Title: Programming Clifford Unitaries with Symplectic Types

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Collection: Foundations of Quantum Computational Advantage

Date: May 01, 2024 - 11:15 AM

URL: https://pirsa.org/24050006

Abstract: This talk will present work-in-progress towards a new programming methodology for Cliffords, where n-ary Clifford unitaries over qudits can be expressed as functions on compact Pauli. Inspired by the fact that projective Cliffords correspond to center-fixing automorphisms on the Pauli group, we develop a type system where well-typed expressions correspond to symplectic morphisms---that is, linear transformations that respect the symplectic form. This language is backed up by a robust categorical and operational semantics, and well-typed functions can be efficiently simulated and synthesized into circuits via Pauli tableaus.

Foundations of Quantum Computational Advantage
May 1, 2024

Programming Clifford Unitaries with Symplectic Types

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intel

Quantum Programming Languages

Gate-based programming:

- Qiskit, Circ, Q#, tket, Intel Quantum SDK

```
quantum_kernel void measZAll() {
  for (int Index = 0; Index < N; Index++)
    MeasZ(QubitReg[Index], CReg[Index]); // Apply measurement gates
}
```

Beyond gate-based programming:

- Identify mathematical abstractions
- Build a language that harnesses those abstractions
- Express algorithms naturally and enable new ideas

Cliffords as automorphisms on the Pauli group

Unitary matrices U satisfying

$$\forall P \in \mathcal{P}_n,$$
$$UPU^\dagger \in \mathcal{P}_n$$

Pauli group ($\mathcal{P}$)

$$X \times X = I$$
$$X \times Y = iZ$$
$$X \times Z = -iY$$
$$X \times I = X$$

Projective Clifford group:

Automorphisms on the Pauli group
$$P \mapsto P'$$
that fix the center

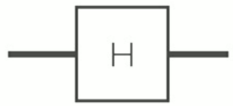
Main Idea

Clifford unitaries

expressed as functions
on qudit Pauli operators

that satisfy certain properties
(center-fixing automorphism)

Example



Idea:
Clifford unitaries
expressed as functions
on Pauli operators
that satisfy certain properties

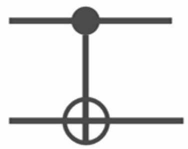
```
h (P : PauliType) : Phase PauliType =
    case P of
        inX -> ?
        inZ -> ?
```

`case ? of …`
=
break up the input into basis elements

`PauliType`
=
type of single-qubit Pauli encodings

`inX/inZ`
=
syntax referring to X/Z Paulis

Example



```
cnot (P : PauliType ⊕ PauliType) : Phase (PauliType ⊕ PauliType) =
    case P of
        in1 Q -> case Q of
                    inX ->
                    inZ ->
        in2 Q -> case Q of
                    inX ->
                    inZ ->
```

$$CNOT\ (Z \otimes I)\ CNOT = Z \otimes I$$

Desiderata

1. Functions implement Cliffords:
   center-fixing automorphisms on the Pauli group

```
notClifford (P : PauliType) : Phase PauliType =
    case P of
        inX -> <0> inX
        inZ -> <0> inX
```

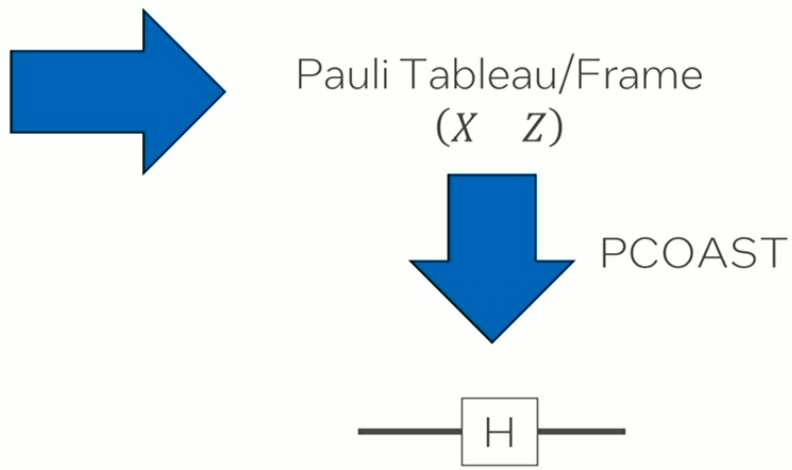
Type-checking Error:
The `inX` and `inZ` branches of the case statement should anticommute.

Type system for ensuring functions are indeed automorphisms.

Desiderata

3. (Qubit) Clifford functions can be compiled to circuits

```
h (P : PauliType) : Phase PauliType =
    case P of
        inX -> <0>inZ
        inZ -> <0>inX
```



Aaronson and Gottesman, “Improved simulation of stabilizer circuits,” 2004.

Paykin, Schmitz, et al. PCOAST: A Pauli-based quantum circuit optimization framework. *QCE 2023*.

Overview

1. Background on encodings of the Pauli group

2. Projective Cliffords as symplectic functions over Pauli encodings

3. Type system for symplectic functions

Background: the Pauli group

Any two Paulis either commute or anti-commute

Single-qubit Paulis (p)

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pauli group $(\mathcal{P})$

$X \times X = I$
$X \times Y = iZ$
$X \times Z = -iY$
$X \times I = X$

...

Symplectic Form
$\omega: \mathcal{P} \otimes \mathcal{P} \to \mathbb{Z}_2$

encodes commutativity of Paulis
$P_1 \times P_2 = (-1)^{\omega(P_1, P_2)} P_2 \times P_1$

$\omega(X, Y) = \omega(Y, Z) = \omega(Z, X) = 1$
$\omega(P, P) = 0$
$\omega(I, P) = 0$

Background: the Pauli algebra

Every member of the Pauli group can be written as

$i^r \Delta_{[x,z]}$

where $r \in \mathbb{Z}_4$ and $x, z \in \mathbb{Z}_2$ and

$\Delta_{[x,z]} = i^{xz} X^x Z^z$

Let's write this $\langle r \rangle [x, z]$.

Symplectic form

$$\omega(\langle r_1 \rangle [x_1, z_1], \langle r_2 \rangle [x_2, z_2]) = x_1 z_2 - z_1 x_2$$

Example:

"Y" $= \langle 0 \rangle [1,1]$

since $Y = iXZ = i^0 i^1 X^1 Z^1$.

Example:

$$\omega(\text{"X"}, \text{"Y"}) = \omega(\langle 0 \rangle [1,0], \langle 0 \rangle [1,1]) = 1 * 1 - 0 * 1 = 1$$

Background: generalizing the Pauli algebra

Generalize to n-qubit Paulis $\mathcal{P}_n$

$$p_0 \otimes \cdots \otimes p_{n-1}$$

Algebra:

$$\langle r \rangle [\vec{x}, \vec{z}]$$

$$= i^r \Delta_{[x_0, z_0]} \otimes \cdots \otimes \Delta_{[x_{n-1}, z_{n-1}]}$$
$$= i^r i^{\vec{x} \cdot \vec{z}} (X^{x_0} \otimes \cdots \otimes X^{x_{n-1}})(Z^{z_0} \otimes \cdots \otimes Z^{z_{n-1}})$$

where $r \in \mathbb{Z}_4$,
$\vec{x} = [x_0, \ldots, x_{n-1}] \in \mathbb{Z}_2^n$
$\vec{z} = [z_0, \ldots, z_{n-1}] \in \mathbb{Z}_2^n$

V = vectors in the Pauli algebra encoding over $\mathbb{Z}_2$
aka $V = \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n$

Symplectic Form
$\omega : V \otimes V \to \mathbb{Z}_2$

$$\omega([\overrightarrow{x_1}, \overrightarrow{z_1}], [\overrightarrow{x_2}, \overrightarrow{z_2}]) = \overrightarrow{x_1} \cdot \overrightarrow{z_2} - \overrightarrow{z_1} \cdot \overrightarrow{x_2}$$

Background: generalizing the Pauli algebra

Generalize to n-qu$\boldsymbol{d}$it Paulis $P_{d,n}$

$$X|r\rangle = |(r+1) \bmod d\rangle$$
$$Z|r\rangle = \zeta^r |r\rangle \qquad \text{where } \zeta^d = 1.$$

Algebra:
$$\langle r\rangle[\vec{x},\vec{z}]$$

$$=\zeta^r \Delta_{[\vec{x},\vec{z}]} = \zeta^r \zeta^{\frac{1}{2}\vec{x}\cdot\vec{z}} X^{\vec{x}} Z^{\vec{x}}$$

where

$$r \in ½\, \mathbb{Z}_{d'}$$
$$d' = \begin{cases} d & d \text{ odd} \\ 2d & d \text{ even} \end{cases}$$
$$\vec{x} = [x_0, \ldots, x_{n-1}] \in \mathbb{Z}_d^n$$
$$\vec{z} = [z_0, \ldots, z_{n-1}] \in \mathbb{Z}_d^n$$

V = vectors in the Pauli algebra encoding over $\mathbb{Z}_d$
aka $V = \mathbb{Z}_d^n \oplus \mathbb{Z}_d^n$

Symplectic Form
$$\omega : V \otimes V \to \mathbb{Z}_d$$

$$\omega([\overrightarrow{x_1}, \overrightarrow{z_1}], [\overrightarrow{x_2}, \overrightarrow{z_2}]) = \overrightarrow{x_1} \cdot \overrightarrow{z_2} - \overrightarrow{z_1} \cdot \overrightarrow{x_2}$$

Theorem

V' = vectors in the Pauli algebra encoding vector space over $R' = \mathbb{Z}_{d'}$

$\frac{1}{2}\mathbb{Z}_{d'}$ = coefficients of ζ in the Pauli algebra encoding where $\zeta^{1/2}$ is a d'-th root of unity

The set of projective Cliffords $PCl'_{d'/d}$

$\cong$

The set of pairs of functions (δ, ϕ) where

- $\delta: V' \to \frac{1}{2}\mathbb{Z}_{d'}$ is a linear transformation;
- $\phi: V' \to V'$ is a symplectomorphism---a linear isomorphism that respects the symplectic form; and
- the function $\Delta_v \mapsto \zeta^{\delta(v)}\Delta_{\phi(v)}$ is right-definite.

$$\Delta_v \mapsto \zeta^{\delta(v)}\Delta_{\phi(v)}$$

Theorem

V = vectors in the Pauli algebra encoding over $\mathbb{Z}_d$
aka $V = \mathbb{Z}_d^n \oplus \mathbb{Z}_d^n$

The set of **projective Cliffords** $PCl'_{d,n}$
$\cong$
Functions over the Pauli algebra where

- $\mu : V \to \mathbb{Z}_d$ is an R-linear map; and
- $\psi : V \to V$ is a symplectomorphism---a linear isomorphism satisfying

$$\omega\big(\psi(P_1), \psi(P_2)\big) = \omega(P_1, P_2)$$

Proof sketch:

Projective Clifford → Encoding (δ, ϕ) over V'
→ Compact encoding (μ, ψ) over V
→ Encoding (δ, ϕ) over V'
→ Projective Clifford

Desiderata

Projective **Cliffords** $PCl'_{d,n}$
$\cong$
Pairs of functions (μ, ψ) where
- $\mu: V \to \mathbb{Z}_d$ is a linear transformation; and
- $\psi: V \to V$ is a symplectomorphism.

1. ~~Functions implement Cliffords: automorphisms on the Pauli group~~
   - ~~Type system for ensuring functions are automorphisms~~

1a. Functions implement (μ, ψ)
   - Type system for ensuring properties are respected

2. ~~All Cliffords can be represented~~

2a. All such functions can be represented

3. All functions can be compiled to circuits

Path towards a type system

Projective **Cliffords** $PCl'_{d,n}$
$\cong$
Pairs of functions (μ, ψ) where
- $\mu: V \to \mathbb{Z}_d$ is a linear transformation; and
- $\psi: V \to V$ is a symplectomorphism.

1. Type system for free modules over a ring, with biproducts

2. Type system for symplectic morphisms—linear transformations that respect the symplectic form

3. Type system for Paulis $\langle r \rangle v$

Defining a type system

1. What are types?

Types: free finitely-generated R-modules

Module Types τ
R
$\tau_1 \oplus \tau_2$

Defining a type system

1. What are types?
2. What are values of a given type?
3. What properties should well-typed expressions satisfy?
4. What are the typing rules for well-typed expressions?

$$e[c_1 \cdot v_1 + c_2 \cdot v_2] \equiv c_1 \cdot e[v_1] + c_2 \cdot e[v_2]$$

Types: free finitely-generated R-modules

Values: vectors in the R-module

Module Types τ	Values
R	Constants $r \in R$
$\tau_1 \oplus \tau_2$	Tuples $[v_1, v_2]$

Expressions: linear transformations

$$x : \tau \vdash e : \tau'$$

1. Expressions

$$\Gamma \vdash e : \tau'$$

$$\Gamma \coloneqq x_1 : \tau_1, \ldots, x_n : \tau_n$$

$$\tau \coloneqq R \mid \tau_1 \oplus \tau_2$$

$e \coloneqq$	$x \mid r \mid [e_1, e_2]$	Variables, scalars, and vectors
	$e_1 + e_2 \mid e_1 \cdot e_2$	Operations on scalars and vectors
	case e of $\{in_1(x_1) \to e_1 \mid in_2(x_2) \to e_2\}$	Vector case analysis

relevant type system:

- contraction: variables **can** be duplicated
- no weakening: variables **cannot** be discarded

Arrighi & Dowek. Lineal: A linear-algebraic lambda-calculus. LMCS 2017.
Díaz-Caro & Dowek. A new connective in natural deduction, and its application to quantum computing. TCS 2023.
Díaz-Caro & Dowek. A linear linear lambda-calculus. MSCS 2024.

1. Semantics

$x : \tau \vdash e : \tau'$

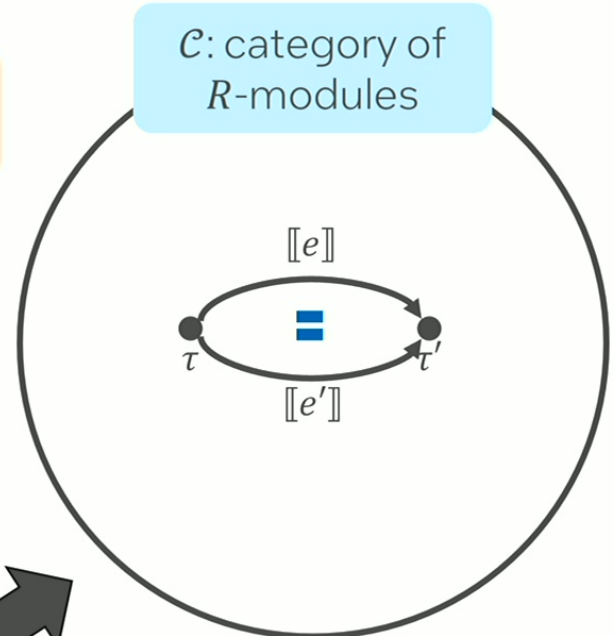
categorical semantics

operational semantics

$x : \tau \vdash e' : \tau'$

equivalence relation

$x : \tau \vdash e \equiv e' : \tau'$



2. Type system for symplectic morphisms

Types: free finitely-generated R-modules for which symplectic form is defined

Values: vectors in the R-module

Symplectic Types $\boldsymbol{\sigma}$	Values
$\boldsymbol{Q} = R \oplus R$	Single-qudit vector $[x, z]$ encoding $\Delta_{[x,z]}$
$\sigma_1 \oplus \sigma_2$	Tuples $[v_1, v_2]$

Path towards a type system

Projective **Cliffords** $PCl'_{d,n}$
$\cong$
Pairs of functions (μ, ψ) where
- $\mu: V \to \mathbb{Z}_d$ is a linear transformation; and
- $\psi: V \to V$ is a symplectomorphism.

1. Type system for free modules over a ring, with biproducts

2. **Type system for symplectic morphisms—linear transformations that respect the symplectic form**

3. Type system for Paulis $\langle r \rangle v$

2. Type system for symplectic morphisms

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Expressions: linear transformations that respect symplectic form

$$x : \sigma \vdash^S e : \sigma'$$

$$\omega(e[v_1], e[v_2]) \equiv \omega(v_1, v_2)$$

2. Symplectic type system

$$a_1 : \tau_1, \ldots, a_n : \tau_n \; ; b : \sigma \vdash^S e : \sigma$$

Linear transformation ($a_1 : \tau_1, \ldots, a_n : \tau_n$)

Respect symplectic form ($b : \sigma$)

Module Types $\boldsymbol{\tau}$	Values
R	Constants $r \in R$
$\tau_1 \oplus \tau_2$	Tuples $[v_1, v_2]$

Symplectic types $\boldsymbol{\sigma}$	Vector spaces used in the Pauli algebra (dimension $\mathbf{2n}$)
$\boldsymbol{Q} = R \oplus R$	Single-qudit vector $[x, z]$ encoding $\Delta_{[x,z]}$
$\sigma_1 \oplus \sigma_2$	Tuple of n-qudit vectors

2. Expressions

Symplectic types $\boldsymbol{\sigma}$	Vector spaces used in the Pauli algebra (dimension $\boldsymbol{2n}$)
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$e \; :=$	$x \mid r \mid [e_1, e_2]$	Variables, scalars, and vectors
	$e_1 + e_2 \mid e_1 \cdot e_2$	Operations on scalars and vectors
*	case e of $\{in_x \to e_x \mid in_z \to e_z\}$	Pauli case analysis
	case e of $\{in_1(x_1) \to e_1 \mid in_2(x_2) \to e_2\}$	Vector case analysis
	$\omega_\sigma(e_1, e_2)$	Symplectic form

$$in_z = [1,0]$$
$$in_x = [0,1]$$

$$\frac{\Gamma; \Delta \vdash^S e : \boldsymbol{Q} \quad \Gamma'; \Delta' \vdash^S e_x : \sigma \quad \Gamma'; \Delta' \vdash^S e_z : \sigma \quad \omega_\sigma(e_x, e_z) \equiv 1}{\Gamma \cup \Gamma'; \Delta, \Delta' \vdash^S \text{case } e \text{ of } \{ in_X \to e_x \mid in_Z \to e_z \} : \sigma}$$

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2. Expressions

Symplectic types $\boldsymbol{\sigma}$	Vector spaces used in the Pauli algebra (dimension $\boldsymbol{2n}$)
$\boldsymbol{Q} = \mathrm{R} \oplus \mathrm{R}$	Single-qudit vector [z,x] encoding $\Delta_{[z,x]}$
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```
notSymplectic(x : QType) : QType =
    case x of
        inX -> inZ
        inZ -> inZ
```

$$\omega(in_Z, in_Z) = \omega([0,1],[0,1]) = 0 - 0 \neq 1$$

$$\frac{\Gamma; \Delta \vdash^S e : \boldsymbol{Q} \quad \Gamma'; \Delta' \vdash^S e_x : \sigma \quad \Gamma'; \Delta' \vdash^S e_z : \sigma \quad \omega_\sigma(e_x, e_z) \equiv 1}{\Gamma \cup \Gamma'; \Delta, \Delta' \vdash^S \text{case } e \text{ of } \{ in_X \to e_x \mid in_Z \to e_z \} : \sigma}$$

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$$\frac{\Gamma; \Delta \vdash^S e : \sigma_1 \oplus \sigma_2 \quad \Gamma'; \Delta', x_1 : \sigma_1 \vdash^S e_1 : \sigma \quad \Gamma'; \Delta', x_2 : \sigma_2 \vdash^S e_2 : \sigma \quad \boxed{\omega_\sigma(e_1, e_2) \equiv 0}}{\Gamma \cup \Gamma'; \Delta, \Delta' \vdash^S \text{case } e \text{ of } \{\, in_1(x_1) \to e_1 \mid in_2(x_2) \to e_2 \,\} : \sigma}$$

Path towards a type system

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3. Type system for Pauli algebra

$$e : \sigma \multimap \mathrm{Phase}(\sigma)$$

$$⇓$$

$$[\![e]\!] : \sigma \rightarrow R \oplus \sigma$$

such that

$$\mu = [\![e]\!] \circ \mathrm{first} : \sigma \rightarrow R$$
$$\psi = [\![e]\!] \circ \mathrm{second} : \sigma \rightarrow \sigma$$

satisfy

- $\mu : \sigma \rightarrow R$ is a linear transformation;
- $\psi : \sigma \rightarrow \sigma$ is a symplectomorphism.

...so what?

- Functions over Paulis as a programming abstraction
  - Data structures, recursion, polymorphism
  - Interactive feedback on what makes a Clifford
  - Quantum algorithms in terms of change-of-basis
  - Alternate bases other than `inX`/`inZ`

Conclusion

- Programming Cliffords as functions over Paulis:
  - Clever encodings and typing rules isolate the functions corresponding to Cliffords
  - Operational and denotational semantics show it is sound
  - Need examples and implementations to show if it is useful

- Type systems can harness mathematical structures into programming abstractions

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Programming Cliffords with Paulis

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