Title: Generalized contextuality as a necessary resource for universal quantum computation

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Collection: Foundations of Quantum Computational Advantage

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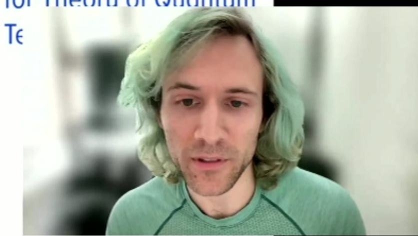
Abstract: A universal and well-motivated notion of classicality for an operational theory is explainability by a generalized-noncontextual ontological model. I will here explain what notion of classicality this implies within the framework of generalized probabilistic theories. I then prove that for any locally tomographic theory, every such classical model is given by a complete frame representation. Using this powerful constraint on the space of possible classical representations, I will then prove that the stabilizer subtheory has a unique classical representation--namely Gross's discrete Wigner function. This provides deep insights into the relevance of Gross's representation within quantum computation. It also implies that generalized contextuality is also a necessary resource for universal quantum computation in the state injection model.

Generalized contextuality as a necessary resource for universal quantum computing



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A puzzle

Gross's discrete Wigner representation has been very useful in studying quantum computation

-Gottesman-Knill theorem extends to all processes represented nonnegatively in Gross's representation

-every state useful for magic state distillation has negativity in Gross's representation

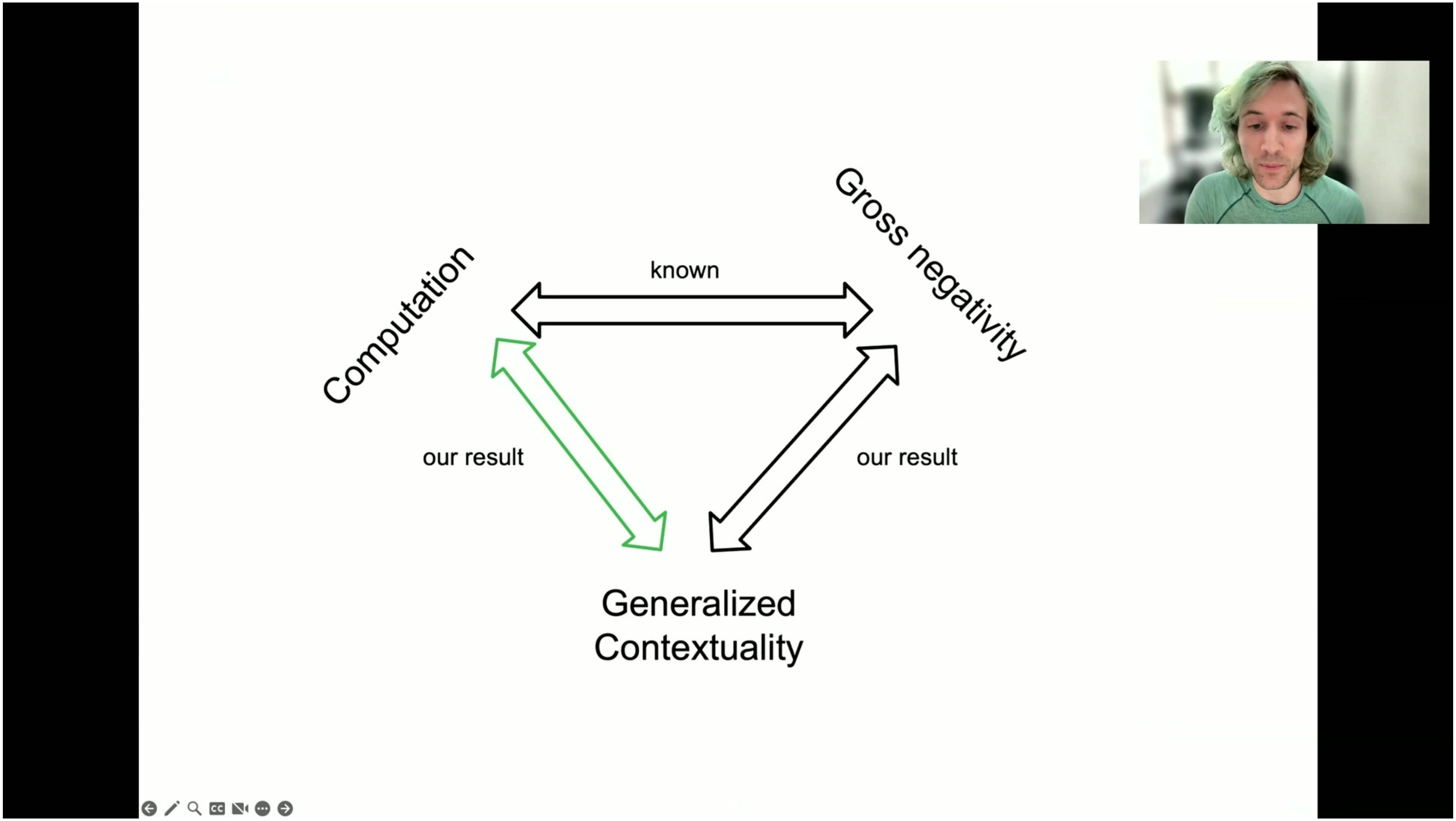
-every state that promotes the stabilizer subtheory to universal quantum computation via magic state distillation must be Kochen-Specker contextual

-negativity in Gross's repn is a necessary resource for computation

But negativity in one particular quasiprobability repn is *not* generally sufficient to establish nonclassicality

Why would negativity in one particular repn (Gross's) be associated to a strong form of nonclassicality (UQC)?

Because Gross's is the unique classical (noncontextual) representation!


Computation
known
Gross negativity
our result
our result
Generalized Contextuality

Process Theory G

collection of processes (on some systems) which is closed under composition

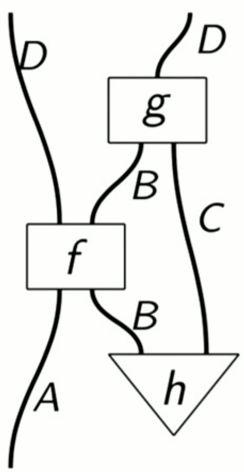
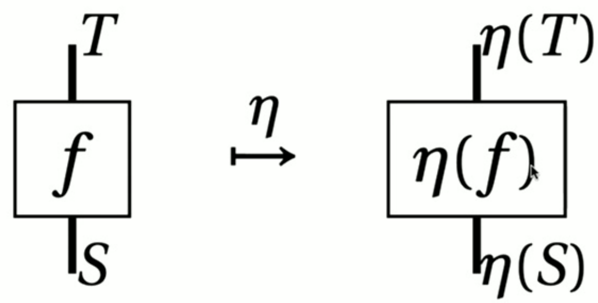


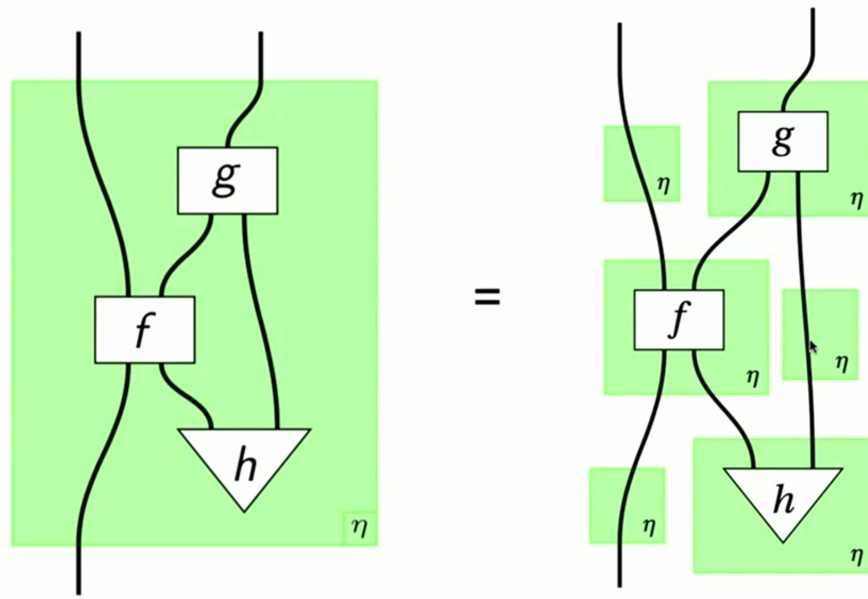
Diagram Preserving m

η : G → G′

takes processes from one th processes of another



commutes with composition


g
f
h
η
=
g
η
η
f
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h
η

Quantum theory as a process theory

systems are vector spaces of Hermitian operators on Hilbe

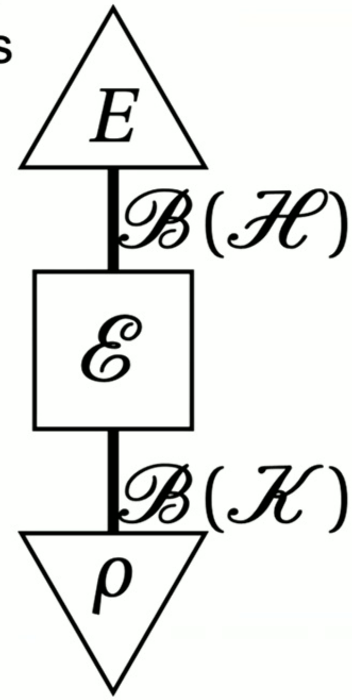
processes are channels

—processes with no inputs are density operators

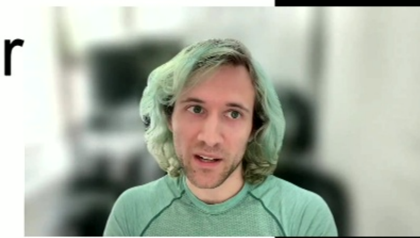
—processes with no outputs are POVM elements

—composition of channels defined as usual

$$\mathrm{Tr}[E\mathcal{E}(\rho)] \quad =$$



Substochastic matrices as a process theor



systems are sets

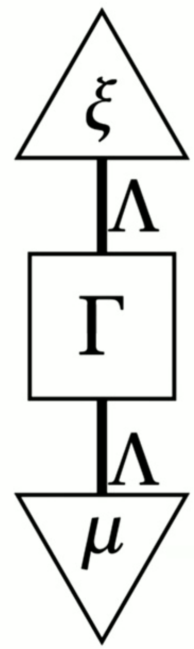
processes are substochastic maps

—processes with no inputs are sub-norm prob distributions

—processes with no outputs are response functions

—composition is matrix multiplication

$$\sum_{\lambda,\lambda'} \xi(\lambda')\Gamma(\lambda'|\lambda)\mu(\lambda) \quad =$$



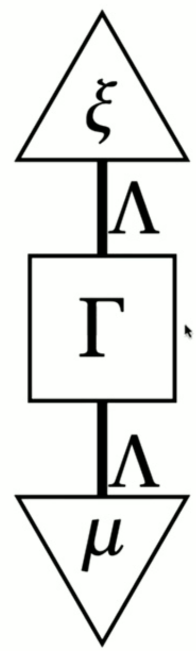
e.g. Louivillian mechanics

QuasiSubstochastic matrices as a process

systems are sets

processes are quasisubstochastic maps

$$\sum_{\lambda,\lambda'} \xi(\lambda')\Gamma(\lambda'|\lambda)\mu(\lambda) \quad =$$



can go negative!

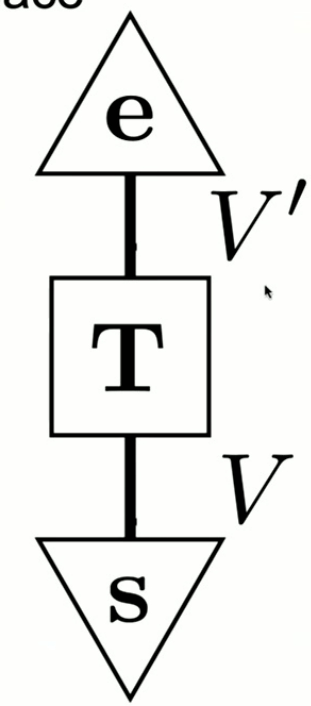
a GPT as a process theory

systems are real vector spaces

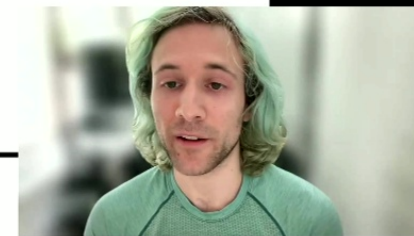
processes are linear maps on the vector space

- —processes with no inputs are vectors in the space
- —processes with no outputs are covectors
- —composition of linear maps defined as usual

$$\mathbf{e}\big(\mathbf{T}(\mathbf{s})\big) =$$

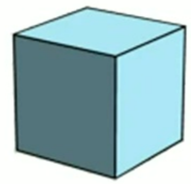


Prepare-measure intuition for a GPT



state space

effect space



inner products give Pr(effect|state)

-observable statistics (and hence the theory) are given by the geometry

-two GPT processes are distinct IFF they give distinct probabilities in at least one circuit

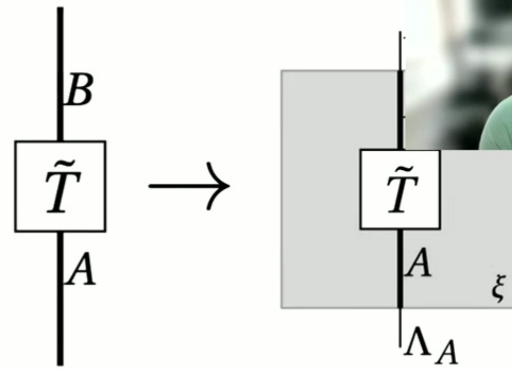
Representations of GPTs

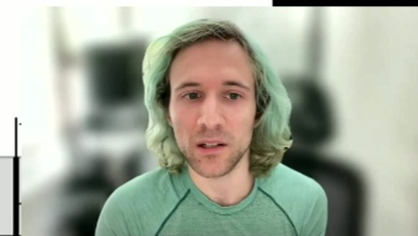
Quantum 8, 1283

Ontological model of a GPT

Linear, diagram-preserving map

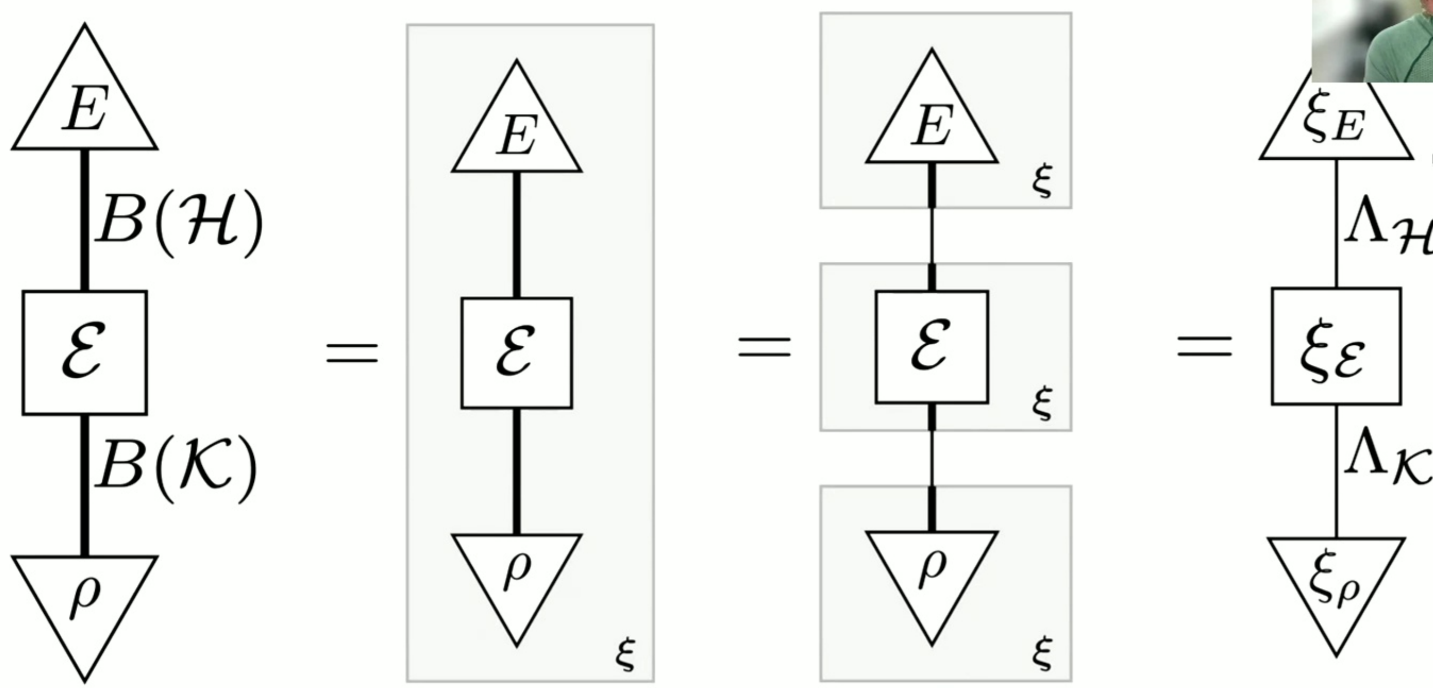
$$\xi : \mathbf{GPT} \to \mathbf{SubStoch}$$





which: 1) reproduces the predictions 2) represents ignoring appropriately

Example:


E
$B(\mathcal{H})$
$\mathcal{E}$
$B(\mathcal{K})$
ρ
=
E
$\mathcal{E}$
ρ
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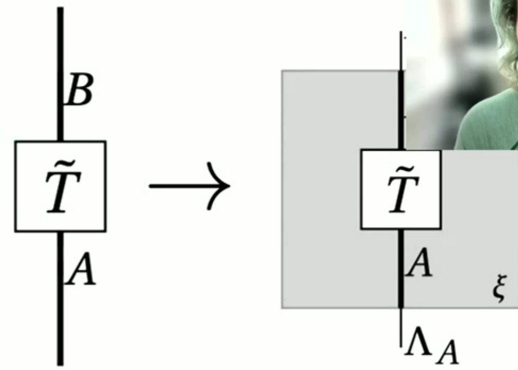
E
ξ
$\mathcal{E}$
ξ
ρ
ξ
=
ξ_E
$\Lambda_{\mathcal{H}}$
$\xi_{\mathcal{E}}$
$\Lambda_{\mathcal{K}}$
ξ_ρ

Quasiprobability repn

~~Ontological model~~ of a GPT

Linear, diagram-preserving map

$$\xi : \mathbf{GPT} \to \cancel{\mathbf{SubStoch}}$$
$$\mathbf{QuasiSubStoch}$$



which: 1) reproduces the predictions 2) represents ignoring appropriately

Clearly, a *positive* quasiprobability representation is just an OM!

Classicality = generalized noncontextuality

-motivated by Leibniz's principle

-equivalent to quantum optics notion of classicality

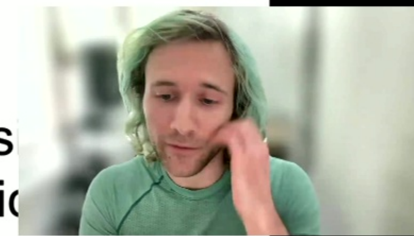
-equivalent to GPT notion of classicality

-emerges in quantum Darwinist limit, or under sufficient noise/coarse-graining

-subsumes KS contextuality, Bell nonlocality, and anomalous weak values

-gen. contextuality is a resource for quantum computation, quantum communication, state discrim., cloning, metrology

We say that an (unquotiented) operational theory is class
explainable if and only if it admits of a noncontextual ontologic

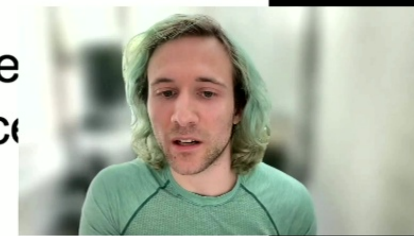


…or equivalently,

if the GPT description of that theory
admits of *any* ontological model

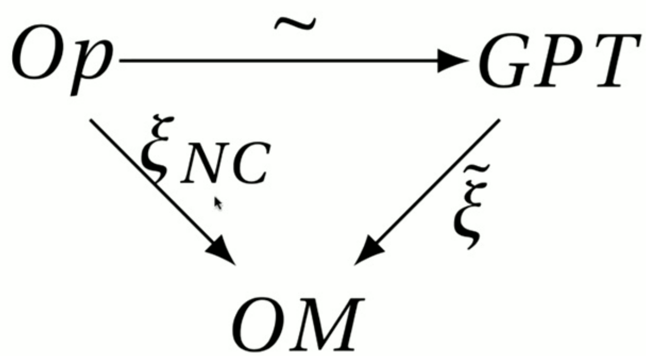
PRX Quantum 2, 010331

Consider an operational theory **Op** and the GPT **G** de
by quotienting it relative to operational equivalenc



Theorem:

There exists a noncontextual ontological model of an operational theory if and only if there exists an ontological model for the GPT associated to it.



Relation to the traditional notion of classicality in GPTs?

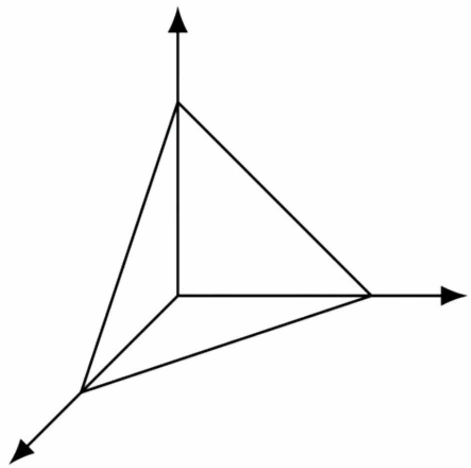
PRX Quantum 2, 010331

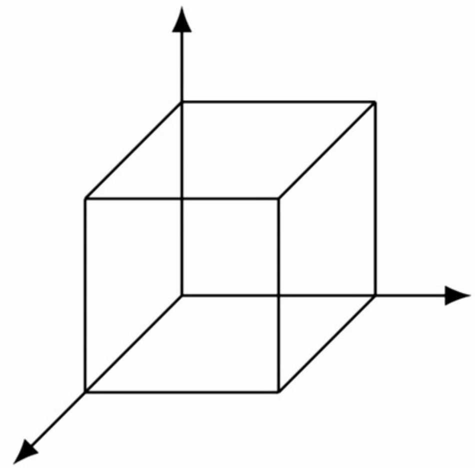
Traditionally, a GPT has been considered classical if it was *simplicial*:

state space: simplex

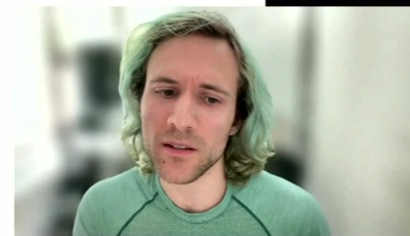
effect space: dual of sin

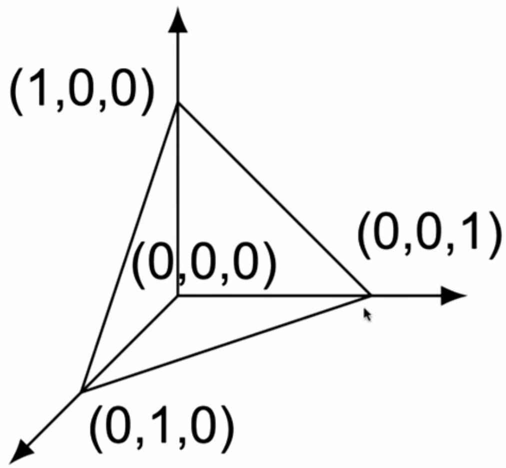
d=3

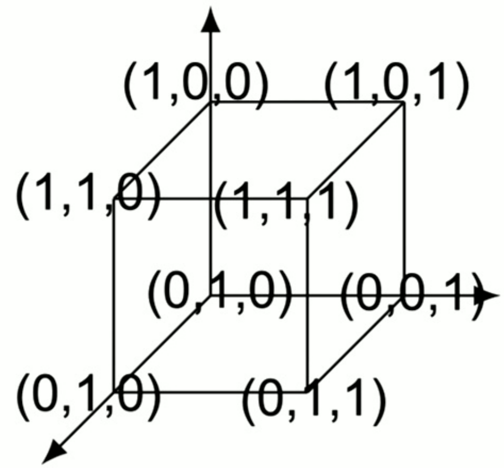




-all states have unique decomposition into pure states
-all measurements are compatible.



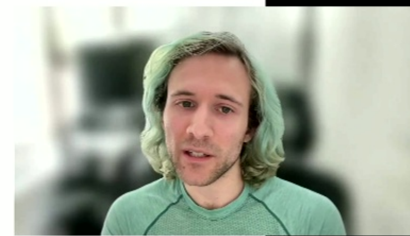




This is just classical probability theory—subStoch!

At this point we have 4 equivalent conditions for a GPT to be classical:

1. the operational theory it came from is noncontextual
2. it admits of an OM
3. it is simplex-embeddable
4. it admits of a positive quasiprobability representation



The structure of (diagram-preserving) quasiprobability representations

Quantum 8, 1283

Structure theorem for DP quasiprobability repns of QT

pick a minimal basis of trace-one Hermitian operators $\{$

compute the unique dual basis satisfying $\mathrm{tr}[D_{\lambda'}F_\lambda] = \delta_{\lambda,\lambda'}$

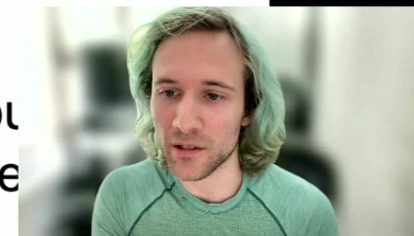
states $\xi_\rho(\lambda) = \mathrm{tr}[D_\lambda \rho]$

effects $\xi_E(\lambda) = \mathrm{tr}[F_\lambda E]$

transformations $\xi_{\mathcal{E}}(\lambda'|\lambda) = \mathrm{tr}[D_{\lambda'}\mathcal{E}(F_\lambda)]$

Extends to all tomographically local GPTs

Every OM of a tomographically local GPT is of analogou
but where everything in the image of the map is positive



Every NCOM of a tomographically local op. theory is of this form, but where one quotients first

$\{F_\lambda\}_\lambda$ is a (non-overcomplete) basis

Powerful tool for studying noncontextuality

Excess baggage theorem ⇒ contextuality

8-state model (Wallman, Bartlett) for stabilizer qubits is contextual

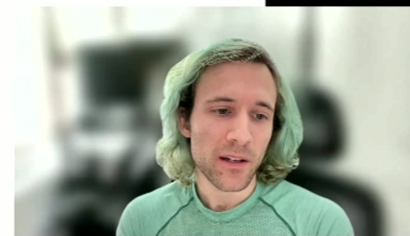
The Stabilizer Subtheory

The stabilizer subtheory can be implemented fault-tolerar

Some nonstabilizer states promote it to universality.

$\Rightarrow$ State injection model for quantum computation.

$$\{|x\rangle\}_{1,2,\ldots,d} \qquad \omega = e^{\frac{2\pi i}{d}}$$

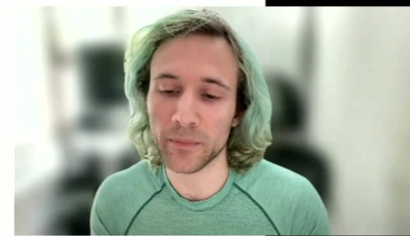


$$X|x\rangle = |x+1\rangle$$ position translator

$$Z|x\rangle = \omega^{x}|x\rangle$$ momentum translator

Weyl operators: $W_{p,q} := Z^{p}X^{q} \qquad p,q \in \mathbb{Z}_d$

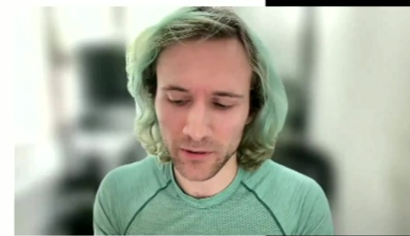
states: eigenstates of these



allowed transformations ("Clifford unitaries")
are those that preserve Weyl operators:

$$U W_{p,q} U^\dagger \propto W_{p',q'}$$

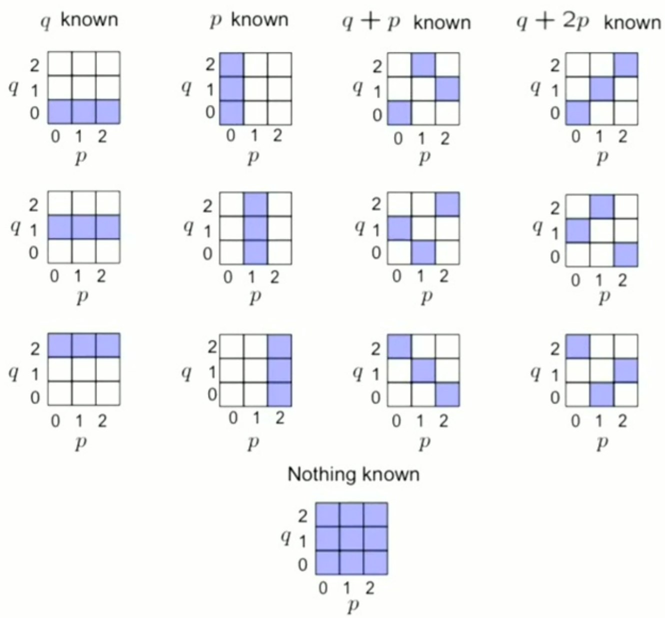
closed under composition

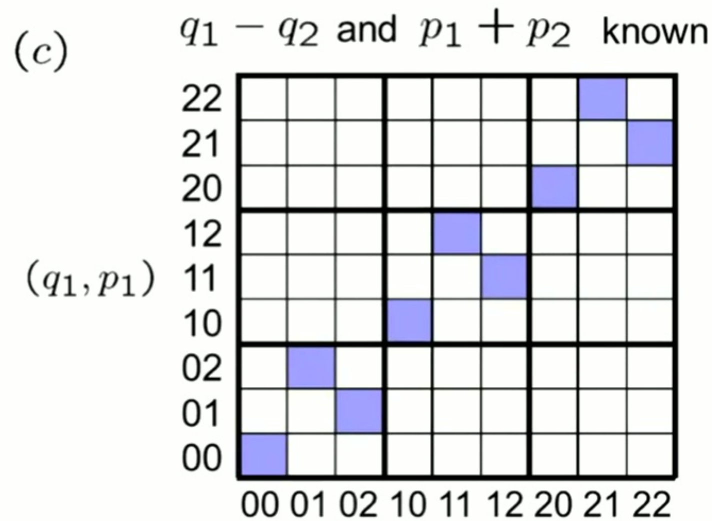


Gross's Representation

repns of stabilizer states

$$\xi_{\rho_{\mathrm{stab}}}(p,q) = \mathrm{tr}[A_{p,q}\rho_{\mathrm{stab}}]$$





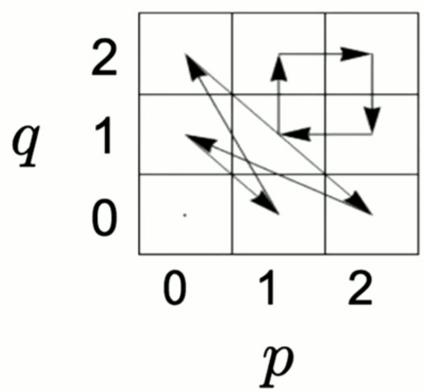
These are all valid probability distributions

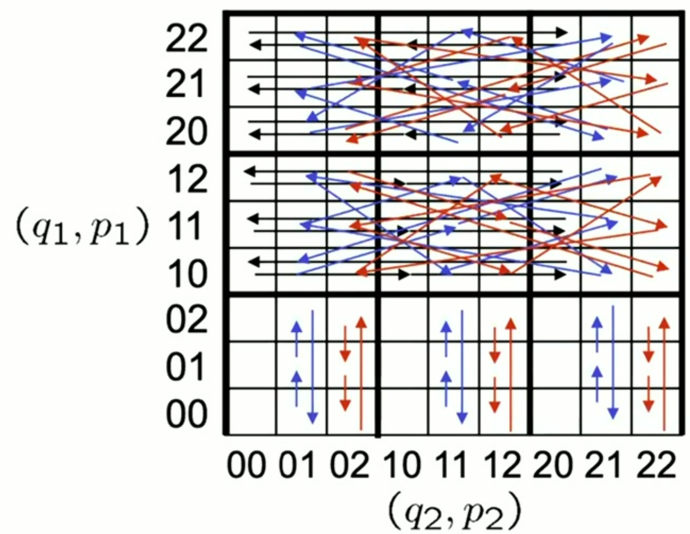
repns of stabilizer unitaries

$$\xi_{U_{\text{stab}}}(p', q' | p, q) = \text{tr}[A_{p',q'} U_{\text{stab}} A_{p,q} U_{\text{stab}}^\dagger]$$

$$\mathrm{q} \mapsto \mathrm{p}$$
$$\mathrm{p} \mapsto -\mathrm{q}$$

$$\mathrm{q}_1 \mapsto \mathrm{q}_1$$
$$\mathrm{p}_1 \mapsto \mathrm{p}_1 - \mathrm{p}_2$$
$$\mathrm{q}_2 \mapsto \mathrm{q}_1 + \mathrm{q}_2$$
$$\mathrm{p}_2 \mapsto \mathrm{p}_2$$





arXiv:2101.06263

Using this structure theorem for quasiprobability representat

In **odd** dimensions, every stabilizer subtheory has a *unique* positive quasiprobability representation, namely Gross's. (Equivalently, there is a unique NCOM.)

Gross's repn $\Leftrightarrow$ Spekkens' toy theory (odd dim)

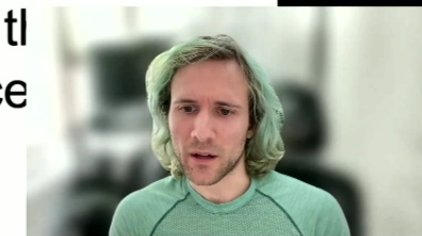
In **even** dimensions, there is no positive quasiprobability representation for any stabilizer subtheory. (Equivalently, there is no NCOM.)

PRL 129 (12), 120403

So there is a *unique* classical explanation of the stabilizer subtheory!

Against the backdrop of the odd dimensional stabilizer subtheory, negativity of a process *in Gross's repn* implies nonclassicality

Gross proved that his repn was the only one among tl
family of “GHW repns” satisfying Clifford covariance



advantages of our uniqueness result over Gross’s:

-generalized noncontextuality is a notion of classicality (covariance is not)

-Gross’s result requires two ad hoc mathematical assumptions (d x d phase space, correct marginal probabilities)

-Gross’s uniqueness result holds only for odd prime, ours holds for all odd dimensions

(similarly for Zhu’s results)

Any state which promotes the stabilizer subtheory to UQC must have negativity in its Gross repn.

Negativity in Gross repn of a state ⇒ KS contextual

Howard et. al. (Nature 2014)

Negativity in Gross repn of a state ⇒ generalized contextuality

our result

So generalized contextuality (like KS contextuality) is necessary for UQC in this model

Is generalized contextuality sufficient for computatio

-not without caveats, at least
(even-dimensional stabilizer subtheories are efficiently simulable)

…but maybe if we consider *quantifying* contextuality?
…scaling of contextuality?
...?

Thank you!

The stabilizer subtheory has a unique noncontextual model
PRL 129 (12), 120403

A structure theorem for generalized-noncontextual ontological models
Quantum 8, 1283