Title: Efficiently achieving fault-tolerant qudit quantum computation via gate teleportation

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Collection: Foundations of Quantum Computational Advantage

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Abstract: Quantum computers operate by manipulating quantum systems that are particularly susceptible to noise. Classical redundancy-based error correction schemes cannot be applied as quantum data cannot be copied. These challenges can be overcome by using a variation of the quantum teleportation protocol to implement those operations which cannot be easily done fault-tolerantly. This process consumes expensive resources called 'magic states'. The vast quantity of these resources states required for achieving fault-tolerance is a significant bottleneck for experimental implementations of universal quantum computers.

I will discuss a program of finding and classifying those quantum operations which can be performed with efficient use of magic state resources. I will focus on the understanding of not just qubits but also the higher-dimensional 'qudit' case. This is motivated by both practical reasons and for the resulting theoretical insights into the ultimate origin of quantum computational advantages. Research into these quantum operations has remained active from their discovery twenty-five years ago to the present. Our approach introduces the novel use of tools from algebraic geometry.

The results in this talk will include joint work with Chen, Lautsch, and Bampounis-Barbosa.

Efficiently achieving fault-tolerant quantum computation via gate teleportation

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May 2, 2024

Setting

- Setting: fault-tolerant quantum computation
- Magic states: 'resources' for FTQC
- Higher-dimensional qudit settings
- Mathematical physics

Overview

- Quantum error correction and fault-tolerance
- Teleportation and gate teleportation
- Mathematical background
- Results:
  - more efficient protocols for key operations of QC
  - classifying the resources that drive QC
  - novel mathematical methods
- Tools: linear algebra, abstract algebra, algebraic geometry

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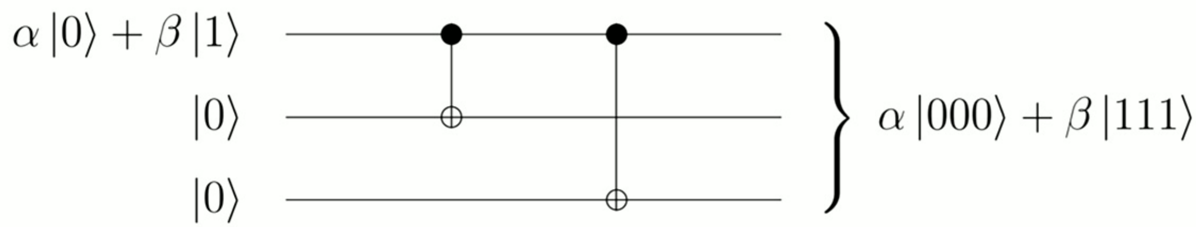
Quantum error correction

Classical computers protect errors using redundancy
e.g. encode 0 as 000 and 1 as 111.

The no-cloning theorem forbids this approach for quantum data.

We can still encode quantum data in a larger physical system.



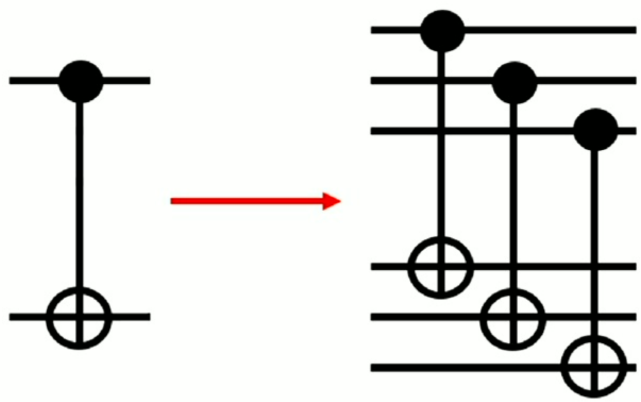
We cannot peek at encoded data without destroying it.

We can ask: *how has the data been corrupted?* and apply appropriate corrections.

Fault tolerance

We have to compute directly on encoded data.

The easiest way to perform fault-tolerant gates is transversally; errors do not propagate.



Not all gates can be performed transversally (Eastin-Knill, 2009).

Stabiliser formalism: Symplectic vector spaces

For $(\vec{p}, \vec{q}) \in \mathbb{Z}_d^{2n}$, $W(\vec{p}, \vec{q}) = Z^{p_1} X^{q_1} \otimes \ldots \otimes Z^{p_n} X^{q_n}$

Stabiliser formalism: Pauli gates

The basic Pauli gates $Z, X \in M_d(\mathbb{C})$ are, for any odd prime d,

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & \omega^{d-1} \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/d}$.

These unitaries satisfy the Weyl canonical commutation relations:

$$Z^d = X^d = \mathbb{I} \qquad ZX = \omega XZ.$$

Stabiliser formalism: Symplectic vector spaces

For $(\vec{p}, \vec{q}) \in \mathbb{Z}_d^{2n}$, $W(\vec{p}, \vec{q}) = Z^{p_1} X^{q_1} \otimes \ldots \otimes Z^{p_n} X^{q_n}$

$W(\vec{p}, \vec{q})$ and $W(\vec{p}\,', \vec{q}\,')$ commute if and only if

$$[(\vec{p}, \vec{q}), (\vec{p}\,', \vec{q}\,')] = \vec{p} \cdot \vec{q}\,' - \vec{p}\,' \cdot \vec{q} = 0.$$

The group of Pauli gates $\mathcal{C}_1 = \{\omega^a W(\vec{p}, \vec{q}) \mid a \in \mathbb{Z}_d, (\vec{p}, \vec{q}) \in \mathbb{Z}_d^{2n}\}$.

Quantum data is encoded using certain eigenvectors of Pauli gates: stabiliser states.

Stabiliser formalism: Transversal gates

Which gates can easily be performed fault-tolerantly (e.g. transversally)?

The group of Clifford gates $\mathcal{C}_2 = \{G \mid G\mathcal{C}_1 G^* \subseteq \mathcal{C}_1\}$.

The normaliser of the Pauli group within the unitary group.

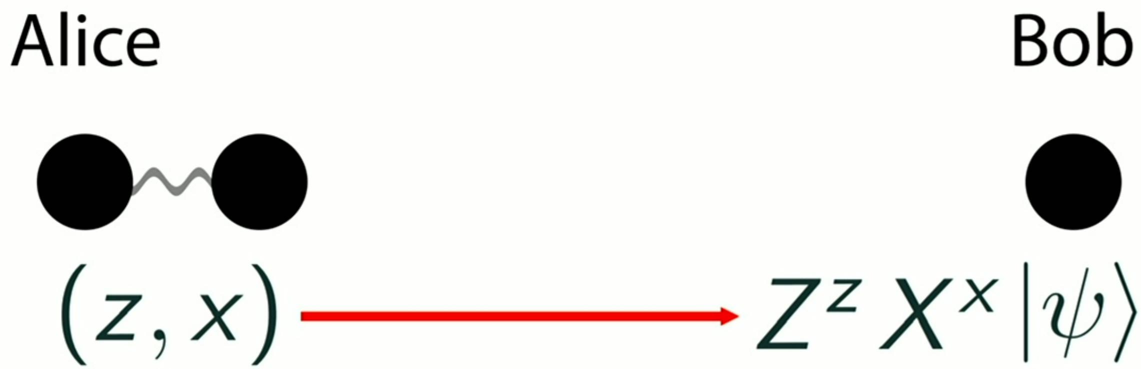
$$\mathcal{C}_2/\mathbb{T} \cong Sp(n, \mathbb{Z}_d) \ltimes \mathbb{Z}_d^{2n}$$

It is a maximal nondense subgroup of the unitaries.

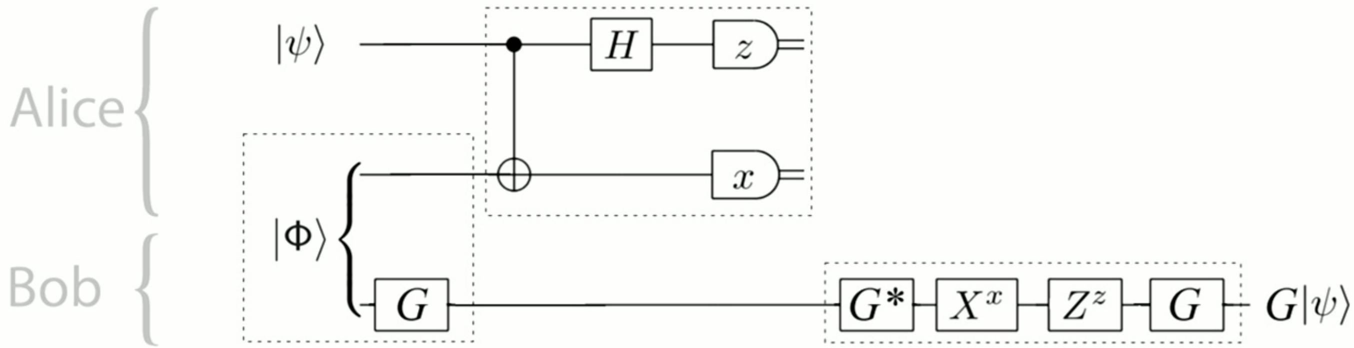
We need to be able to fault-tolerantly perform a non-Clifford gate to achieve universal quantum computation.

Quantum teleportation

Quantum teleportation



Gate teleportation



Third level: $\mathcal{C}_3 = \{G \mid G\mathcal{C}_1 G^* \subseteq \mathcal{C}_2\}$.

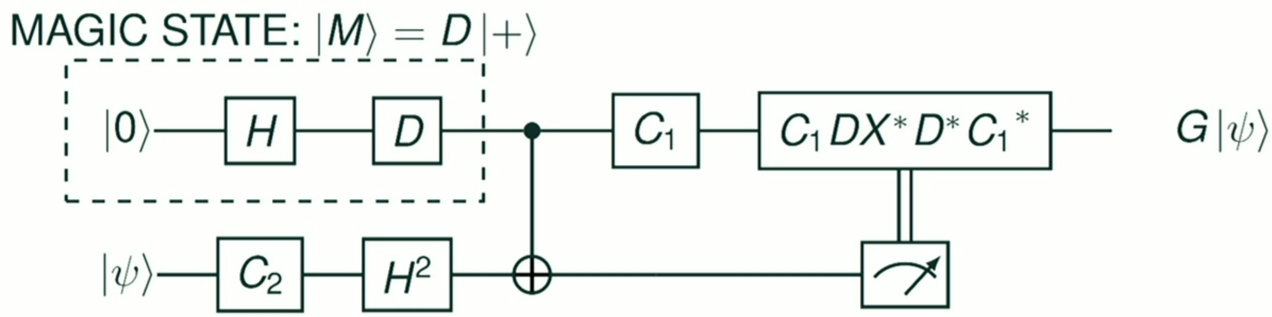
The Clifford hierarchy is defined inductively:
$\mathcal{C}_k = \{G \mid G\mathcal{C}_1 G^* \subseteq \mathcal{C}_{k-1}\}$.

The levels form a nested sequence of sets: $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \ldots$

Compact gate teleportation

The semi-Clifford gates are 'diagonal up to Clifford':

$G = C_1 D C_2$ for $C_1, C_2 \in \mathcal{C}_2$, D is diagonal.



Recap

The Pauli group $\mathcal{C}_1$ of gates are built from Z, X.

The Clifford group $\mathcal{C}_2$ of gates can be *easily* performed fault-tolerantly. These aren't enough!

Third-level (and higher) gates can be performed fault-tolerantly with access to suitable magic states.

Semi-Clifford gates can be performed with efficient magic states.

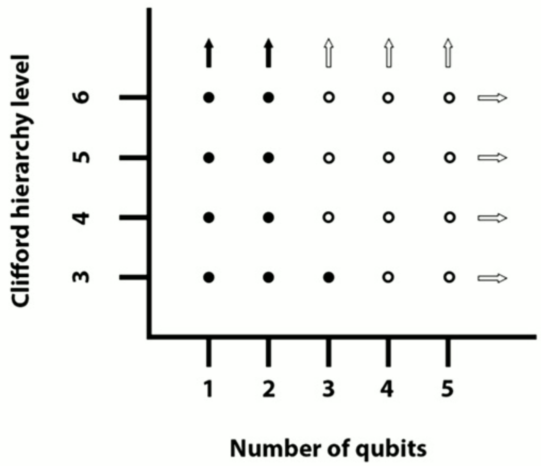
Question 1: What are the gates of the Clifford hierarchy?

Question 2: Which hierarchy gates are semi-Clifford?

Known results: qubit case

For one- or two-qubit gates, all gates of the Clifford hierarchy are semi-Clifford (Zeng-Chen-Chuang, 2008).

For $n > 2, k > 3$ or $n > 3$, *not all* k-th level gates are semi-Clifford. (Beigi-Shor and Gottesman-Mochon, 2009)



The Stone-von Neumann theorem

The Stone-von Neumann theorem (1931): unifies the matrix and wave mechanics pictures of quantum theory.

Roughly: any two representations of the CCRs are unitarily equivalent.

Perfectly suited to systematically studying the Clifford hierarchy (2020).

The discrete Stone-von Neumann theorem

Theorem.
Suppose U, V satisfy the CCRs:

1. $U^d = \mathbb{I}$ *and* $V^d = \mathbb{I}$,
2. $UV = \omega VU$.

There is a gate G such that $U = GZG^*$ and $V = GXG^*$.

A bijection between unitaries G (mod phase) and pairs (U, V).

Theorem (–, 2020).
G is k-th level $\iff$ U and V generate $(k-1)$-th level gates.

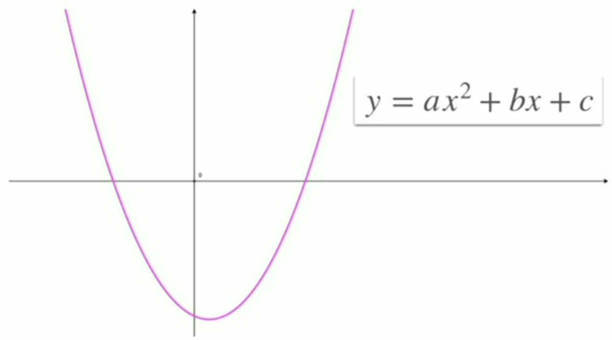
Some new results (–, 2020)

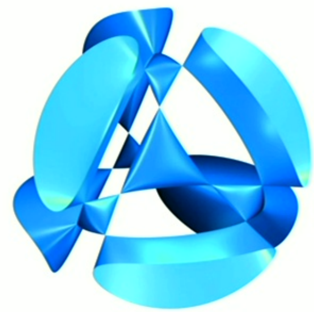
A mathematical framework for studying the Clifford hierarchy via the Stone-von Neumann theorem.

- Generating the Clifford hierarchy
- Recognising and diagonalising semi-Clifford gates.
- All third-level gates of one qudit (any prime d) are semi-Clifford.
- All third-level gates of two qutrits ($d = 3$) are semi-Clifford.

Algebraic geometry

Study algebra via geometry and vice versa.





Understanding the space of solutions to a family of polynomial equations requires algebra ("equations") and geometry ("space").

Hilbert's Nullstellensatz: radical ideals $\iff$ algebraic sets

Semi-Clifford gates via algebraic sets

Main result

Theorem (Chen, –).
For any odd prime dimension d, every two-qudit third-level gate is semi-Clifford.

Proof strategy:

- characterise third-level gates and semi-Clifford third-level gates using polynomial equations over $\mathbb{Z}_d$
- show that, for each d, the two sets of solution coincide
- treating all d with one calculation requires invoking Grothendieck's theory of schemes

Step 1: Third-level gates as an algebraic set

Suppose G is a third-level gate: $G\mathcal{C}_1 G^* \subset \mathcal{C}_2$.

Then $G(Z \otimes \mathbb{I})G^*$, $G(X \otimes \mathbb{I})G^*$, $G(\mathbb{I} \otimes Z)G^*$, $G(\mathbb{I} \otimes X)G^* \in \mathcal{C}_2$.

WLOG each such Clifford gate is of the simplified form:

$$\omega^c D_\Phi Z^{\vec{p}} X^{\vec{q}}.$$

Φ is a 2×2 symmetric matrix over $\mathbb{Z}_d$.
D_Φ is a $d^2 \times d^2$ diagonal gate.

Why? Every d-Sylow subgroup of $Sp(n, \mathbb{Z}_d)$ contains a *unique* maximal abelian subgroup (Barry, 1979).

Step 1: Third-level gates as an algebraic set

Each (simplified) third-level gate corresponds to a solution to the following set of polynomial equations for $1 \leq i < j \leq 4$:

$$\Phi_i \vec{q}_j = \Phi_j \vec{q}_i$$

$$\vec{q}_i^{\,t} \Phi_j \vec{q}_i - \vec{q}_j^{\,t} \Phi_i \vec{q}_j + \vec{p}_i \cdot \vec{q}_j - \vec{p}_j \cdot \vec{q}_i = c_{ij}$$

where

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \{(1,2),(3,4)\} \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Semi-Clifford gates as an algebraic set

WLOG $\Phi_1 = 0$.

A (simplified) third-level gate is semi-Clifford if and only if its corresponding solution also satisfies:

$$\Phi_{31}\Phi_{42} - \Phi_{32}\Phi_{41} = 0$$
$$\Phi_{31}\Phi_{43} - \Phi_{33}\Phi_{41} = 0$$
$$\Phi_{32}\Phi_{43} - \Phi_{33}\Phi_{42} = 0$$

Step 3: Checking equivalence

For a fixed d, the two radical ideals corresponding to these two algebraic sets are the same.

Construct two corresponding schemes and reduce modulo d to check that their algebraic sets of $\mathbb{Z}_d$-rational points coincide.

In principle, this can be checked algorithmically by a computer algebra system: `Magma`.

In practice, the equations are far too complex and involve too many variables.

Simplifications

$$\begin{pmatrix} \vec{q_2}^{\,t} & -\vec{q_1}^{\,t} & 0 & 0 \\ \vec{q_3}^{\,t} & 0 & -\vec{q_1}^{\,t} & 0 \\ \vec{q_4}^{\,t} & 0 & 0 & -\vec{q_1}^{\,t} \\ 0 & \vec{q_3}^{\,t} & -\vec{q_2}^{\,t} & 0 \\ 0 & \vec{q_4}^{\,t} & 0 & -\vec{q_2}^{\,t} \\ 0 & 0 & \vec{q_4}^{\,t} & -\vec{q_3}^{\,t} \end{pmatrix} \begin{pmatrix} \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \end{pmatrix} = \begin{pmatrix} \vec{q_2}^{\,t}\,\Phi_1\,\vec{q_2} - \vec{q_1}^{\,t}\,\Phi_2\,\vec{q_1} + 1 \\ \vec{q_3}^{\,t}\,\Phi_1\,\vec{q_3} - \vec{q_1}^{\,t}\,\Phi_3\,\vec{q_1} \\ \vec{q_4}^{\,t}\,\Phi_1\,\vec{q_4} - \vec{q_1}^{\,t}\,\Phi_4\,\vec{q_1} \\ \vec{q_3}^{\,t}\,\Phi_2\,\vec{q_3} - \vec{q_2}^{\,t}\,\Phi_3\,\vec{q_2} \\ \vec{q_4}^{\,t}\,\Phi_2\,\vec{q_4} - \vec{q_2}^{\,t}\,\Phi_4\,\vec{q_2} \\ \vec{q_4}^{\,t}\,\Phi_3\,\vec{q_4} - \vec{q_3}^{\,t}\,\Phi_4\,\vec{q_3} + 1 \end{pmatrix}$$

Replace satisfaction of this system with consistency.

Eliminate $\vec{p_i}$ variables at the cost of enlarging our algebraic sets.

We show that the new components added to our schemes are extraneous: they do not correspond to actual gates. □

Beyond the third level

Use a purely algebraic approach.

Theorem (–, Lautsch).
Any single-qudit Clifford gate C is expressed uniquely as

$$C = DPM$$

where $D \in \mathcal{D}_2, P$ is a Clifford permutation, and either $M = I$ or $M = HE^c$ for some $c \in \mathbb{Z}_d$ and a fixed $E \in \mathcal{D}_2$.

Theorem (–, Lautsch).
Single-qutrit gates $(n = 1, d = 3)$ of any level of the Clifford hierarchy are semi-Clifford.

Canonical anticommutation relations

Noninteracting fermions are modelled by the canonical anticommutation relations: for $\mu, \nu \in \{1, \ldots, 2n\}$,

$$\{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{I} \qquad \text{where} \qquad \{c_\mu, c_\nu\} := c_\mu c_\nu + c_\nu c_\mu.$$

This leads to an alternative factorisation of quantum gates into 'easy' and 'hard' gates: the matchgate formalism (Valiant, 2001).

With Bampounis and Barbosa, we define an analogous matchgate hierarchy that yields new families of magic states and gate teleportation protocols.

Significance

More efficient paths to achieving universal quantum computation.

Deeper structural understanding of magic states relevant to the theoretical study of quantum advantage.

First application of a key set of mathematical tools within quantum information.

A corollary

The qudit dimension $d \not\equiv 1 \pmod 3$ is prime.

Theorem (–, 2018).
Resourcefully optimal two-qudit magic states $|\Phi\rangle$ are strongly contextual with respect to stabiliser measurements.