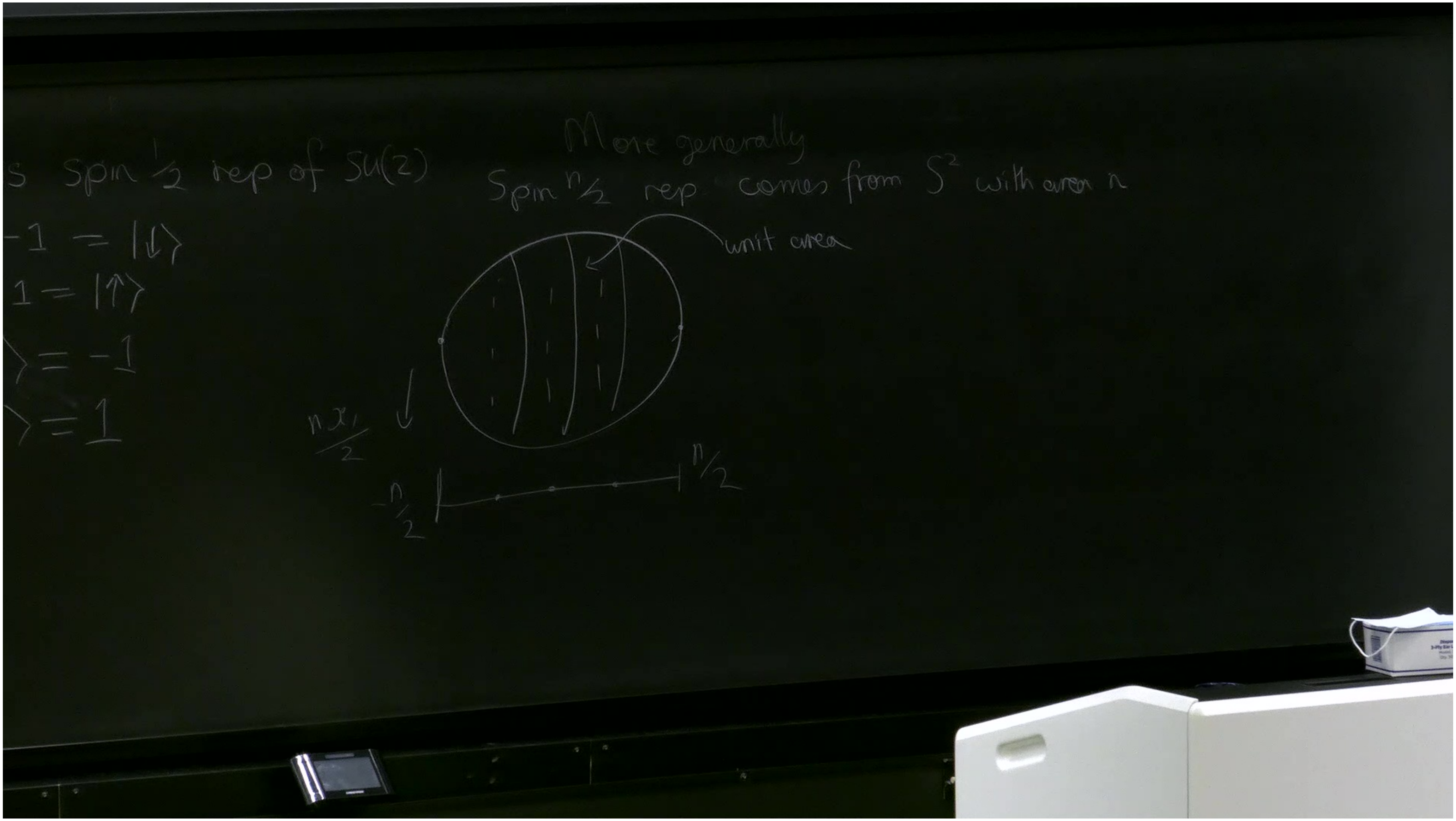
This is spin $\frac{1}{2}$ rep of

$x_1 = -1 = |\downarrow\rangle$

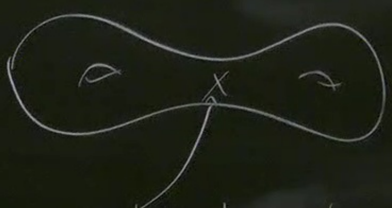
$x_1 = 1 = |\uparrow\rangle$

$x_1 |\downarrow\rangle = -1$

$x_1 |\uparrow\rangle = 1$


s spin 1/2 rep of SU(2)
More generally
Spin n/2 rep comes from S^2 with area n
-1 = |↓⟩
1 = |↑⟩
⟩ = -1
⟩ = 1
unit area
n x_1 / 2
-n/2
n/2

What is phase space in presence of Wilson line?

$\Sigma =$ 

p Wilson line

comes from G action on M, given by Hamiltonians μ_a

Action is like

$$\int_{\Sigma\times\mathbb{R}} n\,CS(A) + \int_{p\times\mathbb{R}} A_t^a \mu_a ,$$

Solve EOM vary A, u

$k=0$

...ce in

...ine?

...n

...s μ_a

Action is like

$$\int_{\Sigma\times\mathbb{R}} n\, CS(A) + \int_{p\times\mathbb{R}} A_t^a \mu_a, \quad a \text{ are gauge indices.}$$

Solve EOM vary A, we find

$$n F(A)_a = \delta_p \mu_a$$

Ex

In $U(1)$ CS level n, Wilson line of charge n is trivial. Why?

Answer: It is trivial if there is a gauge invariant junction

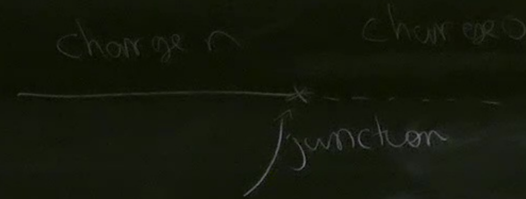
charge n

μ_a

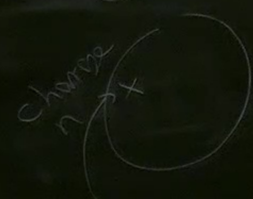
) CS level n, Wilson line of
is trivial. Why?

It is trivial

e is a gauge
t junction

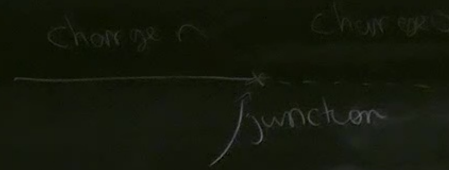


State-operator:
Such junctions = Hilbert space
on S^2 with a Wilson line of charge



... action on ... given by Hamiltonians μ_a

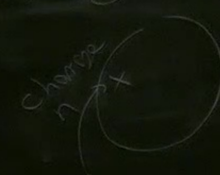
$F(A)_a = \partial_p \mu_a$



U(1) CS level n, Wilson line of charge n is trivial. Why?

Answer: It is trivial if there is a gauge invariant junction

State-operator:
Such junctions = Hilbert space on S^2 with a Wilson line of charge n



Phase space for a particle
of charge n

$$n\frac{F(A)}{2\pi i} = \delta_p \mu$$

If we have a state of charge n,
$\mu = n$ So eq^n is

$$n\frac{F(A)}{2\pi i} = \delta_p n$$

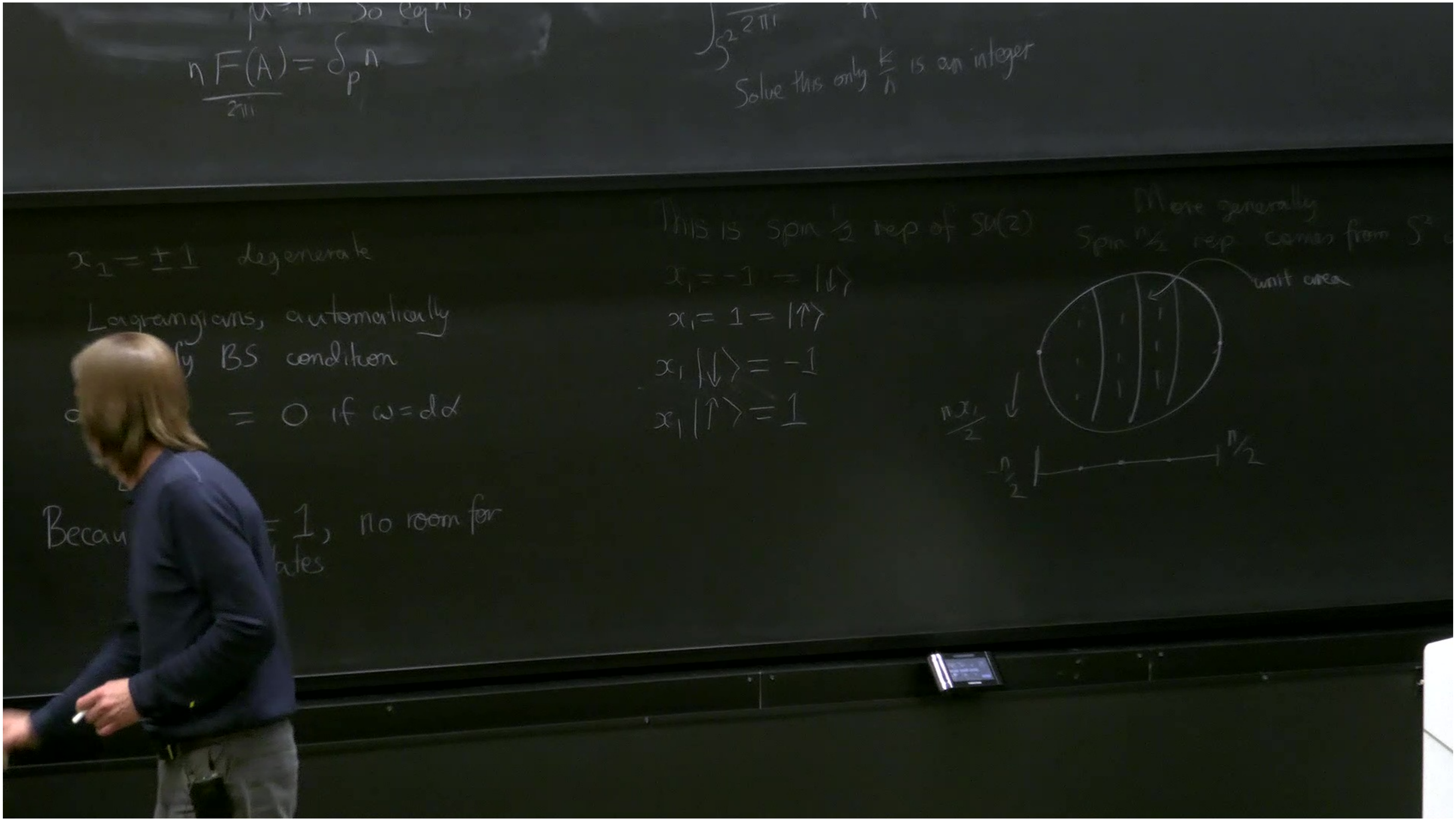
This can be solved
exactly because

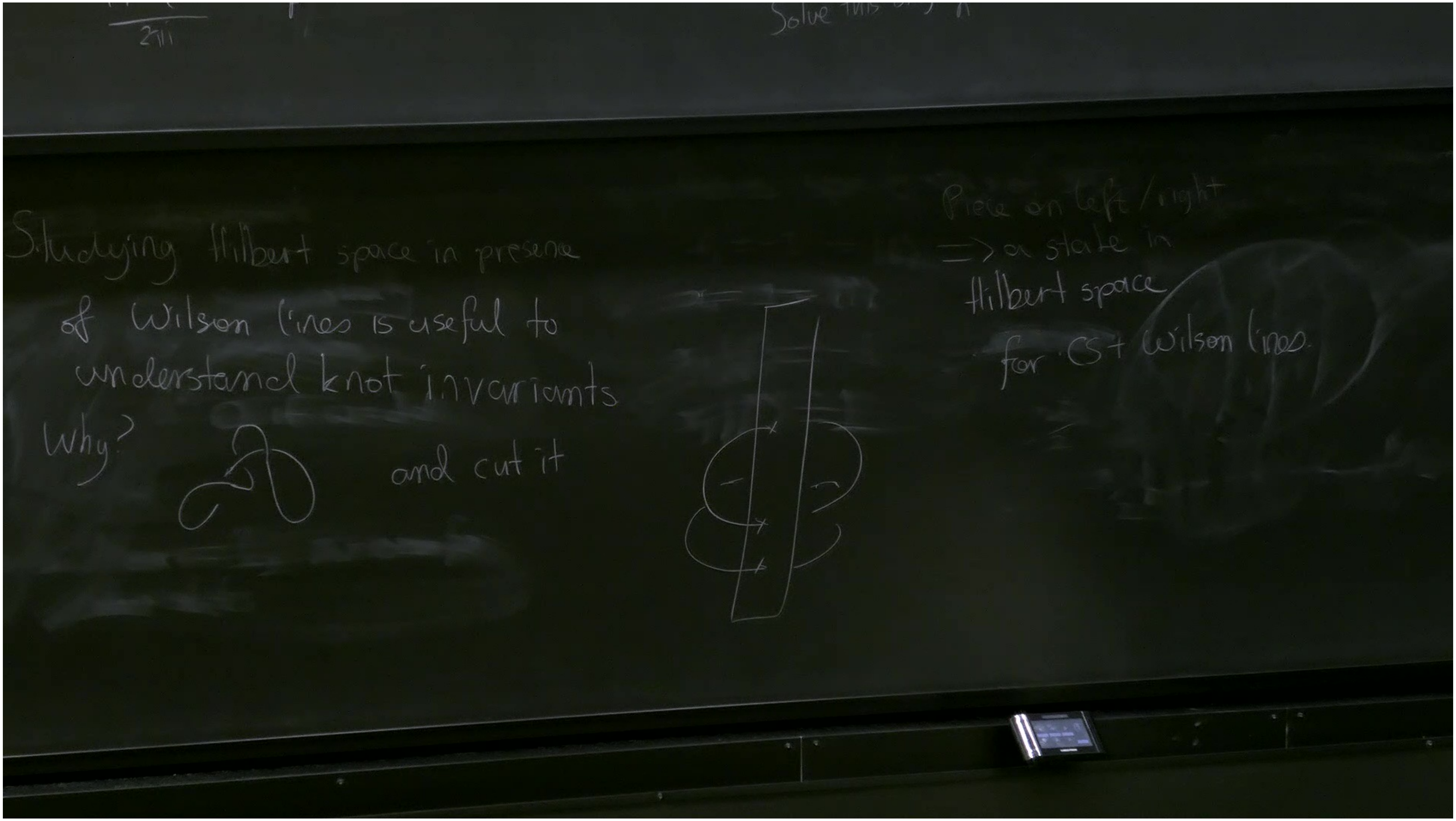
$$\int_{S^2} \frac{F(A)}{2\pi i} = \text{an integer}$$

Wilson line of charge k,

$$\int_{S^2} \frac{F(A)}{2\pi i} = \frac{k}{n}$$

Solve this only $\frac{k}{n}$ is an integer


nF(A)/2πi = δ_p n
Solve this only k/n is an integer
x_1 = ±1 degenerate
Lagrangians, automatically
BS condition
= 0 if ω = dα
= 1, no room for
tates
This is spin 1/2 rep of SU(2)
x_1 = −1 = |↓⟩
x_1 = 1 = |↑⟩
x_1 |↓⟩ = −1
x_1 |↑⟩ = 1
More generally
Spin n/2 rep comes from S^2
unit area
nx_1/2
−n/2
n/2

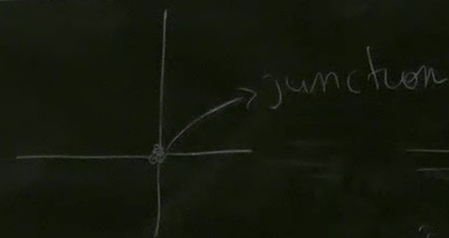

2πi
Solve this
Studying Hilbert space in presence
of Wilson lines is useful to
understand knot invariants
Why?
and cut it
Piece on left / right
⇒ a state in
Hilbert space
for CS + Wilson lines.

Fact

Consider
SU(2) CS
Hilbert space
on S^2 with
4 Wilson lines

This Hilbert space is of dimension
2

Why? Hilbert space =
SU(2) inv. junctions

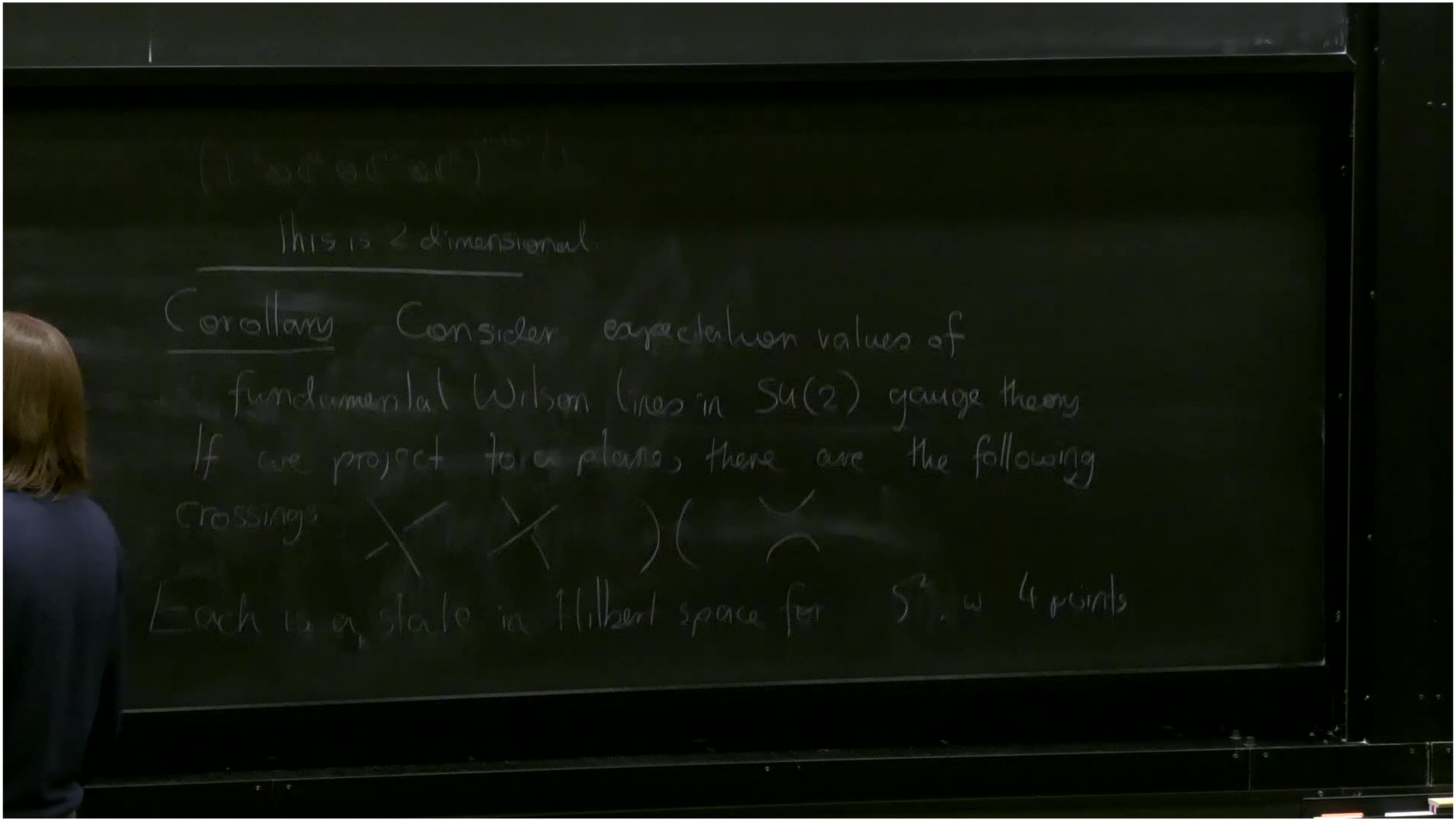


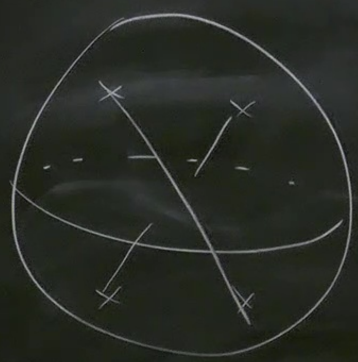
= SU(2) inv. states
in 4 copies of Hilbert
space of Wilson line

$$\left(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\right)^{SU(2)}$$

This is 2 dimensional.

Corollary Consider expectation values of fundamental Wilson lines in $SU(2)$ gauge theory. If we project to a plane, there are the following crossings:


This is 2 dimensional.
Corollary Consider expectation values of
fundamental Wilson lines in SU(2) gauge theory
If we project to a plane, there are the following
crossings
Each is a state in Hilbert space for S², w 4 points



Each crossing
fills in S^2 w. 4 points

This Hilbert space is 2 dimensional,
there are relations!
These are Kauffman / Skein relations

$$= q \;) (\; + q^{-1}$$

$$q = e^{2\pi i/n}$$

This relation is enough to prove that CS knot invariants for SU(2) = Jones polynomial (another famous knot invariant)