Title: Stabilizer operators and Barnes-Wall lattices

Speakers: Vadym Kliuchnikov

Collection: Foundations of Quantum Computational Advantage

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Abstract: We give a simple description of rectangular matrices that can be implemented by a post-selected stabilizer circuit. Given a matrix with entries in dyadic cyclotomic number fields $\mathbb{Q}(\exp(i\frac{2\pi}{2^m}))$, we show that it can be implemented by a post-selected stabilizer circuit if it has entries in $\mathbb{Z}[\exp(i\frac{2\pi}{2^m})]$ when expressed in a certain non-orthogonal basis. This basis is related to Barnes-Wall lattices. Our result is a generalization to a well-known connection between Clifford groups and Barnes-Wall lattices. We also show that minimal vectors of Barnes-Wall lattices are stabilizer states, which may be of independent interest. Finally, we provide a few examples of generalizations beyond standard Clifford groups.

Joint work with Sebastian Schonnenbeck

Stabilizer operators and Barnes-Wall lattices

Wednesday, May 1, 2024

Vadym Kliuchnikov[1], Sebastian Schönnenbeck[2]

[arXiv:2404.17677]

Foundations of Quantum Computational Advantage
https://events.perimeterinstitute.ca/event/71/overview

[1] Microsoft Quantum
[2] RWTH Aachen University

Motivation

$$\mathbb{Z}\left[\frac{1}{\sqrt{2}}, i\right] = \left\{\frac{1}{2^n}\left(a + bi + c\sqrt{2} + di\sqrt{2}\right): a, b, c, d \in \mathbb{Z}\right\}$$

[arXiv:1212.0506] Giles, Selinger

Corollary 2. *Let U be a unitary $2^n \times 2^n$ matrix. Then the following are equivalent:*

(a) U can be exactly represented by a quantum circuit over the Clifford+T gate set on n qubits with no ancillas.

(b) The entries of U belong to the ring $\mathbb{Z}[\frac{1}{\sqrt{2}}, i]$, and:

- $\det U = 1$, *if* $n \geq 4$;
- $\det U \in \{-1, 1\}$, *if* $n = 3$;

When unitary *U*
is a Clifford unitary ?

[arXiv:1908.06076] Amy, Glaudell, Ross : $\mathbb{Z}\left[\frac{1}{2}, i\right], \mathbb{Z}\left[\frac{1}{2}\right], \mathbb{Z}\left[\frac{1}{\sqrt{2}}\right], \mathbb{Z}\left[\frac{1}{i\sqrt{2}}\right]$

Background

- Pauli Matrices

  1 qubit: $I, X, Y, Z,$ n qubits: $\{I, X, Y, Z\}^{\otimes n}$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

- Clifford unitaries

  For any n-qubit Pauli matrix P : $CPC^{-1} = \pm Q$ where Q is an n-qubit Pauli matrix

- Example of Clifford unitaries

  $CNOT, S,$ phase-adjusted Hadamard $\widetilde{H}$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \widetilde{H} = \frac{1}{1+i}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Why the separation of Clifford vs non-Clifford gates is interesting ?

- There is more and more evidence that we need fault-tolerant quantum computers to solve useful problems on quantum computers
- Non-Clifford resources (T gates, CCZ gates) are like a fuel that fault-tolerant quantum computers use

[arXiv:2211.07629]

Assessing requirements to scale to practical quantum advantage

Michael E. Beverland, Prakash Murali, Matthias Troyer, Krysta M. Svore, Torsten Hoefler, Vadym Kliuchnikov, Guang Hao Low, Mathias Soeken, Aarthi Sundaram, Alexander Vaschillo

[arXiv:2401.16317]

Assessing the Benefits and Risks of Quantum Computers

Travis L. Scholten, Carl J. Williams, Dustin Moody, Michele Mosca, William Hurley ("whurley"), William J. Zeng, Matthias Troyer, Jay M. Gambetta

It is useful to know the number of non-Clifford gates needed to solve a problem

- Allows for a simple lower-bound on the time to solve a problem instance
- Space-time volume (#physical qubits x time) per T/CCZ state is a good rough metric for comparing different QC architecture

We need to better understand various aspects of stabilizer circuits to minimizes non-Clifford resources needed to solve problems on quantum computers

[arXiv:1904.01124]
Lower bounds on the non-Clifford resources for quantum computations
Michael Beverland, Earl Campbell, Mark Howard, Vadym Kliuchnikov

[arXiv:2403.18900]
Minimal entanglement for injecting diagonal gates
Vadym Kliuchnikov, Eddie Schoute

Background

- How do know that a Clifford unitary is a Clifford unitary ?

(A) Check condition:

$CPC^{-1} = \pm Q$ for $P = X_1, \dots, X_n, Z_1, \dots, Z_n$

(B) Use a more efficient algorithm

Fast algorithms for classical specifications of stabiliser states and Clifford gates

Nadish de Silva, Wilfred Salmon, Ming Yin

[arXiv:2311.10357]

Background

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Fast algorithms for classical specifications of stabiliser states and Clifford gates

Nadish de Silva, Wilfred Salmon, Ming Yin

[arXiv:2311.10357]

(C) Can we recognize a Clifford unitary by looking at matrix entries ?

Corollary 2. *Let U be a unitary $2^n \times 2^n$ matrix. Then the following are equivalent:*

(a) U can be exactly represented by a quantum circuit over the Clifford+T gate set on n qubits with no ancillas.

(b) The entries of U belong to the ring $\mathbb{Z}[\frac{1}{\sqrt{2}}, i]$, and:

- $\det U = 1$, *if* $n \geq 4$;
- $\det U \in \{-1, 1\}$, *if* $n = 3$;

A special case of the main result

A $2^n \times 2^n$ unitary U with entries in $\mathbb{Z}\left[i, \frac{1}{2}\right] = \left\{\frac{a+bi}{2^n} : a, b \in \mathbb{Z}\right\}$ is a Clifford unitary

if and only if

$\left(B^{\otimes n}\right)^{-1} U(B^{\otimes n})$ has entries in $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$

where $B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$

Where did $\sqrt{2}$ go ?

$\widetilde{H} = \frac{1}{1+i}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Remark: for n = 1, no basis change is needed Theorem 5.1 in [arXiv:1501.04944]

A special case of the main result

A $2^n \times 2^n$ unitary U with entries in $\mathbb{Q}(i) = \{a + ib: a, b \in \mathbb{Q}\}$ is a Clifford unitary

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where $B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$

An application example

- How "non-Clifford" is unitary U with entries in $\mathbb{Z}\left[i, \frac{1}{2}\right] = \left\{\frac{a+bi}{2^n} : a, b \in \mathbb{Z}\right\} = \left\{\frac{a+bi}{(1+i)^n} : a, b \in \mathbb{Z}\right\}$?

- Maximum power of $(1+i)^n$ in the denominator of entries of $\left(B^{\otimes n}\right)^{-1} U (B^{\otimes n})$ where $B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$

- Similar to dyadic monotone in

Lower bounds on the non-Clifford resources for quantum computations

Michael Beverland, Earl Campbell, Mark Howard, Vadym Kliuchnikov

Generalizations

1. Entries in $\mathbb{Q}(i) = \{a + ib : a, b \in \mathbb{Q}\}$

   ⇓

   Entries in $\mathbb{Q}(\zeta_{2^m}) = \left\{\sum_{j=1}^{2^{m-1}} a_j \, \zeta_{2^m}^j : a_j \in \mathbb{Q}\right\}, \zeta_{2^m} = e^{\frac{2\pi i}{2^m}}$

2. $2^n \times 2^n$ Clifford unitaries

   ⇓

   $2^{n'} \times 2^n$ Linear operators implemented by a post-selected stabilizer circuit

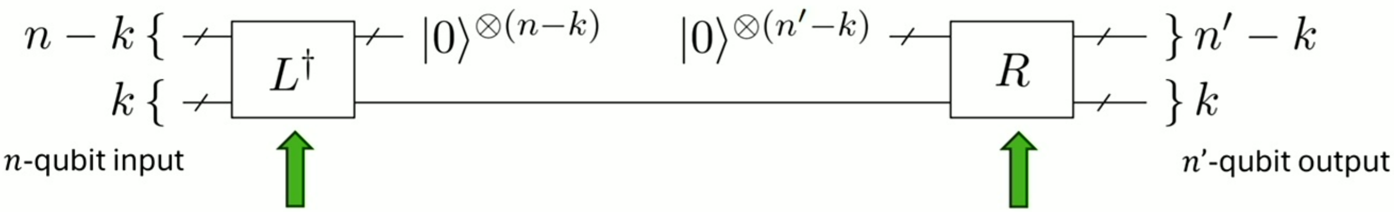
3. Partial results for rebits, qudits with d=3,5 [computational]
4. Other fields beyond $\mathbb{Q}(\zeta_{2^m})$ [computational]

Post-selected stabilizer circuit:

- Allocation of zero states,
- Clifford unitaries,
- post-selected measurement of Pauli observables,
- post-selected destructive Pauli Z measurement

Post-selected stabilizer circuits

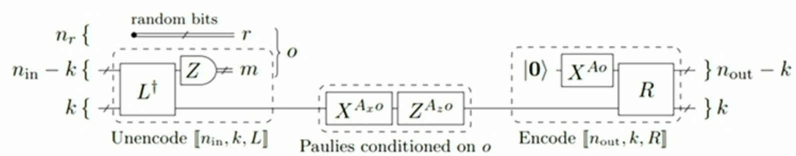
- General form of a post-selected stabilizer circuit



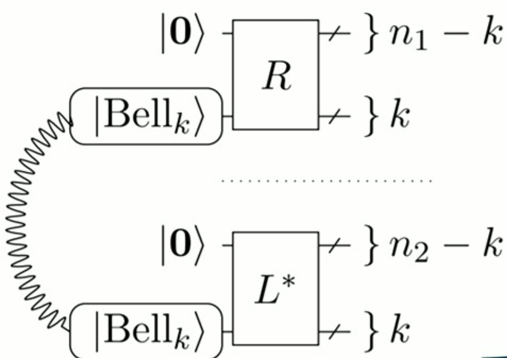
Generalizes to certain class of stabilizer circuits, see [arXiv:2309.08676]

Stabilizer circuit verification

Vadym Kliuchnikov, Michael Beverland, Adam Paetznick



Entanglement structure of stabilizer states



Stabilizer states

Entanglement on mixed stabilizer states: normal forms and reduction procedures

Koenraad M R Audenaert[1] and Martin B Plenio[1]

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New Journal of Physics, Volume 7, January 2005

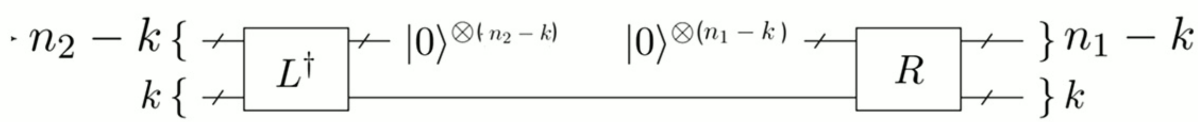
Citation Koenraad M R Audenaert and Martin B Plenio 2005 *New J. Phys.* **7** 170

DOI 10.1088/1367-2630/7/1/170

$$\sum_{j\in\{0,1\}^n} |j\rangle \otimes B|j\rangle \longleftrightarrow B$$

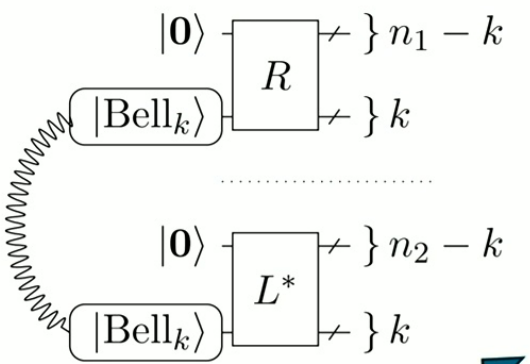
Operator-vector correspondence

(Choi–Jamiołkowski isomorphism)



Stabilizer operators

Entanglement structure of stabilizer states



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$$B$$

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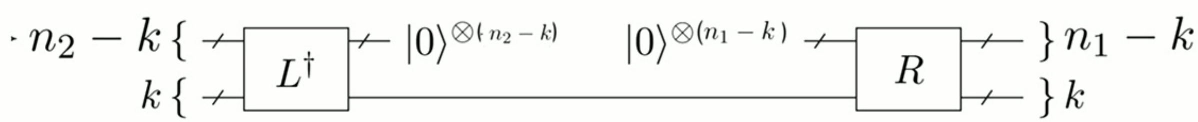

Stabilizer operators

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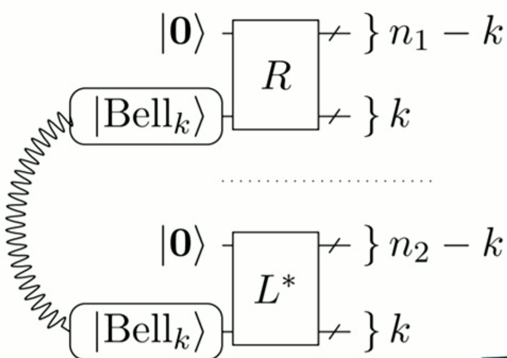

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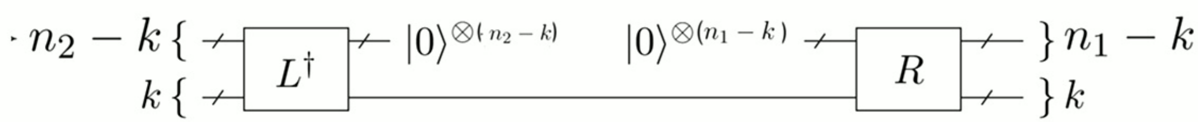


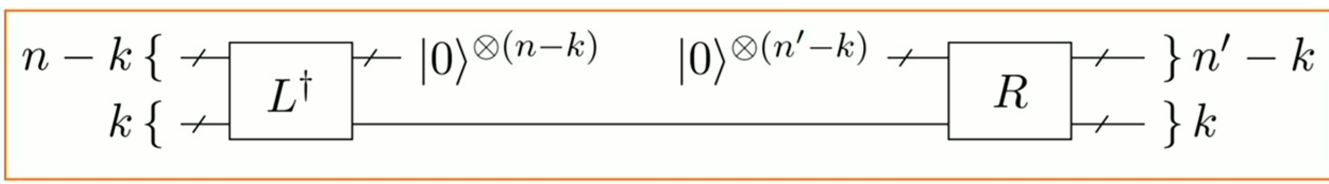
Stabilizer operators

The main result

Theorem 4.3 (Basis and trace condition)**.** *Consider a $2^{n'}$ by 2^n matrix A with entries in $E = \mathbb{Q}(\zeta_{2^m})$ $(m \geq 2)$ such that $\mathrm{Tr}(A^\dagger A) = 2^{n'}$ and such that the matrix $((B^{-1})^{\otimes n'})A(B^{\otimes n})$ has entries in $\mathcal{O}_E$, the ring of integers of E. There exist unitaries L, R from the n'-qubit, n-qubit Clifford group, an integer $k \leq \min(n, n')$ and an integer j such that*

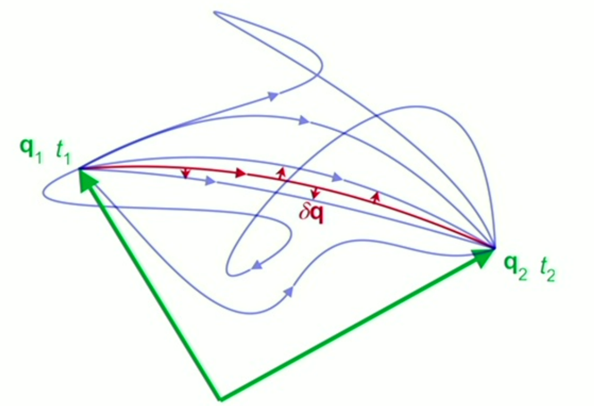
$$A = \zeta_{2^m}^j (1+i)^{n'-k} R \cdot \left(|0\rangle^{\otimes(n'-k)} \otimes I_{2^k}\right) \cdot \left(\langle 0|^{\otimes(n-k)} \otimes I_{2^k}\right) \cdot L^\dagger,$$

that is A is a linear operator that can be implemented by a post-selected stabilizer circuit When A is an isometry, $k = n$ and no post-selection is necessary.



Key technical tool

Theorem 3.4 (Minimal vectors). *The minimal vectors of the n-qubit Barnes-Wall lattice over the cyclotomic field* $E = \mathbb{Q}(\zeta_{2^m})$ $(m \geq 2)$ *are stabilizer states. Furthermore, these minimal vectors can always be represented as* $\zeta_{2^m}^k \cdot (1+i)^n \cdot C|0\rangle^{\otimes n}$ *for some unitary* C *from the Clifford group and some integer* $k \in [2^m]$.



https://commons.wikimedia.org/wiki/File:Least_action_principle.svg

n-qubit **stabilizer state** is a non-zero state that is a ± 1 eigenstate of n independent commuting Pauli matrices

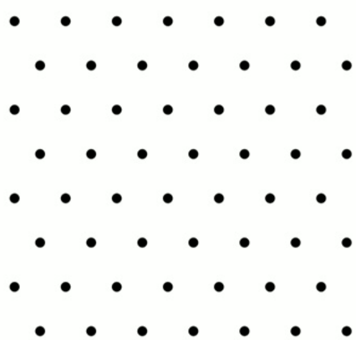
Barnes-Wall lattices

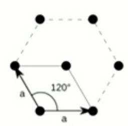
Lattices

https://en.wikipedia.org/wiki/Lattice_(group)

$$\Lambda = \left\{ \sum_{i=1}^{n} a_i v_i \,\middle|\, a_i \in \mathbb{Z} \right\}$$

$\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$





https://commons.wikimedia.org/wiki/File:Equilateral_Triangle_Lattice.svg
https://commons.wikimedia.org/wiki/File:2d_hp.svg

Hermitian Lattices

[Very Informal]
Hermitian lattices are a generalization of lattices where $\mathbb{R}^n$ is replaced with E^n for number field E and $\mathbb{Z}$ is replaced with the ring of integers $\mathcal{O}_E$ of E

$$E = \mathbb{Q}(\zeta_{2^m}) \qquad B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$$

1-qubit Barnes-Wall lattice:

$$B\mathcal{O}_E^2$$

$$\{z_1 \cdot (1+i)|0\rangle + z_2 \cdot (|0\rangle + |1\rangle) : z_1, z_2 \in \mathcal{O}_E\}\,.$$

n-qubit Barnes-Wall lattice:

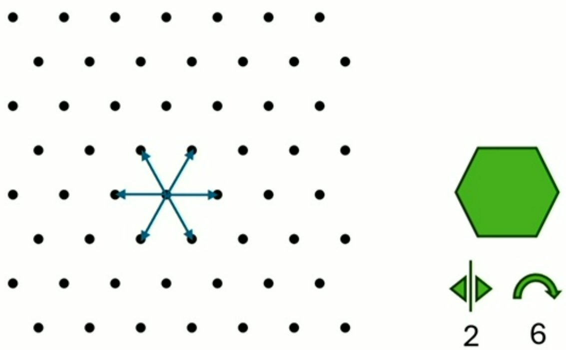
$$B^{\otimes n}\mathcal{O}_E^{2^n}$$

$$|\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle, \text{ where } |\phi_s\rangle \in \{(1+i)|0\rangle, |0\rangle + |1\rangle\}$$

Minimal vectors

Lattices

Minimal vectors are
shortest non-zero vectors



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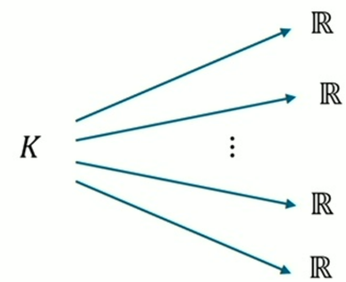
Hermitian Lattices

Measure the length of the vectors using

$\mathrm{Tr}_{K/\mathbb{Q}} \langle\psi|\psi\rangle$

(sum over all embeddings of the real subfield K of E)

$$\min_{|\psi\rangle\in L, |\psi\rangle\neq 0} \mathrm{Tr}_{K/\mathbb{Q}} \langle\psi|\psi\rangle,$$



Automorphism group: orthogonal/unitary matrices that map lattice to itself

Related work on Barnes-Wall lattices

Quantum Error Correction via Codes over GF(4) August 27, 1997 A. R. Calderbank (1), E. M Rains (2), P. W. Shor (1), N. J. A. Sloane (1),	Barnes-Wall lattices and quantum computing
[NRS01] Gabriele Nebe, E. M. Rains, and N. J. A. Sloane. "The Invariants of the Clifford Groups". In: *Designs, Codes and Cryptography* 24 (1 2001), pp. 99–122. ISSN: 09251022. DOI: 10.1023/A:1011233615437.	Tensor product construction $\mathbb{Z}[\sqrt{2}], \mathbb{Z}[\zeta_8]$. Clifford groups are the automorphism groups.
[BE73] Michel Broue and Michel Enguehard. "Une famille infinie de formes quadratiques entières; leurs groupes d'automorphismes". fre. In: *Annales scientifiques de l'Ecole Normale Superieure* 6.1 (1973), pp. 17–51. URL: http://eudml.org/doc/81910.	Transitive action of automorphism group on minimal vectors of classical Barnes-Wall lattices
[CN12] Renaud Coulangeon and Gabriele Nebe. *The unreasonable effectiveness of the tensor product.* 2012. DOI: 10.48550/arXiv.1201.1832. arXiv: 1201.1832 [math.NT].	Tools to extend to other base fields
[Cou00] Renaud Coulangeon. "Tensor products of hermitian lattices". eng. In: *Acta Arithmetica* 92.2 (2000), pp. 115–130. URL: http://eudml.org/doc/207374.	$\mathbb{Z}[i]$ version of Barnes-Wall lattices
[MN08] Daniele Micciancio and Antonio Nicolosi. "Efficient bounded distance decoders for Barnes-Wall lattices". In: IEEE, July 2008, pp. 2484–2488. ISBN: 978-1-4244-2256-2. DOI: 10.1109/ISIT.2008.4595438.	Approximating vectors with vectors in Barnes-Wall lattice, in l_2 norm

Our work: self-contained, proofs from the first principles, using stabilizer formalism

Beyond standard Clifford groups

Theorem A.1. *Suppose that U is an $N \times N$ unitary matrix with entries in field E such that $\tilde{B}^{-1}UB$ has entries in $\mathcal{O}_E$, where $N, E, \tilde{B}$ are given in Table 1. Matrix U is a product of generators of group G times the scalar matrix from C, where G, C are given in corresponding rows of Table A.1. That is U is in G up to a global phase.*

Group G	Dim. N	Basis change $\tilde{B}$	Number field E	Center C
Clifford group (A.1.1)	2	$B_{\mathbb{C},1}$ (10)	$\mathbb{Q}(\zeta_{4m}), m \in [2,8]$	$\zeta_{4m}^j I, j \in [4m]$
	4	$B_{\mathbb{C},2}$ (10)	$\mathbb{Q}(\zeta_{4m}), m \in [2,8]$	$\zeta_{4m}^j I, j \in [4m]$
Real Clifford group (A.1.2)	2	$B_{\mathbb{R},1}$ (11)	$\mathbb{Q}(\zeta_{8m}) \cap \mathbb{R}, m \in [1,4]$	$\pm I$
	4	$B_{\mathbb{R},2}$ (11)	$\mathbb{Q}(\zeta_{8m}) \cap \mathbb{R}, m \in [1,4]$	$\pm I$
	8	$B_{\mathbb{R},3}$ (11)	$\mathbb{Q}(\zeta_8) \cap \mathbb{R} = \mathbb{Q}(\sqrt{2})$	$\pm I$
Rational subgroup of Clifford group (A.1.3)	2	$B_{\mathbb{Q},1}$ (12)	$\mathbb{Q}$	$\pm I$
	4	$B_{\mathbb{Q},2}$ (12)	$\mathbb{Q}$	$\pm I$
Qutrit Clifford group (A.1.4)	3	$B_1^{(3)}$ (13)	$\mathbb{Q}(\zeta_{3m}), m \in [1,9]$	$\pm\zeta_{3m}^j I$
	9	$B_2^{(3)}$ (14)	$\mathbb{Q}(\zeta_{3m}), m \in [1,9]$	$\pm\zeta_{3m}^j I$
Qudit Clifford group, $d = 5$ (A.1.4)	5	$B_1^{(5)}$ (15)	$\mathbb{Q}(\zeta_{5m}), m \in [1,3]$	$\pm\zeta_{5m}^j I$

How to find basis change (a.k.a Hermitian lattice) starting from a finite unitary group G ?

Step 1: Pick number field E so you can write all unitaries as matrices with entries in E (up to global phase)

Step 2: Construct lattice, find its basis **(if it exists)**

$$\mathrm{G}|0\rangle = \{\mathrm{g}|0\rangle : g \in \mathrm{G}\} = \{\mathrm{v}_1, \ldots, v_N\} \quad L = \left\{ \sum_{j \in [N]} a_j v_j : a_j \in O_E \right\}$$

Step 3: Compute automorphism group of the lattice

Step 4: Check if automorphism group is equal to G up to scalars

Beyond standard Clifford groups

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Computational tools

- Basis change/Hermitian lattice can be easily computed with:

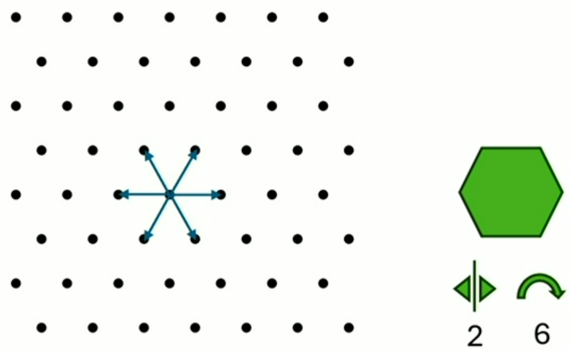
  1. Magma Computational Algebra System
  2. Hecke (Julia package)

- We provide Julia notebook for results in Theorem A.1. using Hecke

Minimal vectors

Lattices

Minimal vectors are
shortest non-zero vectors



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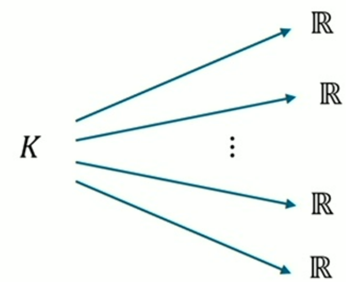


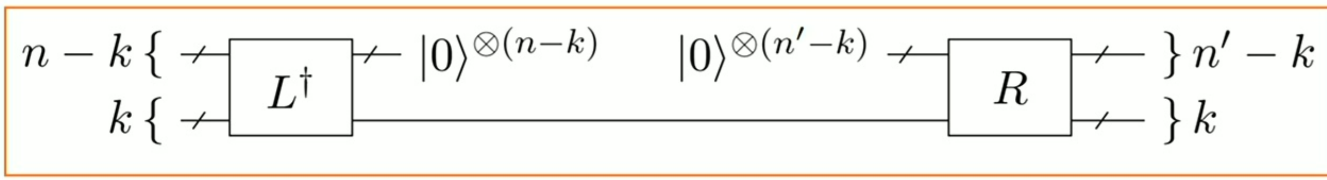
Automorphism group: orthogonal/unitary matrices that map lattice to itself

The main result

Theorem 4.3 (Basis and trace condition)**.** *Consider a* $2^{n'}$ *by* 2^n *matrix* A *with entries in* $E = \mathbb{Q}(\zeta_{2^m})$ $(m \geq 2)$ *such that* $\mathrm{Tr}(A^\dagger A) = 2^{n'}$ *and such that the matrix* $((B^{-1})^{\otimes n'})A(B^{\otimes n})$ *has entries in* $\mathcal{O}_E$*, the ring of integers of* E*. There exist unitaries* L, R *from the* n'*-qubit,* n*-qubit Clifford group, an integer* $k \leq \min(n, n')$ *and an integer* j *such that*

$$A = \zeta_{2^m}^j (1+i)^{n'-k} R \cdot \left(|0\rangle^{\otimes(n'-k)} \otimes I_{2^k}\right) \cdot \left(\langle 0|^{\otimes(n-k)} \otimes I_{2^k}\right) \cdot L^\dagger,$$

that is A *is a linear operator that can be implemented by a post-selected stabilizer circuit When* A *is an isometry,* $k = n$ *and no post-selection is necessary.*



Related work on Barnes-Wall lattices

Quantum Error Correction via Codes over GF(4) August 27, 1997 A. R. Calderbank (1), E. M Rains (2), P. W. Shor (1), N. J. A. Sloane (1),	Barnes-Wall lattices and quantum computing
[NRS01] Gabriele Nebe, E. M. Rains, and N. J. A. Sloane. "The Invariants of the Clifford Groups". In: *Designs, Codes and Cryptography* 24 (1 2001), pp. 99–122. ISSN: 09251022. DOI: 10.1023/A:1011233615437.	Tensor product construction $\mathbb{Z}[\sqrt{2}], \mathbb{Z}[\zeta_8]$. Clifford groups are the automorphism groups.
[BE73] Michel Broue and Michel Enguehard. "Une famille infinie de formes quadratiques entières; leurs groupes d'automorphismes". fre. In: *Annales scientifiques de l'Ecole Normale Superieure* 6.1 (1973), pp. 17–51. URL: `http://eudml.org/doc/81910`.	Transitive action of automorphism group on minimal vectors of classical Barnes-Wall lattices
[CN12] Renaud Coulangeon and Gabriele Nebe. *The unreasonable effectiveness of the tensor product.* 2012. DOI: 10.48550/arXiv.1201.1832. arXiv: `1201.1832 [math.NT]`.	Tools to extend to other base fields
[Cou00] Renaud Coulangeon. "Tensor products of hermitian lattices". eng. In: *Acta Arithmetica* 92.2 (2000), pp. 115–130. URL: `http://eudml.org/doc/207374`.	$\mathbb{Z}[i]$ version of Barnes-Wall lattices
[MN08] Daniele Micciancio and Antonio Nicolosi. "Efficient bounded distance decoders for Barnes-Wall lattices". In: IEEE, July 2008, pp. 2484–2488. ISBN: 978-1-4244-2256-2. DOI: 10.1109/ISIT.2008.4595438.	Approximating vectors with vectors in Barnes-Wall lattice, in l_2 norm

Our work: self-contained, proofs from the first principles, using stabilizer formalism

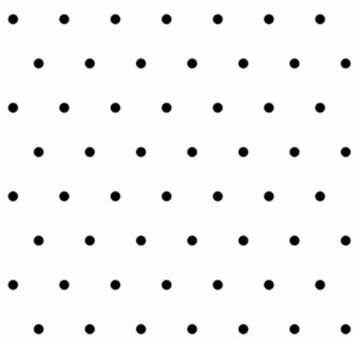
Barnes-Wall lattices

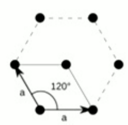
Lattices

https://en.wikipedia.org/wiki/Lattice_(group)

$$\Lambda = \left\{ \sum_{i=1}^{n} a_i v_i \,\middle|\, a_i \in \mathbb{Z} \right\}$$

$\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$





https://commons.wikimedia.org/wiki/File:Equilateral_Triangle_Lattice.svg
https://commons.wikimedia.org/wiki/File:2d_hp.svg

Hermitian Lattices

[Very Informal]
Hermitian lattices are a generalization of lattices where $\mathbb{R}^n$ is replaced with E^n for number field E and $\mathbb{Z}$ is replaced with the ring of integers $\mathcal{O}_E$ of E

$$E = \mathbb{Q}(\zeta_{2^m}) \qquad B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$$

1-qubit Barnes-Wall lattice:

$B\mathcal{O}_E^2$

$$\{z_1 \cdot (1+i)|0\rangle + z_2 \cdot (|0\rangle + |1\rangle) : z_1, z_2 \in \mathcal{O}_E\}.$$

n-qubit Barnes-Wall lattice:

$B^{\otimes n}\mathcal{O}_E^{2^n}$

$$|\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle, \text{ where } |\phi_s\rangle \in \{(1+i)|0\rangle, |0\rangle + |1\rangle\}$$

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An application example

- How "non-Clifford" is unitary U with entries in $\mathbb{Z}\left[i, \frac{1}{2}\right] = \left\{\frac{a+bi}{2^n} : a, b \in \mathbb{Z}\right\} = \left\{\frac{a+bi}{(1+i)^n} : a, b \in \mathbb{Z}\right\}$?

- Maximum power of $(1+i)^n$ in the denominator of entries of $\left(B^{\otimes n}\right)^{-1} U (B^{\otimes n})$ where $B = \begin{pmatrix} 1+i & 1 \\ 0 & 1 \end{pmatrix}$

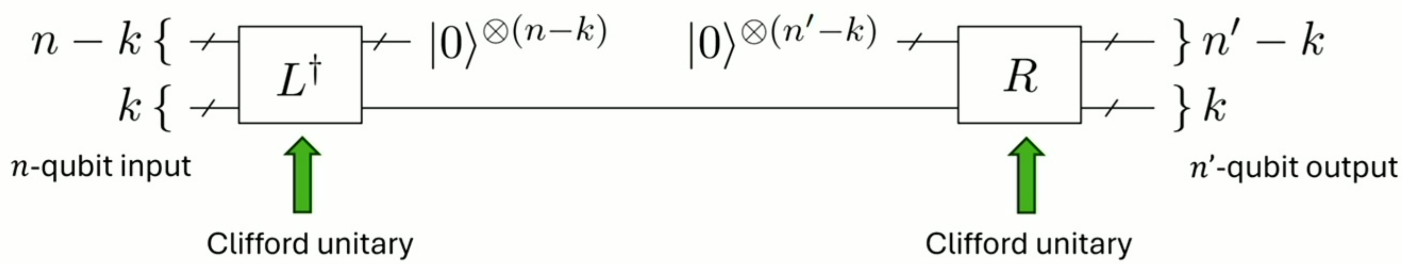
- Similar to dyadic monotone in

Lower bounds on the non-Clifford resources for quantum computations

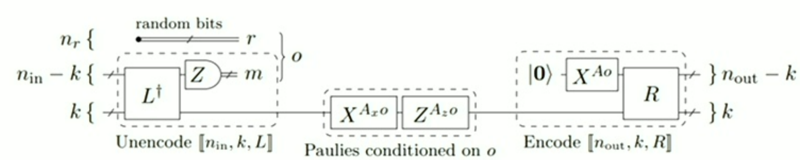
Michael Beverland, Earl Campbell, Mark Howard, Vadym Kliuchnikov

Post-selected stabilizer circuits

- General form of a post-selected stabilizer circuit



Generalizes to certain class of stabilizer circuits, see [arXiv:2309.08676]



Stabilizer circuit verification

Vadym Kliuchnikov, Michael Beverland, Adam Paetznick

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