Title: GOLD-PLATED SICS

Speakers: Ingemar Bengtsson

Collection: Foundations of Quantum Computational Advantage

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Abstract: There are well established conjectures about the symmetries of SIC-POVMs, and the number fields needed to construct them. If the dimension is of the form n^2 + 3 there is also an algorithm that allows us to calculate them, making use of Stark units in a subfield of the full number field. The algorithm works in the 72 dimensions where it has been tested.

Joint work with (among others) Markus Grassl and Gary McConnell

GOLD-PLATED SICS

- SIC-POVM =
  = maximal equiangular tight frame in $\mathbb{C}^d$

$$\sum_{i=1}^{d^2} |\psi_i\rangle\langle\psi_i| = \alpha_1 \mathbb{1}$$

$$|\langle\psi_i|\psi_j\rangle|^2 = \begin{cases} 1 & \text{if } i=j \\ \alpha_2 & \text{if } i \neq j \end{cases}$$

$$\alpha_1 = d \qquad \alpha_2 = \frac{1}{d+1}$$

- Desirable objects — Feng + 14, arXiv: 2310.08838
- Orbits under Weyl-Heisenberg group — Zauner; Renes et al.
  Samuel & Gedik, arXiv: 2401.11026
- Unitary symmetry of order 3 — Zauner; Appleby
- Many detailed observations for $d \leq 67$ ($d \leq 193$) — Scott & Grassl

  If $d = n^2 + 3$ SICs with Anti-Unitary symmetry exist

Warning

To finish in 45 minutes I will slur over some details

Full details in

arXiv: 2112.0552 by Appleby, Bengtsson, Grassl, Harrison, McConnell

arXiv: 2403.02872 by Bengtsson, Grassl, McConnell

What mathematics gives rise to SICs ?

$$|\psi_{i,j}\rangle = X^i Z^j |\psi\rangle, \quad 0 \le i,j < d$$

M. Grassl

$$
\begin{aligned}
v_1 := \Big(&(336(\sqrt{7}-\sqrt{21})\theta_1 - 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} - 378)\theta_2^2\\
&+(56(3\sqrt{7}-2\sqrt{3}+3)\theta_1 + 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63)\theta_2\\
&+(168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 6\Big)i\\
&+(336(\sqrt{7}+\sqrt{21})\theta_1 + 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 378)\theta_2^2\\
&+(56(3\sqrt{7}-2\sqrt{3}-3)\theta_1 - 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} - 63)\theta_2\\
&+(24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7} - 168)\theta_1 - 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} + 6,\\
v_2 := \Big(&(672(\sqrt{7}-\sqrt{21})\theta_1 - 168\sqrt{3} + 504)\theta_2^2\\
&+(28(3\sqrt{21}+5\sqrt{3}-3\sqrt{7}-15)\theta_1 - 42\sqrt{3} + 126)\theta_2\\
&+(336 - 48\sqrt{21} - 112\sqrt{3} + 48\sqrt{7})\theta_1 - 12\sqrt{21} - 12\sqrt{3} + 12\sqrt{7} + 36\Big)i\\
&-(84\sqrt{21} - 252\sqrt{3} - 252\sqrt{7} + 252)\theta_2^2\\
&+(84(\sqrt{21}+\sqrt{3}-3\sqrt{7}-1)\theta_1 - 6\sqrt{21} + 18\sqrt{7})\theta_2 - 24\sqrt{3} + 24,\\
v_3 := \ &6(\sqrt{7}-\sqrt{3})i + 6\sqrt{21} + 12\sqrt{3} - 12\sqrt{7} - 18\\
v_4 := \Big(&(336(\sqrt{7}-\sqrt{21})\theta_1 + 126\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 126)\theta_2^2\\
&+(56(6-3\sqrt{21}-2\sqrt{3}+3\sqrt{7})\theta_1 - 9\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63)\theta_2\\
&+((168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 54)\Big)i\\
&+(336(\sqrt{21}-3\sqrt{7})\theta_1 + 42\sqrt{21} - 378\sqrt{3} - 126\sqrt{7} + 378)\theta_2^2\\
&+(168(\sqrt{3}-1)\theta_1 - 3\sqrt{21} + 63\sqrt{3} + 9\sqrt{7} - 63)\theta_2\\
&+(24\sqrt{21} + 168\sqrt{3} - 72\sqrt{7} - 168)\theta_1 + 6 - 6\sqrt{21} - 6\sqrt{3} + 18\sqrt{7},\\
v_5 := \Big(&(672\sqrt{7}\theta_1 + 84\sqrt{21} - 168\sqrt{3} + 252)\theta_2^2\\
&-((84\sqrt{21} - 140\sqrt{3} + 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} + 42\sqrt{3})\theta_2\\
&-(112\sqrt{3}\theta_1 - 48\sqrt{7}\theta_1 + 12\sqrt{3} - 12\sqrt{7} + 24)\Big)i\\
&+(672\sqrt{7}\theta_1 - 84\sqrt{21} - 168\sqrt{3} - 252)\theta_2^2\\
&+((84\sqrt{21} + 140\sqrt{3} - 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} - 42\sqrt{3})\theta_2\\
&-112\sqrt{3}\theta_1 + 48\sqrt{7}\theta_1 - 12\sqrt{3} + 12\sqrt{7} + 24,\\
v_6 := \ &6(\sqrt{7}-\sqrt{3})i - 6\sqrt{21} + 18.
\end{aligned}
$$

Fig. 1. Explicit solution for the initial state $|\phi_0\rangle = \theta_3(v_1, v_2, v_3, v_4, v_5, v_6)^t$ of a SIC-POVM in dimension $d = 6$.

WH group $\quad ZX = \omega XZ$

$X^d = Z^d$

$$\omega = e^{2\pi i/d}$$

$$Z|r\rangle = \omega^r |r\rangle$$

$$X|r\rangle = |r+1\rangle$$

Preferred basis !

In that basis, components belong to abelian extension of

$$\mathbb{K} = \mathbb{Q}(\sqrt{D})$$

- $D = \mathrm{sqf}\ (d+1)(d-3)$

Appleby, Yadsan-Appleby, Zauner

SIC SYMMETRIES

$N = \begin{cases} d & \text{if } d \text{ odd} \\ 2d & \text{if} \ldots \end{cases}$

$$U_G X^{i} Z^{j} U_G^{-1} \sim X^{i'} Z^{j'}$$

$SL(2, \mathbb{Z}_N)$

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}, \quad \alpha\beta - \gamma\delta = \pm 1$$

integers mod N

(factor group of "Clifford")

Rep. of WH fixed $\Rightarrow$ rep of $SL(2, \mathbb{Z}_N)$ fixed

Ex: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to$ Fourier matrix

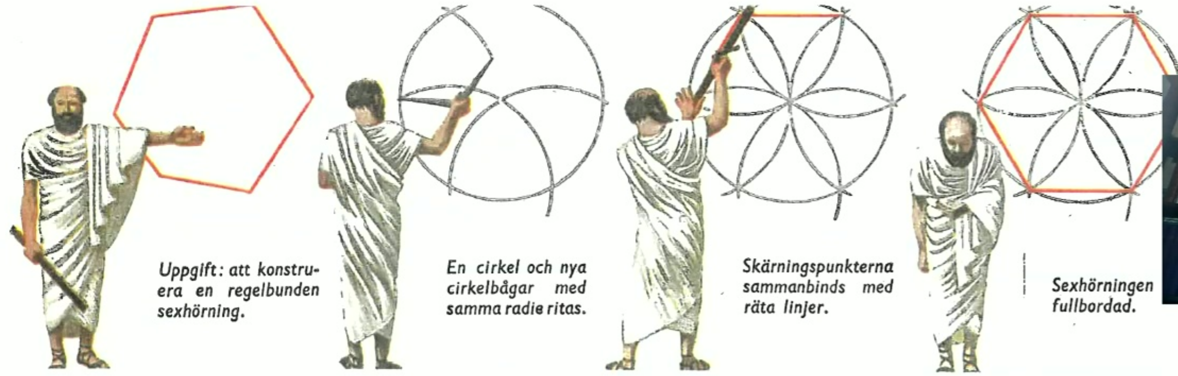
$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \to$ permutation matrix !

NB: $d = p_1^{k_1} p_2^{k_2} \cdots \Rightarrow WH(d) = WH(p_1^{k_1}) \times WH(p_2^{k_2})$,

$SL(2, \mathbb{Z}_d) = SL(2, \mathbb{Z}_{p_1^{k_1}}) \times SL(2, \mathbb{Z}_{p^{k_2}}) \times \ldots$

Order 3 diagonal? $\alpha^3 = 1 \mod p \Rightarrow p = 3k + 1$

Number



Base field $\mathbb{Q}$

$\Downarrow$

$\mathbb{Q}(\omega)$

algebraic unit

modulus

$\omega = e^{2\pi i/d}$

Cyclotomic fields
- general solution

Kronecker-Weber

"Ray class fields" over a base field give the general abelian extension, depending on a "modulus"

$$\boxed{\text{Appleby, Flammia, McConnell, Yard 2016}}$$

Base field $\mathbb{Q}(\sqrt{D})$

$\Downarrow$

abelian extensions classified (cf. Magma)

"Nice" description missing
Hilbert, 1900

Stark units are Stark, 70ies
conjectural analogues of roots of unity

$(d+1)(d-3) = f^2 D$ has ∞ solutions d_ℓ if D is fi

"Dimension towers"

$D = 2$: $d_\ell = 7, 35, 199, (1155), 6727, \ldots$

$D = 5$: $d_\ell = 4, 8, 19, 48, 124, 323, 844, (2208), 5779, \ldots, 39604, \ldots$

$D = 13$: $d_\ell = 12, 120, 1299, \ldots$

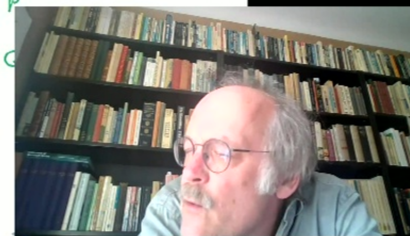
$\boxed{\text{Symmetry of order } 3\ell}$

$D = 3$: $d_\ell = 5, 53, 195, (725), \ldots$

$D = 6$: $d_\ell = 11, 99, \ldots$

$\boxed{\text{If } d_1 = n^2+3, \text{ so is } d_\ell \text{ if } \ell \text{ is odd}}$

SICs from Stark units using $\begin{cases} \text{a/ overlap phases} & \text{Kopp} \\ \text{b/ fiducial vectors} & \text{AB} \end{cases}$



$$\overset{!}{d = } n^2+3 = 4^{e_1} \times 3^{e_2} \times p_1^{k_1} \times p_2^{k_2} \times \dots$$

$e_1, e_2 = 0$ or 1

$p_1, p_2, \dots \overset{!}{=} 1 \mod 3$

Suppose first $d = n^2+3 = p$ (Hardy & Littlewood)

(Arrange so that) order 3 symmetry $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha^3 = 1 \mod p$

$U_{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}}$ is a permutation matrix

Centralizer of this "Zauner" is $U_{\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}}$, $\theta^{p-1} = 1 \mod p$

Also a permutation matrix.

Anti-unitary symmetry $U_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ "=" complex conjugation

Scott: There exists a real fiducial vector !

Apply DFT, $|\psi_C\rangle = F\,|\psi_R\rangle$

components

Einstein: $\langle\psi_R|X^i|\psi_R\rangle = \langle\psi_C|F X^i F^{-1}|\psi_C\rangle = \langle\psi_C|Z^i|\psi_C\rangle = \sum_k \omega^{ik}\,|a$

Roy: $|a_k|^2 = \frac{1}{d}\sum_i \omega^{-ki}\langle\psi_R|X^i|\psi_R\rangle \overset{\text{SIC}}{=} \frac{1}{d}\left(1 + \frac{1}{\sqrt{d+1}}\sum_{i=1}^{d-1}\omega^{-ki}\right)$ "almost flat"

$$\psi = N\begin{pmatrix} \sqrt{x_0} \\ e^{i\varphi_1} \\ e^{i\varphi_2} \\ \vdots \\ e^{i\varphi_{d-1}} \end{pmatrix} \qquad x_0 = -2-\sqrt{d+1}$$

$d-1/3e$ distinct phase factors

$U_{\begin{pmatrix}\Theta & \\ & \Theta^{-1}\end{pmatrix}}$ acts by permutations,

$$a_{\Theta^0} \to a_{\Theta} \to a_{\Theta^2} \to \dots$$

$$a_{\Theta^{(d-1)/3e}} = a_{\Theta^0}$$

$\boxed{\text{All we need is one number, and a cyclic Galois transformation of order } (d-1)/3e}$

Apply DFT, $|\psi_C\rangle = F\,|\psi_R\rangle$

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$d-1/3e$ distinct phase factors

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All we need is one number, and a cyclic Galois transformation of order $d-1/3e$

Base field $\mathbb{Q}(\sqrt{D})$, ray class field with conductor d, deno[...] ^{d}

$$D = \text{square-free part of } (d+1)(d-3)$$

$$d = n^2+3 \Rightarrow d+1 = f^2 D$$

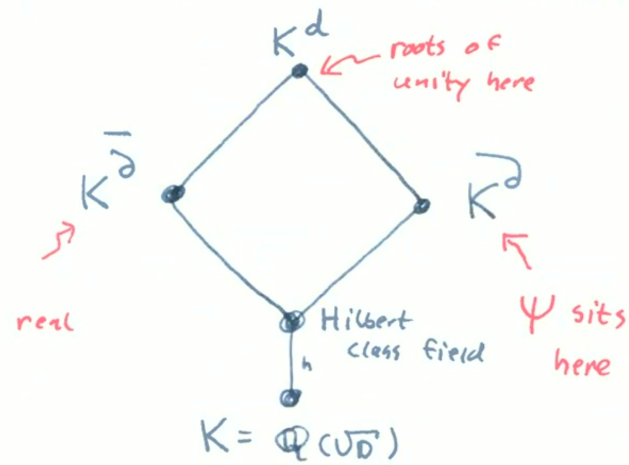
$$d = f^2D - 1 = (f\sqrt{D}+1)(f\sqrt{D}-1) \equiv \partial\bar{\partial}$$

conductor for SIC field

The prime d splits over $K = \mathbb{Q}(\sqrt{D})$

$$\partial \mid d \Rightarrow K^{\partial} \subset K^{d} \quad \text{- a subfield}$$

Use $\partial = f\sqrt{D}+1$ as conductor for "fiducial field" – roots of unity have "decoupled" from ψ



$$\text{Degree}(K^d) = h \times \frac{(d-1)^2}{3\ell}$$

$$\boxed{\text{Degree}(K^{\partial}) = h \times \frac{d-1}{3\ell}, \text{ cyclic Galois group}}$$

Harold Stark gave conjectural formula for algebraic units in our ray class fields

$$\epsilon_\sigma = e^{\delta'(s,\sigma)}\Big|_{s=0}$$

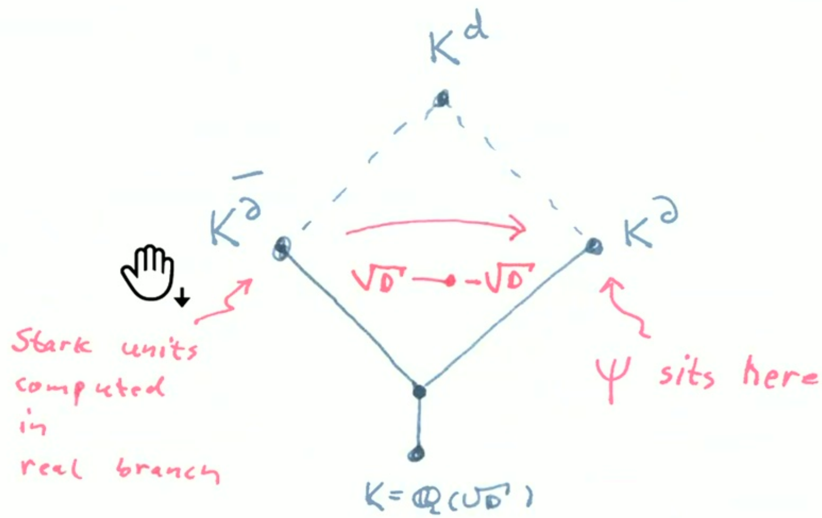
σ = element of Galois (ray class) group

$\delta(s)$ is an analytic function, difference between zeta-functions

This can be evaluated numerically (detour involving Hecke L-functions).

We need enough precision to determine the minimal polynomial

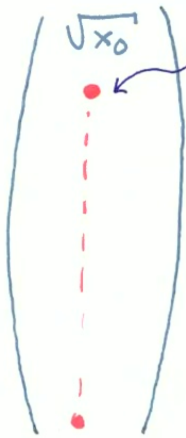
CPU time 82 days for $d = 19603$



- $\epsilon_\sigma = e^{\delta'(s,\sigma)}\big|_{s=0} > 0$, so their square roots should

- When moved from $K^{\bar{d}}$ to K^{d} the Stark units become phase

$\Psi = N$



Place the square root of a complex Stark unit here, and act with Galois group

It's done, with an algorithm

- If the dimension is odd and composite, story is messier, but similar, with two surprises

- If the dimension is even, we need a (lovely) trick

- We tested all $d = n^2+3$ with $n \leq 53$ + 19 additional cases

  It always works, once the surprises are handled

(But: why is this a SIC-vector ?)

$d = n^2 + 3 = 4 \times \text{odd}$

No order 3 symmetry acts through permutations in the dimension-4 factor.

Change basis! Diagonalize

$$X^2, \quad Z^2, \quad X^2 Z^2$$

in dimension-4 factor.

Then the entire Clifford group is represented by monomial matrices

Appleby + 4, some time ago

$$U_Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \text{permutation}$$

$$\Psi = \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \\ \underline{v}_4 \end{pmatrix}$$

$d-1/e$ Stark u
from K

$d-1/3e$ Stark units
from K^∂

Nice feature:

$$d = n^2 + 3 = 4p$$

$\Rightarrow$ n odd and

$$p - 1 = \frac{n^2 - 1}{4} = \frac{n-1}{2} \times \frac{n+1}{2}$$

Why is it a SIC - part II

(How to prove the
Stark conj

a) The checking boils down to identities for Stark units

Nice example:

square roots of Stark units

Stark unit

$$\sqrt{d+1}\,\langle\psi|\,X^{-2j}\,|\psi\rangle = -\left(\frac{a_j}{|a_j|}\right)^2 = \epsilon_j$$

permutation matrix

b) Different perspective on the number theory? Appleby, Flammia, Kopp, Lagarias, to appear

c) Some completely new idea?

Meanwhile, our algorithm gives SICs in dimensions where numerical searches are impossible (and incidentally, Stark units in number fields where no other way is possible)

Warning

To finish in 45 minutes I will slur over some details

Full details in

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