Title: Quantum metrological limits in noisy environments

Speakers: Sisi Zhou

Collection: Foundations of Quantum Computational Advantage

Date: May 03, 2024 - 9:15 AM

URL: https://pirsa.org/24050015

Abstract: The Heisenberg limit (HL) and the standard quantum limit (SQL) are two fundamental quantum metrological limits, which describe the scalings of estimation precision of an unknown parameter with respect to N, the number of one-parameter quantum channels applied. In the first part, we show the HL (1/N) is achievable using quantum error correction (QEC) strategies when the "Hamiltonian-not-in-Kraus-span" (HNKS) condition is satisfied; and when HNKS is violated, the SQL (1/N^1/2) is optimal and can be achieved with repeated measurements. In the second part, we identify modified metrological limits for estimating one-parameter qubit channels in settings of restricted controls where QEC cannot be performed. We prove unattainability of the HL and further show a "rotation-generators-not-in-Kraus-span" (RGNKS) condition that determines the achievability of the SQL.
Quantum metrological limits in noisy environments

Sisi Zhou

May 03, 2024

Foundations of Quantum Computational Advantage @ PI
Background and Motivation
Quantum metrology is the science of estimation in quantum systems.

Optical interferometry  Atomic clock  Nitrogen-vacancy centers

Quantum metrology enhanced by quantum controls?

Opportunities and Challenges
Mach-Zehnder Interferometry

\[ |1,0\rangle \rightarrow \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}} \rightarrow e^{i\phi} \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}} \rightarrow \cos \frac{\phi}{2} |1,0\rangle + \sin \frac{\phi}{2} |0,1\rangle \]

The probability of detecting the photon in the upper (lower) port is \( p_\uparrow \) \( p_\downarrow \), like in the biased-coin-tossing experiment where the probability of getting heads is \( p_\uparrow = \cos^2 \frac{\phi}{2} \) and the probability of getting tails is \( p_\downarrow = \sin^2 \frac{\phi}{2} \).
Mach-Zehnder Interferometry

The probability of detecting the photon in the upper (lower) port is $p_\uparrow$ ($p_\downarrow$), like in the biased-coin-tossing experiment where the probability of getting heads is $p_\uparrow = \cos^2 \frac{\varphi}{2}$ and the probability of getting tails is $p_\downarrow = \sin^2 \frac{\varphi}{2}$.

Image Credit: Kolodynski, PhDThesis (2014)
Mach-Zehnder Interferometry with Uncorrelated Photons

|N, 0⟩

Probability of detecting k photons in the upper port:

\[ p_\uparrow^N(k) = \binom{N}{k} p_\uparrow^k (1 - p_\uparrow)^{N-k} \]

with \( p_\uparrow = \cos^2 \frac{\varphi}{2} \),

equivalent to the outcome of k repeated coin-tossing experiments.

Image Credit: Kolodynski, PhDThesis (2014)
Mach-Zehnder Interferometry with Uncorrelated Photons

\[ \frac{N}{N} \text{ uncorrelated photons} \]

\[ |N, 0\rangle \]

50/50 beam splitter

50/50 beam splitter

\[ \delta \varphi = \mathbb{E}[(\hat{\varphi} - \varphi)^2]^{1/2} \] is proportional to \( 1/\sqrt{N} \).

"Standard quantum limit"

Image Credit: Kolodynski, PhDThesis (2014)
Mach-Zehnder Interferometry with NOON States

Probability of detecting even/odd number of photons in the upper port:

\[ p^N_\uparrow (\text{even}) = \cos^2 \frac{N\varphi}{2}, \quad p^N_\uparrow (\text{odd}) = \sin^2 \frac{N\varphi}{2}. \]

Image Credit: Kolodynski, PhDThesis (2014)
Mach-Zehnder Interferometry with NOON States

Probability of detecting even/odd number of photons in the upper port:

\[
p_{\uparrow}^N (\text{even}) = \cos^2 \frac{N\phi}{2}, \quad p_{\uparrow}^N (\text{odd}) = \sin^2 \frac{N\phi}{2}.
\]

The estimation error \( \delta \phi \) is proportional to \( 1/N \). 

“Heisenberg limit”
Mach-Zehnder Interferometry with Photon Losses

Losses are modeled by fictitious beam splitters of transmissivity $\eta$.

\[
\rho_\varphi = \eta^N \left( \frac{e^{iN\varphi} |N,0\rangle + |0,N\rangle}{\sqrt{2}} \right) \left( \frac{e^{-iN\varphi} \langle N,0 | + \langle 0,N |}{\sqrt{2}} \right) + (1 - \eta^N) \rho_0
\]

The estimation error $\delta \varphi$ grows exponentially with $N$, due to quantum noise.

*Can quantum controls, e.g., quantum error correction, help?*

Image Credit: Kolodynski, PhD Thesis (2014)
Quantum Channel Estimation
---with arbitrary quantum controls
Quantum Channel Estimation

- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i} \rho K_{\omega,i}^\dagger$, $\omega \approx 0$.
- **Heisenberg limit (HL):** $\delta \omega \propto 1/N$
- **Standard quantum limit (SQL):** $\delta \omega \propto 1/\sqrt{N}$

Sequential strategy

Parallel strategy
Limits of Quantum Channel Estimation

- **HL vs. SQL**
  - Given an arbitrary quantum channel $\mathcal{E}_\omega$, is it possible to achieve the HL using sequential/parallel strategies?

- **Asymptotic coefficients ($f_{HL}$ and $f_{SQL}$)**
  - What is the ultimately achievable asymptotic estimation precision in both cases?

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SZ & Jiang, PRX Quantum 2, 010343 (2021)
“Hamiltonian-not-in-Kraus-span” (HNKS) Condition

- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^r K_{\omega,i}\rho K_{\omega,i}^\dagger$
- The Heisenberg limit ($\delta \omega \propto 1/N$) is achievable using sequential/parallel strategies IF AND ONLY IF
  
  Hamiltonian $H \notin$ Kraus Span $S$,

where

Hamiltonian (signal): $H(\mathcal{E}_\omega) = i \sum_i K_i^\dagger \partial_{\omega} K_i$,

Kraus span (noise): $S(\mathcal{E}_\omega) = \text{span}\{K_i^\dagger K_j, \forall i, j\}$.

Remark: For unitary channel $U_\omega = e^{-iG_\omega}$ or noisy channel $\mathcal{E}_\omega = \mathcal{N} \circ U_\omega$, the Hamiltonian $H(\mathcal{E}_\omega) = i U_\omega \partial_{\omega} U_\omega = G$.

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SZ & Jiang, PRX Quantum 2, 010343 (2021)
“Hamiltonian-not-in-Kraus-span” (HNKS) Criterion

- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$
- The Heisenberg limit ($\delta \omega \propto 1/N$) is achievable using sequential/parallel strategies if and only if
  \[ \text{Hamiltonian } H \notin \text{Kraus Span } S, \]

Example:

\[ \mathcal{E}_\omega(\rho) = (1 - p)e^{-i\omega Z(\cdot)}e^{i\omega Z} + p E e^{-i\omega Z(\cdot)}e^{i\omega Z}E \]

- Hamiltonian $H = Z$, Error $E = Z$, Kraus span = span\{I, Z\}. The HL is not achievable.
- Hamiltonian $H = Z$, Error $E = X$, Kraus span = span\{I, X\}. QEC can recover the HL.

SZ & Jiang, PRX Quantum 2, 010343 (2021)
Example: Pauli-Z Hamiltonian

\[ \mathcal{E}_\omega(\cdot) = e^{-i\omega Z(\cdot)}e^{i\omega Z} \]

\[ |\psi_0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow |\psi_\omega\rangle = \frac{e^{-i\omega N}|0\rangle + e^{i\omega N}|1\rangle}{\sqrt{2}} \]

*Heisenberg limit:* \( \delta \omega \propto 1/N \)
Example: Pauli-Z Hamiltonian + Bif-flip noise

\[ \mathcal{E}_\omega(\cdot) = (1 - p)e^{-i\omega Z(\cdot)}e^{i\omega Z} + p X e^{-i\omega Z(\cdot)}e^{i\omega Z} X \]

\[ |\psi_0\rangle = \frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}} \Rightarrow |\psi_\omega\rangle = \frac{e^{-i\omega N}|0_L\rangle + e^{i\omega N}|1_L\rangle}{\sqrt{2}} \]

Heisenberg limit: \( \delta \omega \propto 1/N \)

Example: Pauli-Z Hamiltonian + Bif-flip noise

\[ \mathcal{E}_\omega(\cdot) = (1 - p)e^{-i\omega Z(\cdot)}e^{i\omega Z} + p X e^{-i\omega Z(\cdot)}e^{i\omega Z} X \]

\[ |\psi_0\rangle = \frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}} \]

\[ = \frac{|00 \cdots 0\rangle_{S_1 \cdots S_N} + |11 \cdots 1\rangle_{S_1 \cdots S_N}}{\sqrt{2}} \]

*Heisenberg limit: \( \delta \omega \propto 1/N \)*

*Dur et al. PRL 112, 080801 (2014); Kessler et al. PRL 112, 080802 (2014); Arrad et al. PRL 112, 150801 (2014)*
Example: Pauli-Z Hamiltonian + Bif-flip noise

\[ \varepsilon_{\omega}(\cdot) = (1 - p) e^{-i\omega Z(\cdot)} e^{i\omega Z} + p X e^{-i\omega Z(\cdot)} e^{i\omega Z} X \]

Recovery operation (Majority voting):

\[ |00000111\rangle \rightarrow |00000000\rangle \]
\[ |01100111\rangle \rightarrow |11111111\rangle \]

*Heisenberg limit*: \( \delta \omega \propto 1/N \)

Example: Pauli-Z Hamiltonian + Dephasing noise

$$\mathcal{E}_\omega(\cdot) = (1 - p)e^{-i\omega Z(\cdot)}e^{i\omega Z} + p Z e^{-i\omega Z(\cdot)}e^{i\omega Z} Z$$

$$|\psi_0\rangle = \left(\frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}}\right)^\otimes N$$

Standard quantum limit: $\delta \omega \propto 1/\sqrt{N}$

Restricted Quantum Controls

---What if we cannot do QEC?
Metrological Limits with Restricted Controls

Sequential strategy

- Noiseless ancilla
- CPTP controls (mid-circuit measurement)

Parallel strategy

- Large system size
- Long-range entanglement

What happens if QEC is not possible?
Metrological Limits with Restricted Controls

- Ancilla-free sequential strategy, unital controls
  \[
  \rho_0 \xrightarrow{\epsilon_\omega} \mathcal{U}_1 \xrightarrow{\epsilon_\omega} \mathcal{U}_2 \xrightarrow{\ldots} \mathcal{U}_{N-1} \xrightarrow{\epsilon_\omega} \{M_i\}
  \]

- Ancilla-free sequential strategy, CPTP controls
  \[
  \rho_0 \xrightarrow{\epsilon_\omega} \mathcal{C}_1 \xrightarrow{\epsilon_\omega} \mathcal{C}_2 \xrightarrow{\ldots} \mathcal{C}_{N-1} \xrightarrow{\epsilon_\omega} \{M_i\}
  \]
Metrological Limits with Restricted Controls

- Ancilla-free sequential strategy, unital controls

- Bounded-ancilla strategy, unital controls
Classification of Qubit Channels

- **Unitary channels**

  \[ \mathcal{E}_\omega(\cdot) = V_\omega(\cdot)V_\omega^\dagger, \]

  Hamiltonian: \( H = iV_\omega^\dagger \partial_\omega V_\omega \)

- **Dephasing-class channels** (Dephasing channels up to unitary rotations)

  \[ \mathcal{E}_\omega(\cdot) = V_\omega \left( (1 - p_\omega)U_\omega(\cdot)U_\omega^\dagger + p_\omega \mathbf{Z} U_\omega(\cdot)U_\omega^\dagger \mathbf{Z} \right) V_\omega^\dagger, \]

  Unitary rotation generators: \( H_0 = iV_\omega^\dagger \partial_\omega V_\omega, \quad H_1 = iU_\omega^\dagger \partial_\omega U_\omega \)

- **Strictly contractive channels**

  \[ \| \mathcal{E}_\omega(\rho) - \mathcal{E}_\omega(\sigma) \|_1 < \| \rho - \sigma \|_1 \]
# Results: Scalings of QFI

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Sequential, or Parallel</th>
<th>Sequential, Ancilla-free, Unital controls</th>
<th>Sequential, Ancilla-free, CPTP controls</th>
<th>Sequential, Bounded-ancilla, Unital controls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unitary</td>
<td>$\Theta(N^2)$</td>
<td>$\Theta(N^2)$</td>
<td>$\Theta(N^2)$</td>
<td>$\Theta(N^2)$</td>
</tr>
<tr>
<td>Dephasing-class ($H_0$ or $H_1 \notin S$)</td>
<td>$\Theta(N^2)$ (HNKS holds)</td>
<td>$\Theta(N)$</td>
<td>$O(N^{3/2})$, $\Omega(N)$</td>
<td>$\Theta(N)$</td>
</tr>
<tr>
<td>Dephasing-class ($H_{0,1} \in S$)</td>
<td>$\Theta(N)$</td>
<td>$O(1)$</td>
<td>$O(N)$</td>
<td>$O(N)$</td>
</tr>
<tr>
<td>Strictly Contractive</td>
<td>$\Theta(N)$</td>
<td>$O(1)$</td>
<td>$\Theta(1)$</td>
<td>$O(N)$</td>
</tr>
</tbody>
</table>
Summary and Outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerful – recovering the HL, achieving the optimal asymptotic coefficients.
- Sensing limits are compromised with restricted quantum controls---but the dichotomic behavior still exists.
- **Future directions:**
  - Fault-tolerant QEC (state preparation and measurement noise + gate implementation noise)
  - Usefulness of QEC for finite number of channels & copies
  - Other types of noise and signal, e.g., correlated noise, distributed sensing...

Thank you!