Title: Quantum metrological limits in noisy environments

Speakers: Sisi Zhou

Collection: Foundations of Quantum Computational Advantage

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Abstract: The Heisenberg limit (HL) and the standard quantum limit (SQL) are two fundamental quantum metrological limits, which describe the scalings of estimation precision of an unknown parameter with respect to N, the number of one-parameter quantum channels applied. In the first part, we show the HL (1/N) is achievable using quantum error correction (QEC) strategies when the ``Hamiltonian-not-in-Kraus-span" (HNKS) condition is satisfied; and when HNKS is violated, the SQL (1/N^1/2) is optimal and can be achieved with repeated measurements. In the second part, we identify modified metrological limits for estimating one-parameter qubit channels in settings of restricted controls where QEC cannot be performed. We prove unattainability of the HL and further show a ``rotation-generators-not-in-Kraus-span" (RGNKS) condition that determines the achievability of the SQL.

Quantum metrological limits in noisy environments

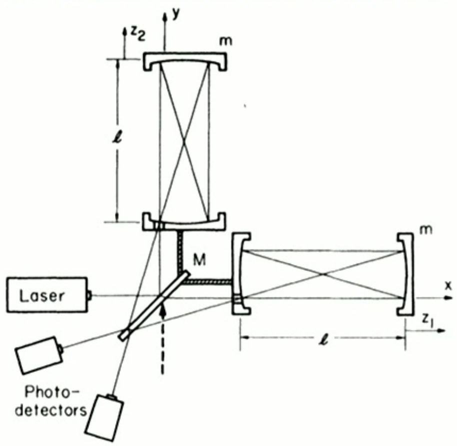
Sisi Zhou

May 03, 2024

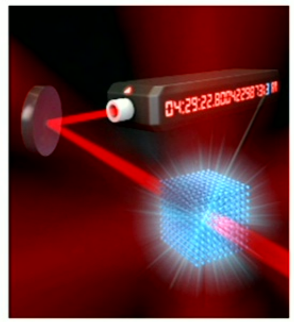
Foundations of Quantum Computational Advantage @ PI

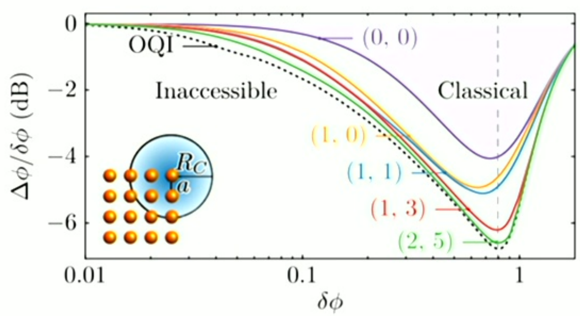
Background and Motivation

Quantum metrology is the science of estimation in quantum systems.

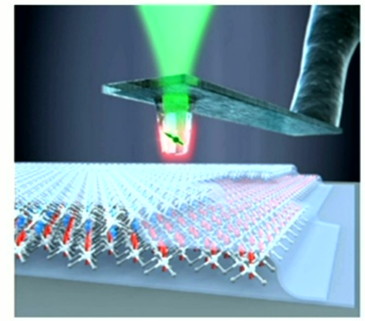


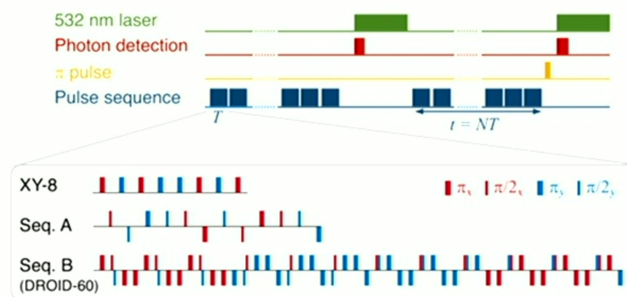
Optical interferometry





Atomic clock

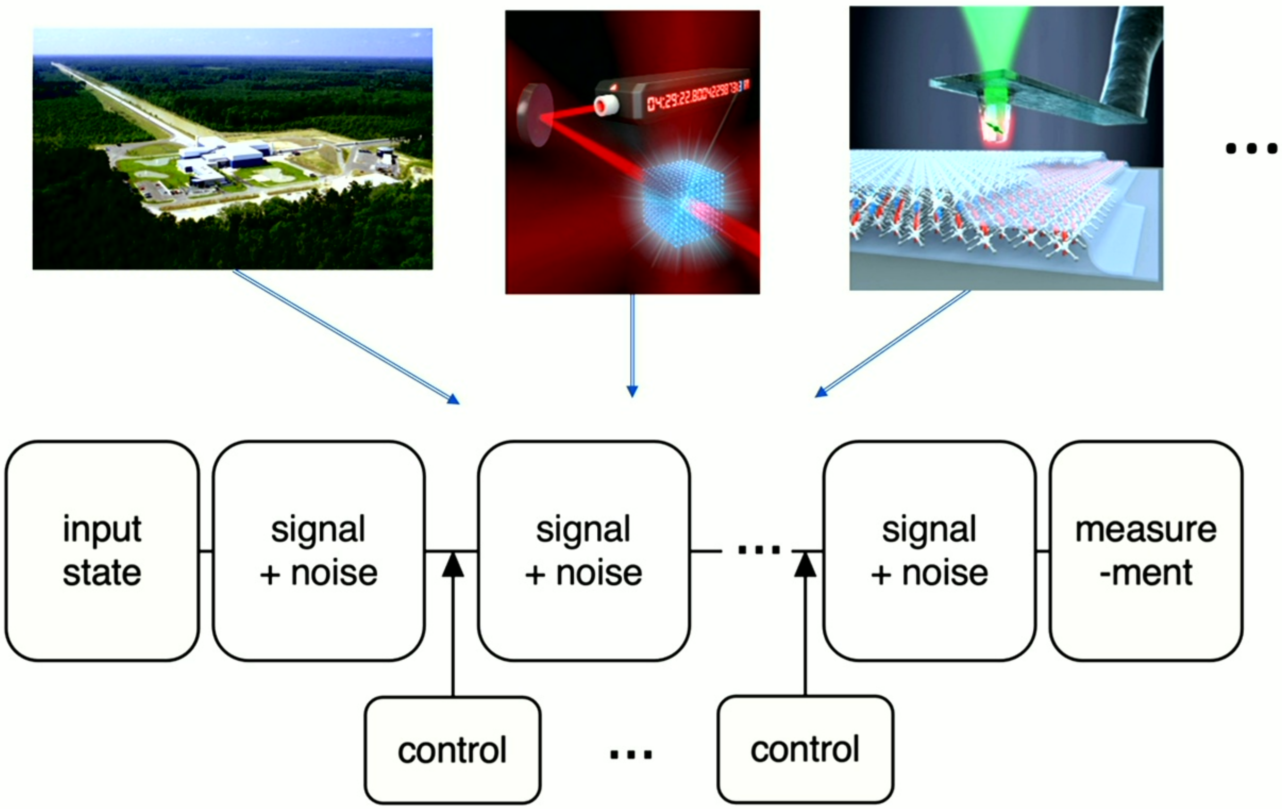




Nitrogen-vacancy centers

Images from: https://www.ligo.caltech.edu/news/ligo20191015, https://www.photonicsviews.com/atomic-clock-with-a-3d-optical-lattice/ https://physicsworld.com/a/diamond-quantum-microscope-images-nanoscale-features-in-2d-magnets/, PhysRevD.23.1693, PhysRevX.11.041045, PhysRevX.10.031003

Quantum metrology enhanced by quantum controls?



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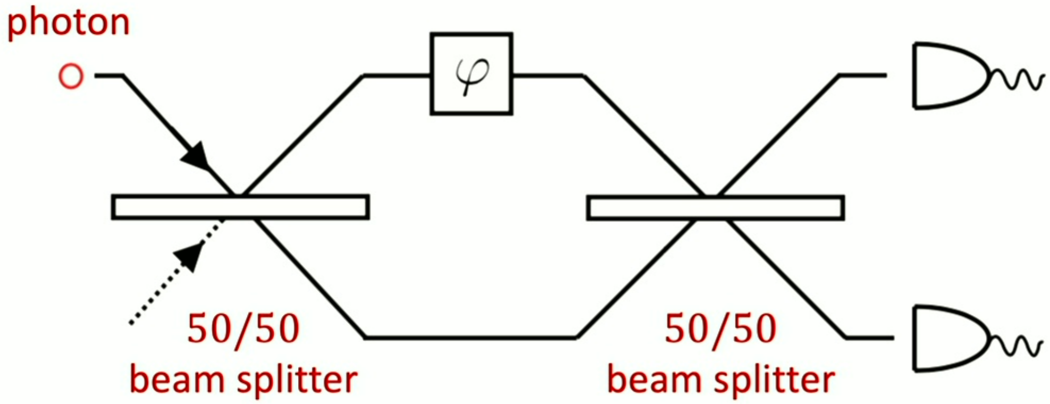
Opportunities and Challenges

Mach-Zehnder Interferometry



$p_{\uparrow} = \cos^2\frac{\varphi}{2}$

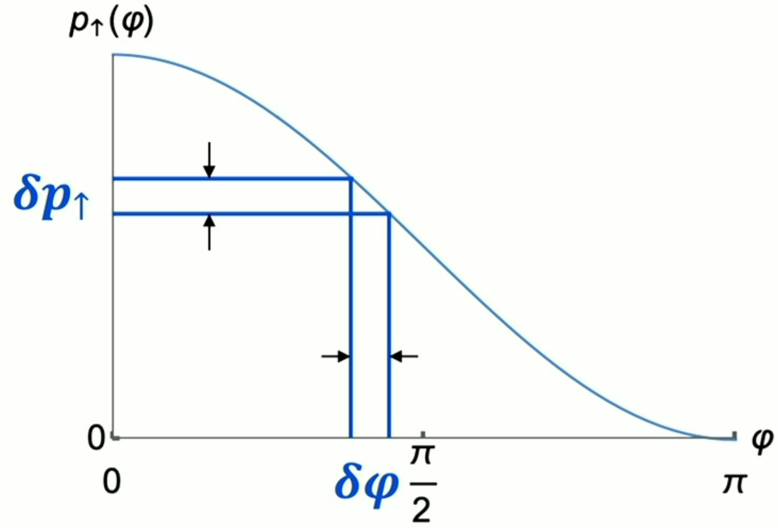
$p_{\downarrow} = \sin^2\frac{\varphi}{2}$

$$|1,0\rangle \;\rightarrow\; \frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}} \;\rightarrow\; \frac{e^{i\varphi}|1,0\rangle + |0,1\rangle}{\sqrt{2}} \;\rightarrow\; \cos\frac{\varphi}{2}|1,0\rangle + \sin\frac{\varphi}{2}|0,1\rangle$$

The probability of detecting the photon in the upper (lower) port is $p_{\uparrow}$ ($p_{\downarrow}$), like in the biased-coin-tossing experiment where the probability of getting heads is $p_{\uparrow} = \cos^2\frac{\varphi}{2}$ and the probability of getting tails is $p_{\downarrow} = \sin^2\frac{\varphi}{2}$.

Image Credit: Kolodynski, PhDThesis (2014)

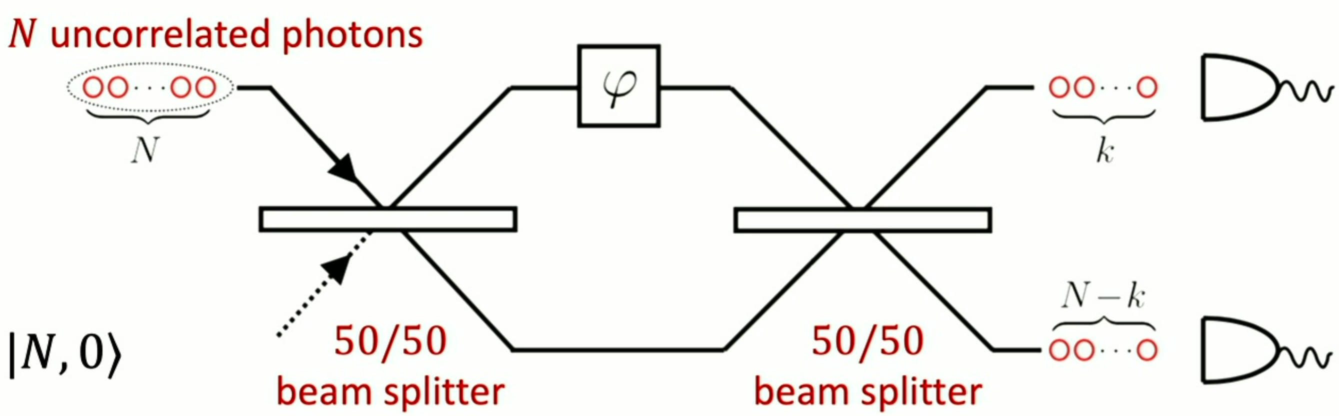
Mach-Zehnder Interferometry



The probability of detecting the photon in the upper (lower) port is $p_\uparrow$ ($p_\downarrow$), like in the biased-coin-tossing experiment where the probability of getting heads is $p_\uparrow = \cos^2\frac{\varphi}{2}$ and the probability of getting tails is $p_\downarrow = \sin^2\frac{\varphi}{2}$.

Image Credit: Kolodynski, PhDThesis (2014)

Mach-Zehnder Interferometry with Uncorrelated Photons



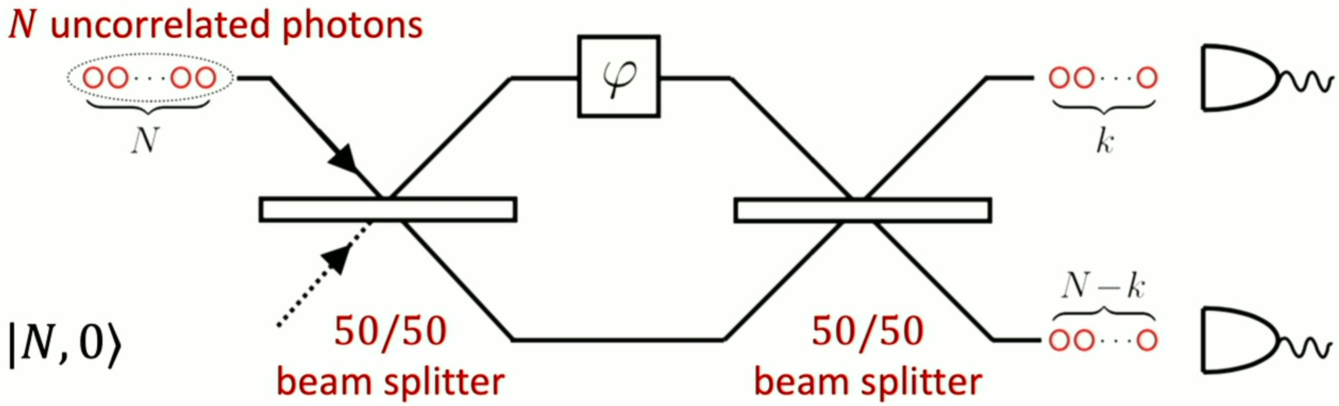
Probability of detecting k photons in the upper port:

$$p_{\uparrow}^{N}(k) = \binom{N}{k} {p_{\uparrow}}^{k} (1 - p_{\uparrow})^{N-k} \quad \text{with } p_{\uparrow} = \cos^2 \frac{\varphi}{2},$$

equivalent to the outcome of k repeated coin-tossing experiments.

Image Credit: Kolodynski, PhDThesis (2014)

Mach-Zehnder Interferometry with Uncorrelated Photons

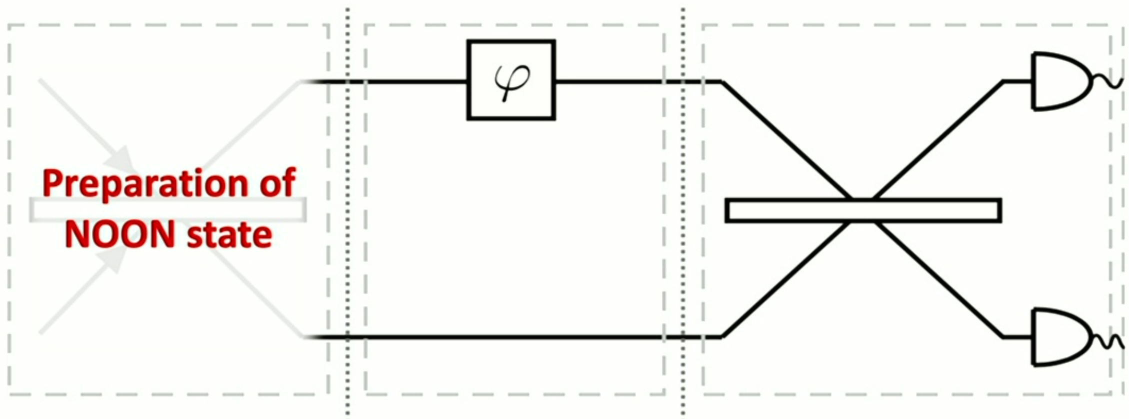


According to the central limit theorem,
the estimation error $\delta\varphi = \mathbb{E}[(\hat{\varphi} - \varphi)^2]^{1/2}$ is proportional to $1/\sqrt{N}$.

"Standard quantum limit"

Image Credit: Kolodynski, PhDThesis (2014)

Mach-Zehnder Interferometry with NOON States

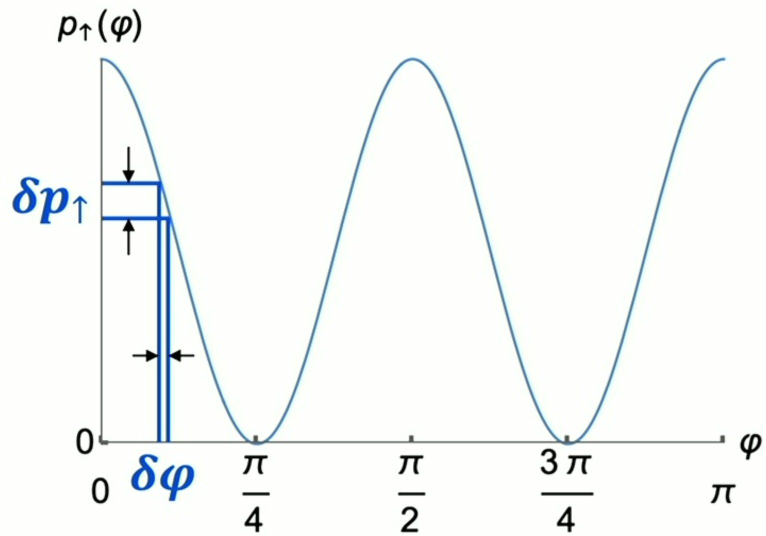


Probability of detecting even/odd number of photons in the upper port:

$$p_{\uparrow}^{N}(\text{even}) = \cos^2 \frac{N\varphi}{2}, \qquad p_{\uparrow}^{N}(\text{odd}) = \sin^2 \frac{N\varphi}{2}.$$

Image Credit: Kolodynski, PhDThesis (2014)

Mach-Zehnder Interferometry with NOON States



Probability of detecting even/odd number of photons in the upper port:

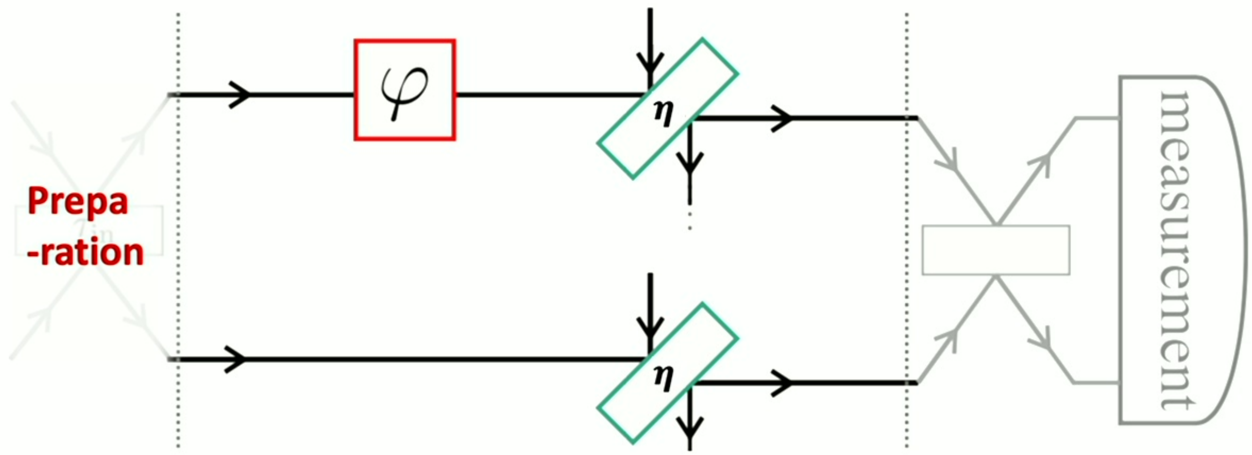
$$p_\uparrow^N(\text{even}) = \cos^2\frac{N\varphi}{2}, \qquad p_\uparrow^N(\text{odd}) = \sin^2\frac{N\varphi}{2}.$$

The estimation error $\delta\varphi$ is proportional to $1/N$. ***"Heisenberg limit"***

Image Credit: Kolodynski, PhDThesis (2014)

Mach-Zehnder Interferometry with Photon Losses

Losses are modeled by fictitious beam splitters of transmissivity η.



$$\rho_\varphi = \eta^N \left(\frac{e^{iN\varphi}|N,0\rangle + |0,N\rangle}{\sqrt{2}}\right)\left(\frac{e^{-iN\varphi}\langle N,0| + \langle 0,N|}{\sqrt{2}}\right) + (1-\eta^N)\rho_0$$

The estimation error $\delta\varphi$ grows exponentially with N, due to quantum noise.

Can quantum controls, e.g., quantum error correction, help?

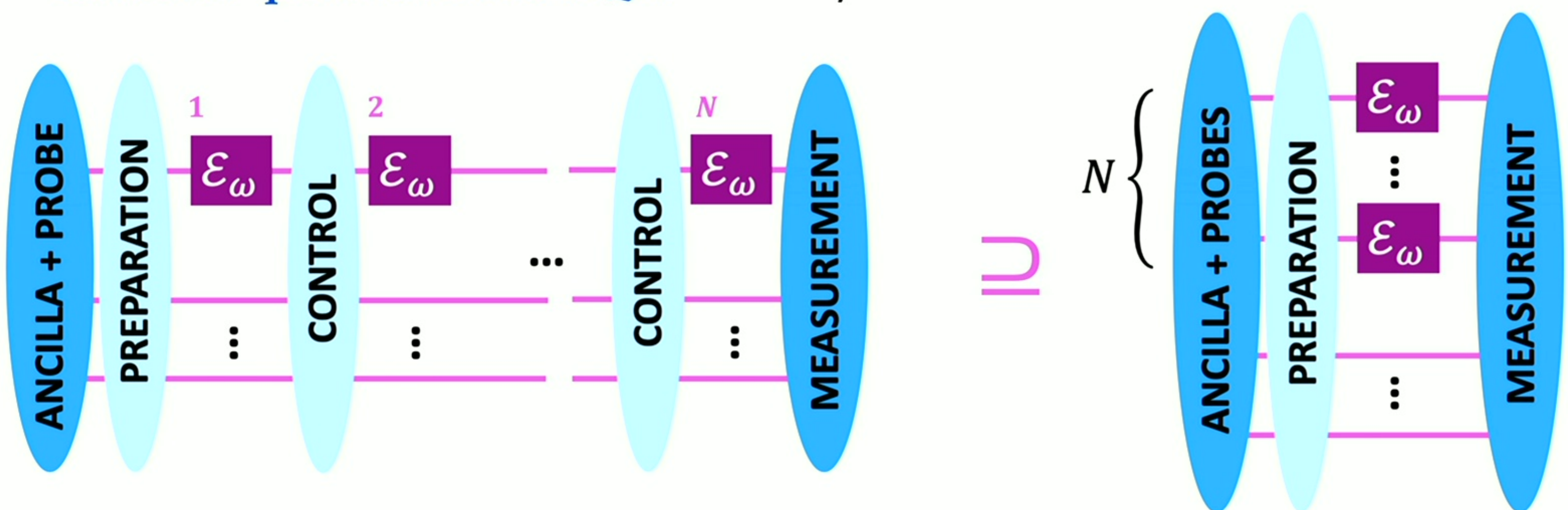
Image Credit: Kolodynski, PhDThesis (2014)

Quantum Channel Estimation

---with arbitrary quantum controls

Quantum Channel Estimation

- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^{\dagger}$, $\omega \approx 0$.
- **Heisenberg limit (HL)**: $\delta\omega \propto 1/N$
- **Standard quantum limit (SQL)**: $\delta\omega \propto 1/\sqrt{N}$



Sequential strategy

Parallel strategy

11

Limits of Quantum Channel Estimation

- **HL vs. SQL**

— Given an arbitrary quantum channel $\mathcal{E}_\omega$, is it possible to achieve the HL using sequential/parallel strategies?

$\delta\omega \propto 1/N$ / $\delta\omega \propto 1/\sqrt{N}$

The HNKS condition

- **Asymptotic coefficients (f_{HL} and f_{SQL})**

— What is the ultimately achievable asymptotic estimation precision in both cases?

$\delta\omega \approx f_{\mathrm{HL}}/N$ / $\delta\omega \approx f_{\mathrm{SQL}}/\sqrt{N}$

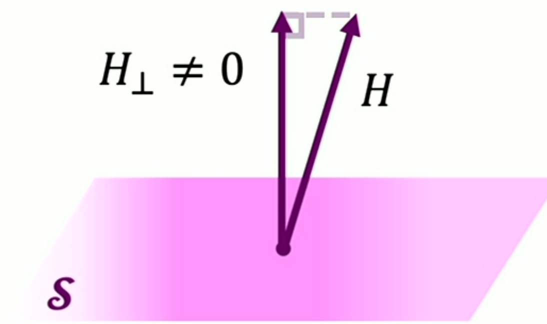
Optimizing over a class of quantum error correction protocols

SZ & Jiang, PRX Quantum 2, 010343 (2021)
SZ, Zhang, Preskill, Jiang, Nature Communications 9:78 (2018); Demkowicz-Dobrzański *et al.* PRX 7, 041009 (2017)

"Hamiltonian-not-in-Kraus-span" (HNKS) Condition

- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$
- The Heisenberg limit $(\delta\omega \propto 1/N)$ is achievable using sequential/parallel strategies IF AND ONLY IF

Hamiltonian $H \notin$ Kraus Span $\mathcal{S}$,



where

Hamiltonian (signal): $H(\mathcal{E}_\omega) = i\sum_i K_i^\dagger \partial_\omega K_i$,

Kraus span (noise): $\mathcal{S}(\mathcal{E}_\omega) = \text{span}\{K_i^\dagger K_j, \forall i,j\}$.

Remark: For unitary channel $U_\omega = e^{-iG\omega}$ or noisy channel $\mathcal{E}_\omega = \mathcal{N} \circ \mathcal{U}_\omega$, the Hamiltonian $H(\mathcal{E}_\omega) = iU_\omega \partial_\omega U_\omega = G$.

SZ & Jiang, PRX Quantum 2, 010343 (2021)
SZ, Zhang, Preskill, Jiang, Nature Communications 9:78 (2018); Demkowicz-Dobrzański *et al.* PRX 7, 041009 (2017)

"Hamiltonian-not-in-Kraus-span" (HNKS) Criterion

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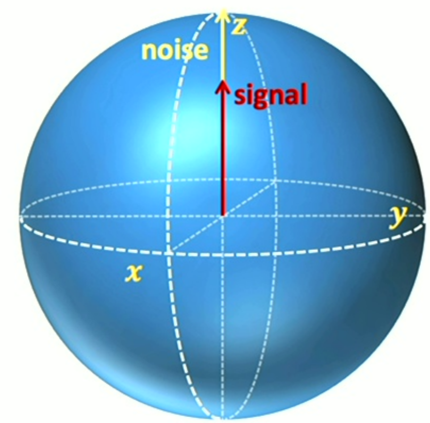


Example:

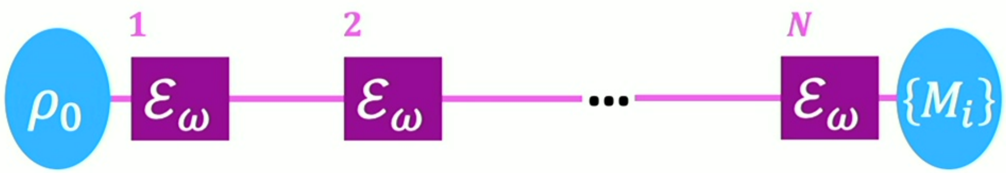
$$\boldsymbol{\mathcal{E}_\omega(\rho) = (1-p)e^{-i\omega Z}(\cdot)e^{i\omega Z} + p\, E\, e^{-i\omega Z}(\cdot)e^{i\omega Z} E}$$

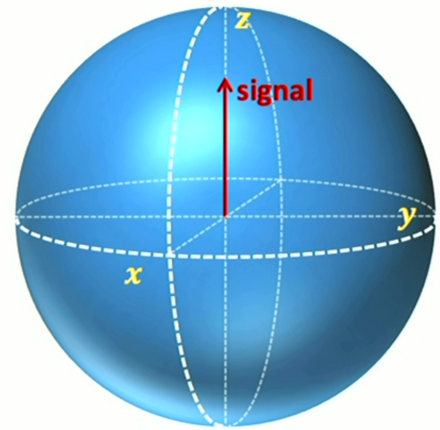
- Hamiltonian $H = Z$, Error $E = Z$, Kraus span = span$\{I, Z\}$. The HL is not achievable.
- Hamiltonian $H = Z$, Error $E = X$, Kraus span = span$\{I, X\}$. QEC can recover the HL.

SZ & Jiang, PRX Quantum 2, 010343 (2021)
SZ, Zhang, Preskill, Jiang, Nature Communications 9:78 (2018); Demkowicz-Dobrzański *et al.* PRX 7, 041009 (2017)

Example: Pauli-Z Hamiltonian

$$\mathcal{E}_\omega(\cdot) = e^{-i\omega Z}(\cdot)e^{i\omega Z}$$



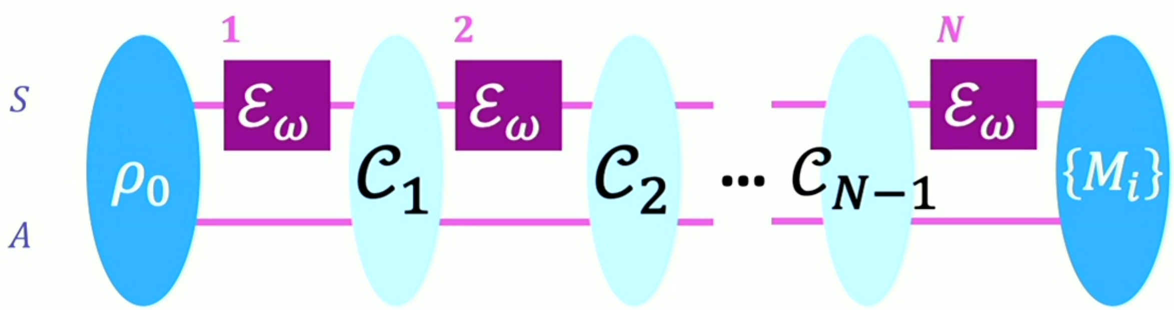


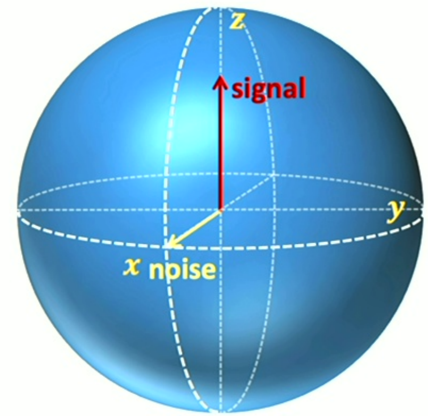
$$|\psi_0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow |\psi_\omega\rangle = \frac{e^{-i\omega N}|0\rangle + e^{i\omega N}|1\rangle}{\sqrt{2}}$$

Heisenberg limit: $\delta\omega \propto 1/N$

Example: Pauli-Z Hamiltonian + Bif-flip noise

$$\mathcal{E}_\omega(\cdot) = (1-p)e^{-i\omega Z}(\cdot)e^{i\omega Z} + p\, X\, e^{-i\omega Z}(\cdot)e^{i\omega Z}\, X$$





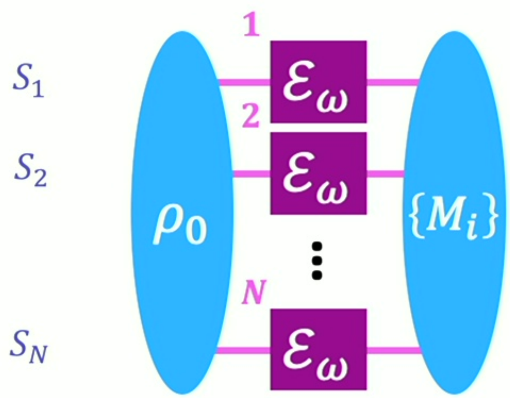
$$|\psi_0\rangle = \frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}} \Rightarrow |\psi_\omega\rangle = \frac{e^{-i\omega N}|0_L\rangle + e^{i\omega N}|1_L\rangle}{\sqrt{2}}$$

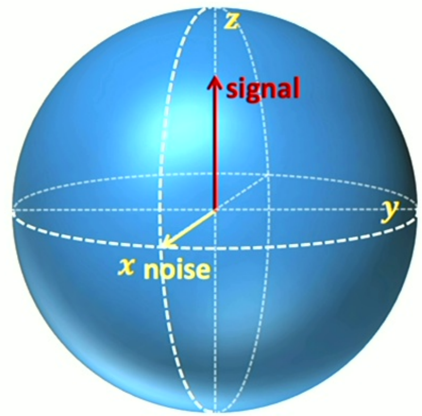
Heisenberg limit: $\delta\omega \propto 1/N$

Dur *et al.* PRL 112, 080801 (2014); Kessler *et al.* PRL 112, 080802 (2014); Arrad *et al.* PRL 112, 150801 (2014)

Example: Pauli-Z Hamiltonian + Bif-flip noise

$$\mathcal{E}_\omega(\cdot) = (1-p)e^{-i\omega Z}(\cdot)e^{i\omega Z} + p\, X\, e^{-i\omega Z}(\cdot)e^{i\omega Z}\, X$$





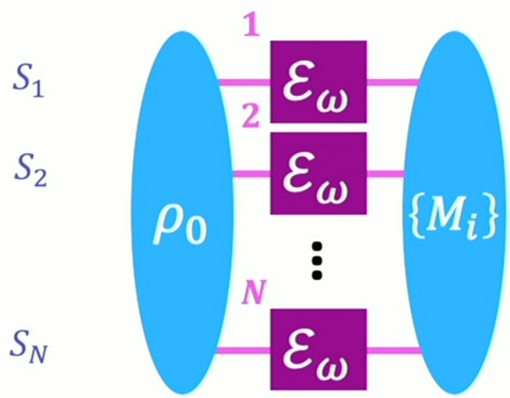
$$|\psi_0\rangle = \frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}} = \frac{|00\cdots0\rangle_{S_1\cdots S_N} + |11\cdots1\rangle_{S_1\cdots S_N}}{\sqrt{2}}$$

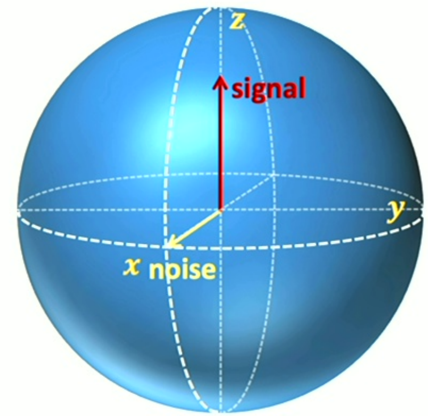
Heisenberg limit: $\delta\omega \propto 1/N$

Dur *et al.* PRL 112, 080801 (2014); Kessler *et al.* PRL 112, 080802 (2014); Arrad *et al.* PRL 112, 150801 (2014)

Example: Pauli-Z Hamiltonian + Bif-flip noise

$$\mathcal{E}_\omega(\cdot) = (1-p)e^{-i\omega Z}(\cdot)e^{i\omega Z} + p\, X\, e^{-i\omega Z}(\cdot)e^{i\omega Z}\, X$$





Recovery operation (Majority voting):

$|00000111\rangle \rightarrow |00000000\rangle$

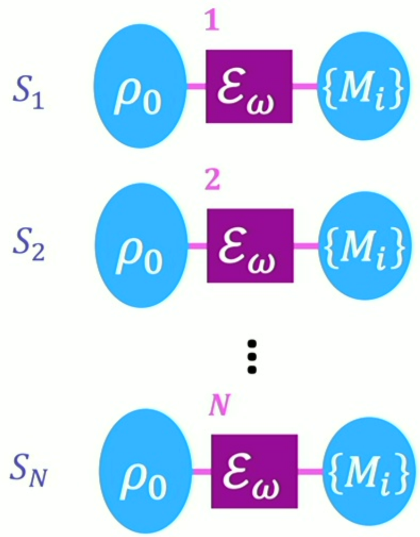
$|01100111\rangle \rightarrow |11111111\rangle$ ……

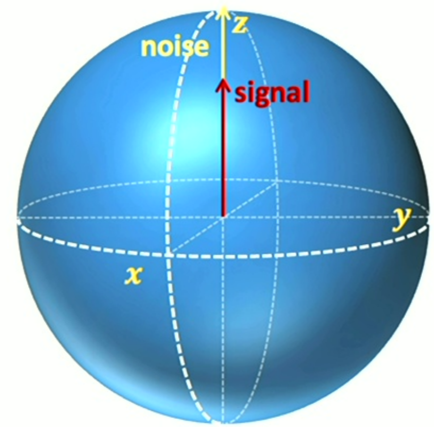
Heisenberg limit: $\delta\omega \propto 1/N$

Dur *et al.* PRL 112, 080801 (2014); Kessler *et al.* PRL 112, 080802 (2014); Arrad *et al.* PRL 112, 150801 (2014)

Example: Pauli-Z Hamiltonian + Dephasing noise

$$\mathcal{E}_{\omega}(\cdot) = (1-p)e^{-i\omega Z}(\cdot)e^{i\omega Z} + p\, Z\, e^{-i\omega Z}(\cdot)e^{i\omega Z}\, Z$$





$$|\psi_0\rangle = \left(\frac{|0_L\rangle + |1_L\rangle}{\sqrt{2}}\right)^{\otimes N}$$

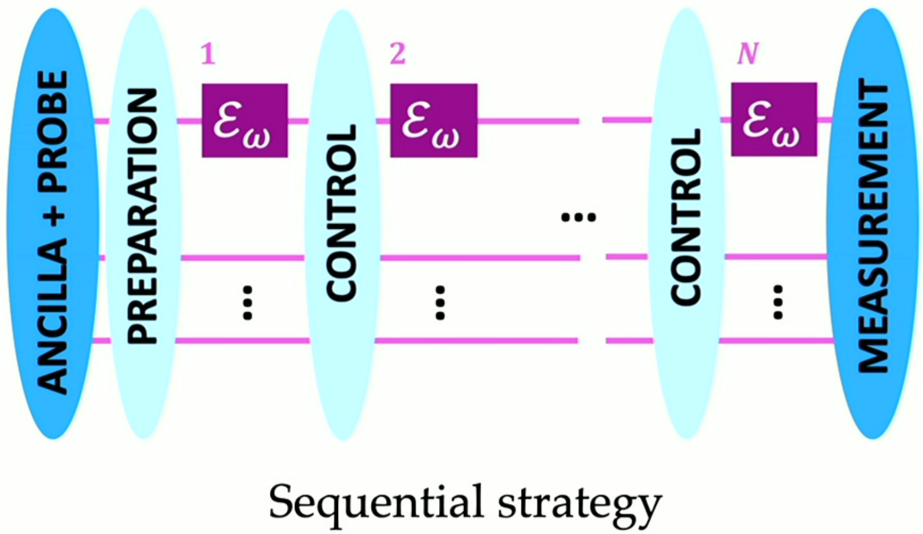
Standard quantum limit: $\delta\omega \propto 1/\sqrt{N}$

Dur *et al.* PRL 112, 080801 (2014); Kessler *et al.* PRL 112, 080802 (2014); Arrad *et al.* PRL 112, 150801 (2014)

Restricted Quantum Controls

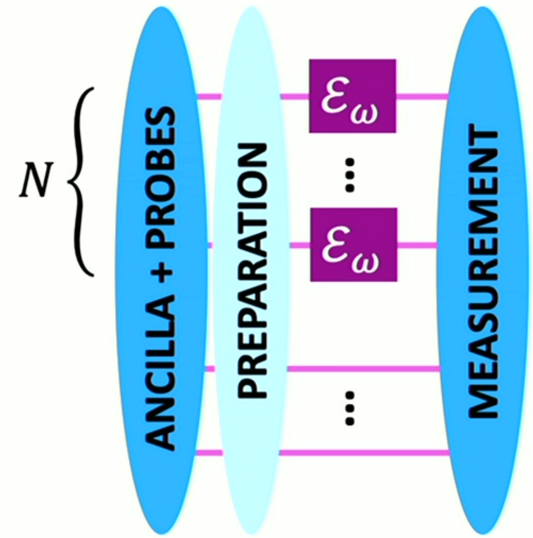
---What if we cannot do QEC?

Metrological Limits with Restricted Controls



Sequential strategy

- Noiseless ancilla
- CPTP controls (mid-circuit measurement)



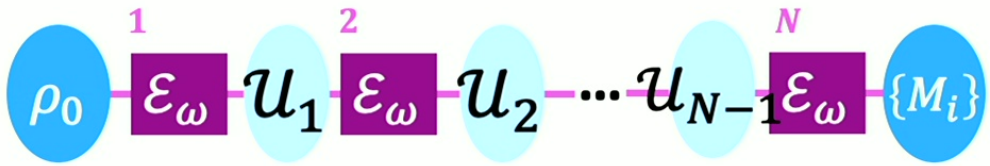
Parallel strategy

- Large system size
- Long-range entanglement

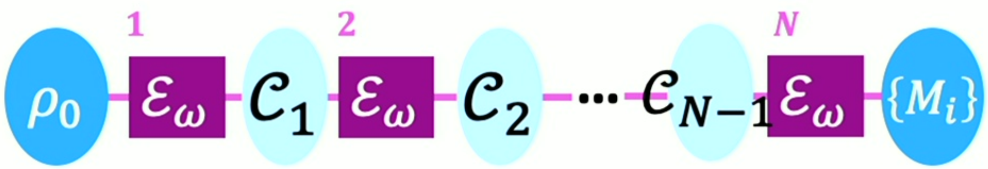
What happens if QEC is not possible?

Metrological Limits with Restricted Controls

- Ancilla-free sequential strategy, unital controls

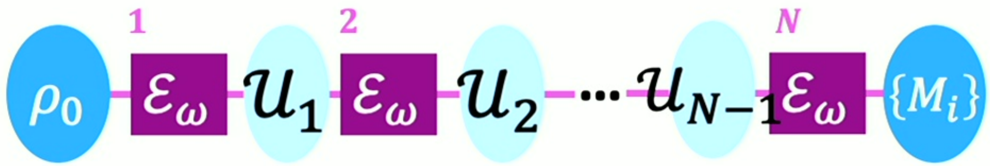


- Ancilla-free sequential strategy, CPTP controls

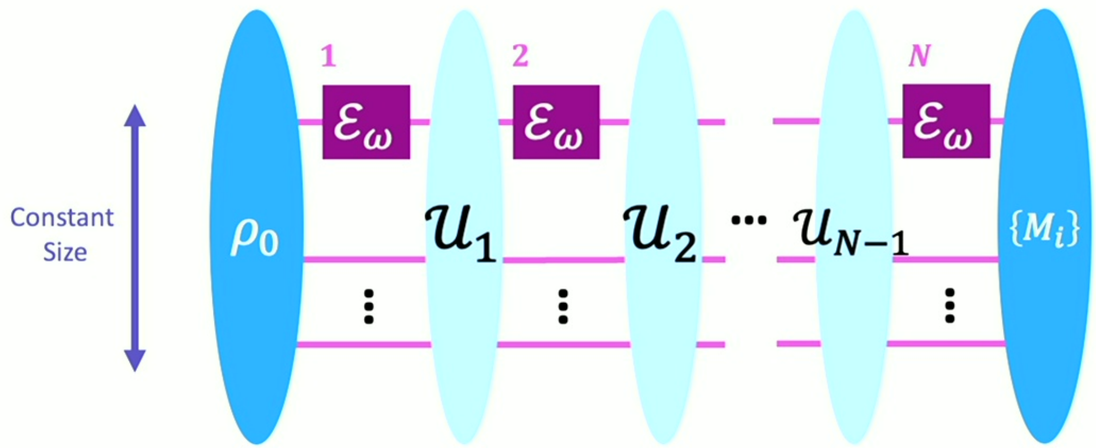


Metrological Limits with Restricted Controls

- Ancilla-free sequential strategy, unital controls



- Bounded-ancilla strategy, unital controls



Classification of Qubit Channels

- **Unitary channels**

$$\mathcal{E}_\omega(\cdot) = V_\omega(\cdot)V_\omega^\dagger,$$

$$\text{Hamiltonian: } H = iV_\omega^\dagger \partial_\omega V_\omega$$

- **Dephasing-class channels** (Dephasing channels up to unitary rotations)

$$\mathcal{E}_\omega(\cdot) = V_\omega\left((1-p_\omega)U_\omega(\cdot)U_\omega^\dagger + p_\omega\, \boldsymbol{Z}\, U_\omega(\cdot)U_\omega^\dagger\, \boldsymbol{Z}\right)V_\omega^\dagger,$$

$$\text{Unitary rotation generators: } \boldsymbol{H_0} = iV_\omega^\dagger \partial_\omega V_\omega,\ \boldsymbol{H_1} = iU_\omega^\dagger \partial_\omega U_\omega$$

- **Strictly contractive channels**

$$\|\mathcal{E}_\omega(\rho) - \mathcal{E}_\omega(\sigma)\|_1 < \|\rho - \sigma\|_1$$

King & Ruskai IEEE-TIT (2001)

Results: Scalings of QFI

Strategies	Sequential, or Parallel	Sequential, Ancilla-free, Unital controls	Sequential, Ancilla-free, CPTP controls	Sequential, Bounded-ancilla, Unital controls
Unitary	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(N^2)$
Dephasing-class (H_0 or $H_1 \notin \mathcal{S}$)	$\Theta(N^2)$ (HNKS holds) $\Theta(N)$ (HNKS fails)	$\Theta(N)$	$O(N^{3/2}), \Omega(N)$	$\Theta(N)$
Dephasing-class ($H_{0,1} \in \mathcal{S}$)	$\Theta(N)$	$O(1)$	$O(N)$	$O(N)$
Strictly Contractive	$\Theta(N)$	$O(1)$	$\Theta(1)$	$O(N)$

SZ, arXiv: 2402.18765

Summary and Outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerful – recovering the HL, achieving the optimal asymptotic coefficients.
- Sensing limits are compromised with restricted quantum controls---but the dichotomic behavior still exists.
- **Future directions:**
  - Fault-tolerant QEC (state preparation and measurement noise + gate implementation noise)
  - Usefulness of QEC for finite number of channels & copies
  - Other types of noise and signal, e.g., correlated noise, distributed sensing…

Thank you!

45