Title: Quantum rainbow codes

Speakers: Arthur Pesah

Series: Perimeter Institute Quantum Discussions

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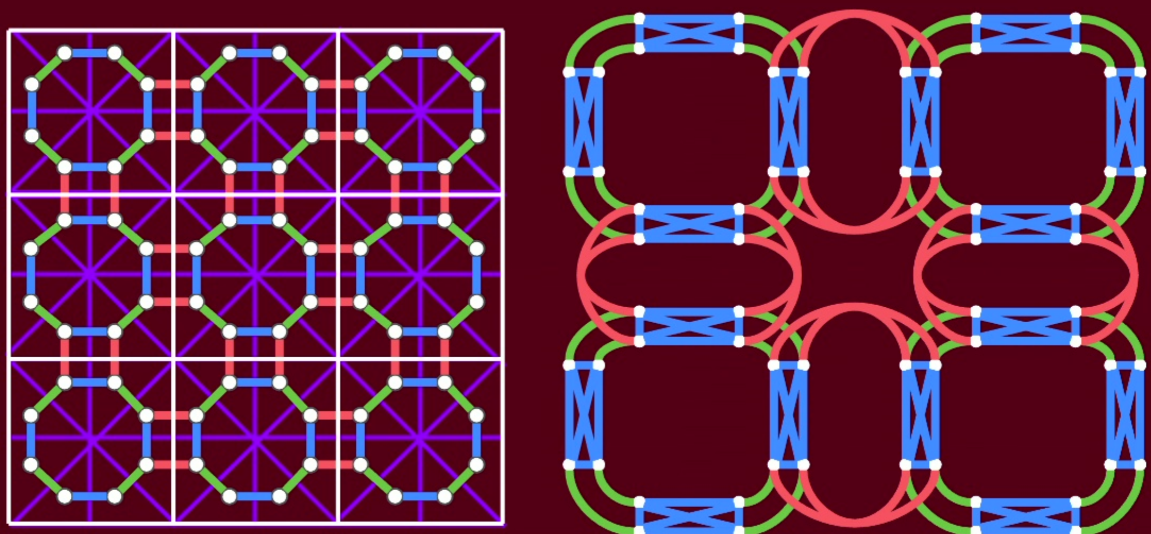
Abstract: With the recent construction of quantum low-density parity-check (LDPC) codes with optimal asymptotic parameters, finding methods to perform low-overhead computation using those constructions has become a central problem of quantum error-correction. In particular, triorthogonal codes---which admit transversal non-Clifford operations---are of particular interest, but few examples of these codes are presently known. In our work, we introduce a new family of codes, the quantum rainbow codes, a generalization of pin codes and color codes, that can be constructed from any chain complex. When applied to the hypergraph product of three complexes, we show that those codes can implement transversal non-Clifford gates and have improved parameters compared to pin codes. Considering expander graphs with large girth as the input complexes, we can for instance obtain families of triorthogonal codes with parameters [[n,?(n^{2/3}),?(log(n))]].

Zoom link

Give colour to your LDPC life with *rainbow* codes™

How generalized colour codes could help us get transversal gates on LDPC codes

Arthur Pesah
University College London (UCL)



Work in progress with Tom Scruby (OIST) and Mark Webster (UCL)

Overview

We found a family of product-based qLDPC codes that supports transversal non-Clifford gates

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① qLDPC codes

Quantum low-density parity-check codes:
→ Codes with sparse connectivity
→ Geometry doesn't matter
→ Guarantees the existence of a threshold

Recent progress:
→ "Good" quantum LDPC codes exist

$$[[n, k, d]]$$

#physical qubits #logical qubits distance

$$k = \Theta(n)$$
$$d = \Theta(n)$$

P. Panteleev, G. Kalachev, 2021

Overview

We found a family of product-based qLDPC codes that supports transversal non-Clifford gates

① qLDPC codes

② Product-based

Hypergraph product code (HGP)
→ Product of classical codes that gives a quantum code
→ Preserve the LDPC property
→ Can give constant-rate codes, i.e. $k=\Theta(n)$

Base ingredient of good qLDPC codes
→ By quotienting a HGP code by a group, choosing the classical codes carefully, and other tricks, we can get good LDPC codes

J.P. Tillich, G. Zémor, 2009

Overview

We found a family of product-based qLDPC codes that supports transversal non-Clifford gates

① qLDPC codes

② Product-based

③ Transversal non-Clifford

S. Kubica, M. Beverland, 2013 (color codes)
S. Nezami, J. Haah, 2021 (triorthogonal codes)

Transversal gate
→ Logical gate obtained by applying a physical gate on every qubit individually
→ Guaranteed fault-tolerance, low-overhead

Non-Clifford gate (e.g. T, CCZ)
→ High-overhead when using non-transversal methods (e.g. magic state distillation)
→ Few codes are known to have them
(3D topo codes, Haah's triorthogonal codes)

Overview

We found a family of product-based qLDPC codes that supports transversal non-Clifford gates

① qLDPC codes

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④ Found a family

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Rainbow codes

→ Generalize colour codes beyond topological manifolds (works for arbitrary chain complex)

→ Generalize pin codes

Our family

→ has a global transversal non-Clifford gate

→ has best parameters (with best rate)

$$k = \Theta\left(n^{2/3}\right) \qquad d = \Theta\left(\log(n)\right)$$

Our approach

3D codes have non-Clifford gates, e.g. transversal CCZ on 3D toric codes, transversal T on 3D colour codes

3D codes have the potential for addressability, e.g. sheets of CZ on 3D toric codes or sheets of S gates on 3D colour codes => exponential number of CZ/S representatives

Observation: gates on color codes are much easier to design than gates on toric codes (no need to find different lattices, just apply a bunch of T and T†)

Recent paper: found that 3D color codes on hyperbolic manifolds can have addressability for Clifford gates, global non-Clifford, and constant rate!

Question: can we improve on those properties by considering color codes beyond manifolds?

Sub-question: what properties do we get if we apply color code ideas to product constructions?

G Zhu, S Sikander, E Portnoy, AW Cross, BJ Brown, 2023

What is a colour code?

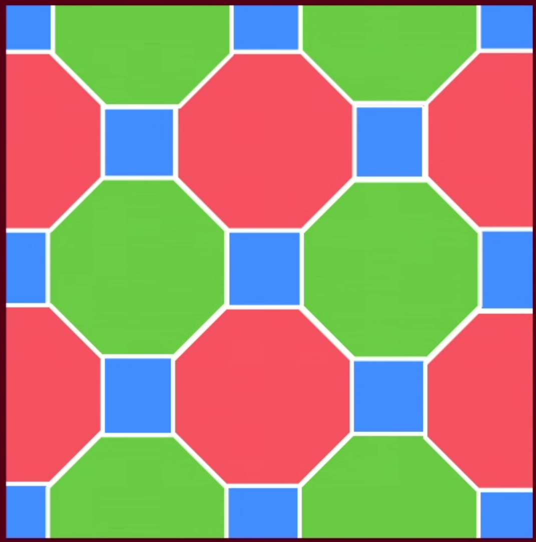
Recipe to build a (D-dimensional) colour code:

Start with a (D+1)-colourable (D+1)-valent cellulation of a manifold

Place qubits on vertices

Place X-stabilizers on 2-dimensional objects

Place Z-stabilizers on D-dimensional objects



What is a colour code?

Recipe to build a (D-dimensional) colour code:

① Start with a (D+1)-colourable (D+1)-valent cellulation of a manifold

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③ Place X-stabilizers on 2-dimensional objects

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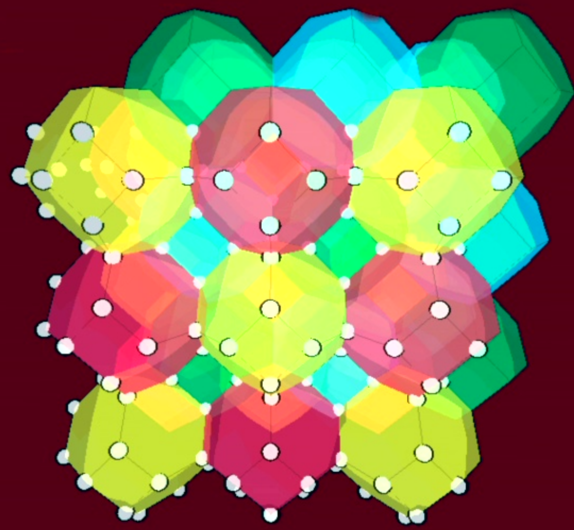


Properties of colour code

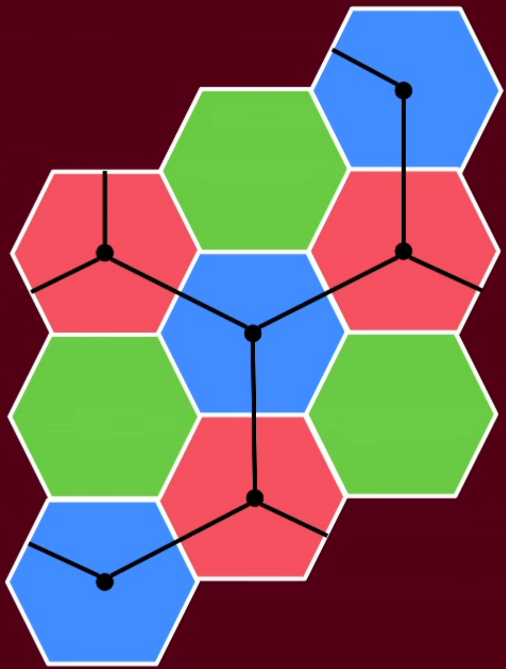
1) It unfolds into D copies of the toric code

That is, there exists a local unitary operator that turns the color codes into D copies of the toric code

Those copies are defined on the so-called shrunk lattices, defined by shrinking cells of all colours but one

⇒ It encodes 2× as many logical qubits as the toric codes

⇒ It can be decoded using a toric code decoder on the shrunk lattices

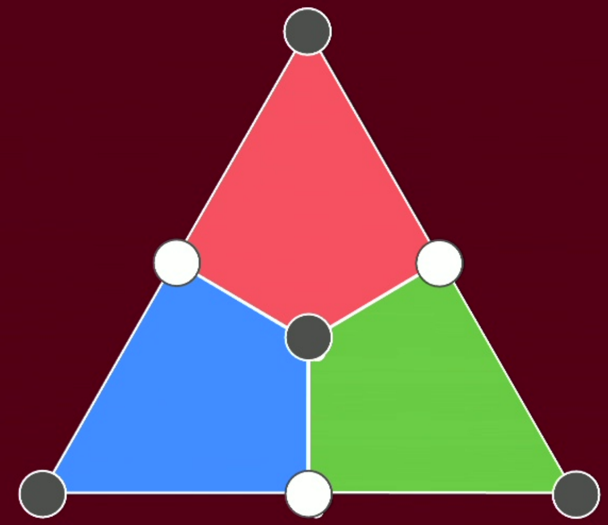


Properties of colour code

2) It has many diagonal transversal gates

More precisely, the D-dimensional colour code has the gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{2(D-1)}} \end{pmatrix}$ (S for D=2 ; T for D=3 ; etc.)

Example: in 2D, we first find a bi-colouring of the vertices, then apply S and S† on black and white vertices respectively.



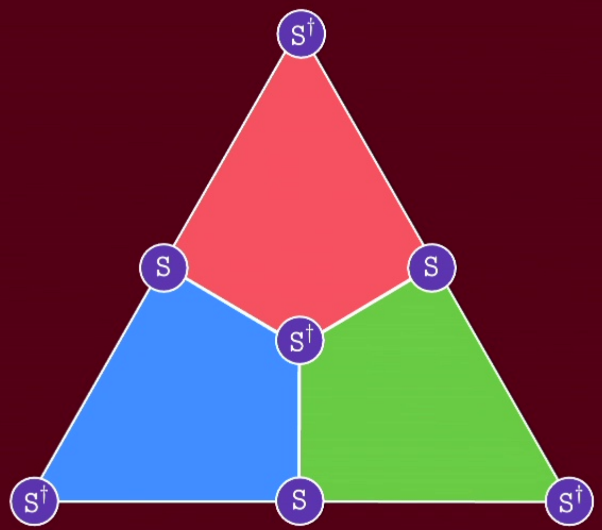
Properties of colour code

2) It has many diagonal transversal gates

Key property: bi-orthogonality

$$|S_1 \cap S_2| = 0 \mod 2$$
$$|S \cap L| = 0 \mod 2$$

For all X-stabilizers S, S_1, S_2, and X-logical L



Interpretation:
1) Applying the S gate on the X stabilizer gives a Y operator on the same support. It must have even intersection with all the X stabilizers to be a stabilizer itself.

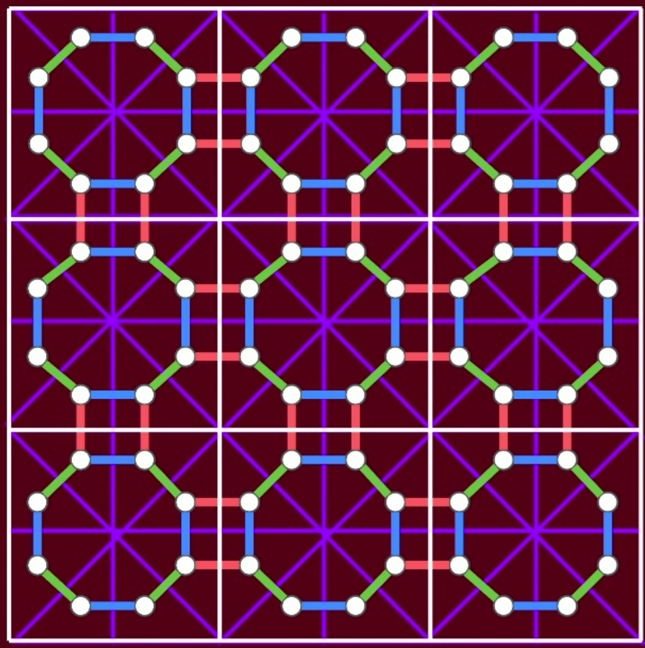
Properties of colour code

2) It has many diagonal transversal gates

For 3D codes: we require tri-orthogonality

$$|S_1 \cap S_2 \cap S_3| = 0 \mod 2$$
$$|S_1 \cap S_2 \cap L| = 0 \mod 2$$
$$|S \cap L_1 \cap L_2| = 0 \mod 2$$

Key property for magic state distillation



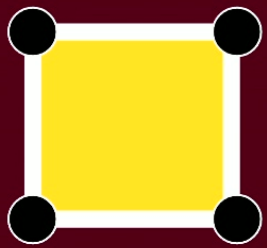
GOING BEYOND MANIFOLDS WITH PIN CODES

Chain complexes

Generalize the notion of manifold cellulation

Two pieces of data:

1) D vector spaces over $\mathbb{Z}_2$: C_0, C_1, C_2, etc., corresponding to "vertices", "edges", "faces", etc.



$$C_0 = \langle v_1, v_2, v_3, v_4 \rangle$$

$$C_1 = \langle e_1, e_2, e_3, e_4 \rangle$$

$$C_2 = \langle f_1 \rangle$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ etc.}$$

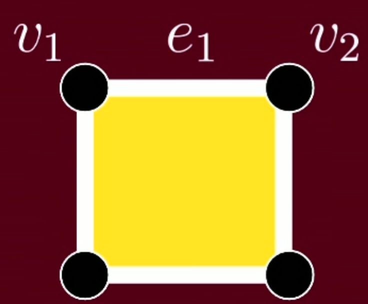
Chain complexes

Generalize the notion of manifold cellulation

Two pieces of data:

1) D vector spaces over $\mathbb{Z}_2$: C_0, C_1, C_2, etc., corresponding to “vertices”, “edges”, “faces”, etc.

2) Boundary operators, giving the incidence relation between objects:



$$C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2$$

Example: $\partial(e_1) = v_1 + v_2$

Chain complexes

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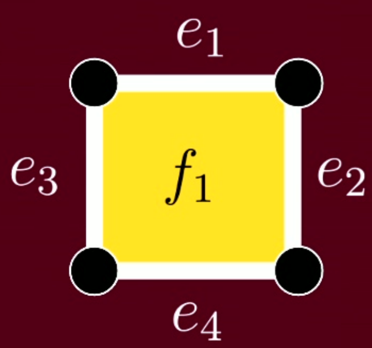

$$C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2$$

Example: $\partial(f_1) = e_1 + e_2 + e_3 + e_4$

Chain complexes

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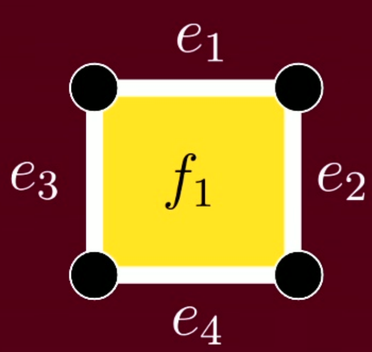


$$C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2$$

Chain complex condition: $\partial_i \circ \partial_{i+1} = 0$

Example: $\partial_0 \circ \partial_1(f_1) = \partial_0(e_1) + \partial_0(e_2) + \partial_0(e_3) + \partial_0(e_4)$

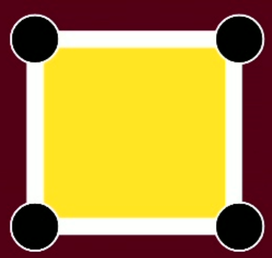
$$= v_1 + v_2 + v_2 + v_3 + v_3 + v_4 + v_4 + v_1$$

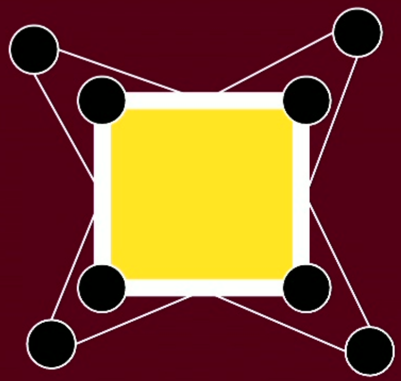
$$= 0$$

Chain complexes

Combinatorial manifold v.s. general chain complex

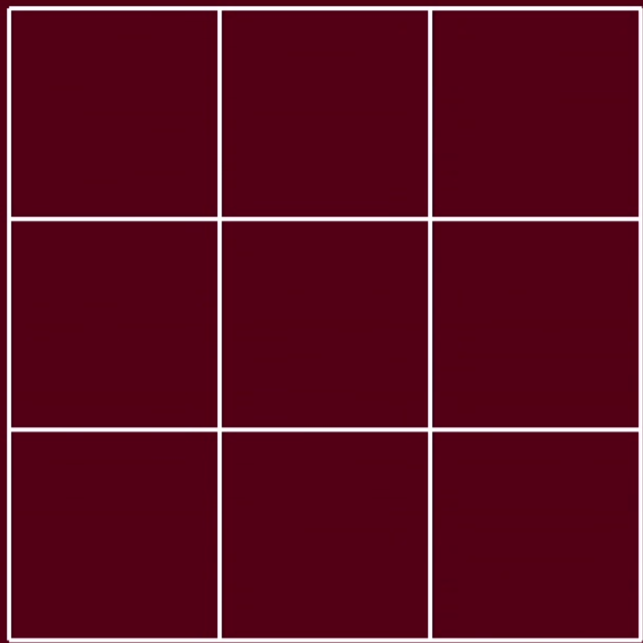
Combinatorial manifold	General chain complex
Each edge is connected to exactly 2 vertices	No such constraint
Intersection of vertex & face = 2 incident edges	
Intersection of edge & cell = 2 incident faces	





Pin codes from chain complexes

Different way to obtain a code from basically any chain complex



C. Vuillot, N. Breuckmann, *Quantum Pin Codes*, 2019

Chain complexes

Examples:

CSS code: general 2-chain complex $\mathcal{S}_{\mathcal{X}} \xleftarrow{H_X} \mathcal{Q} \xleftarrow{H_Z^T} \mathcal{S}_{\mathcal{Z}}$

D-dimensional toric code: part of the D-chain complex of a manifold

$$\boxed{C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2} \xleftarrow{\partial_2} C_3 \xleftarrow{\partial_3} \ldots$$

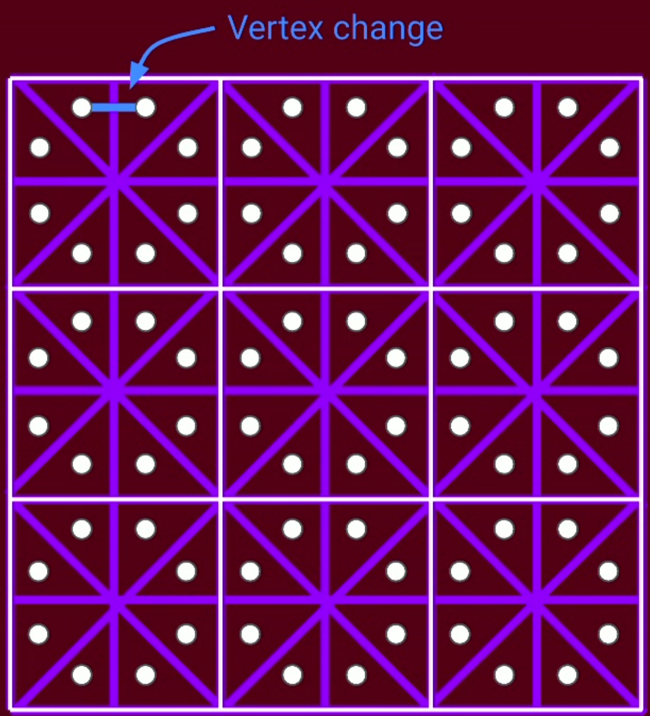
D-dimensional color code: built from a (D+1)-valent (D+1)-colourable D-chain complex of a manifold, with qubits on vertices and stabilizers on faces & D-cells

Pin codes from chain complexes

Different way to obtain a code from basically any chain complex

Associate a qubit to every triplet (vertex, edge, face).
Each triplet is called a flag

Draw an edge between two flags that only differs by one object. Colour the edge depending on the type of the changing object (where it belongs in the chain complex)



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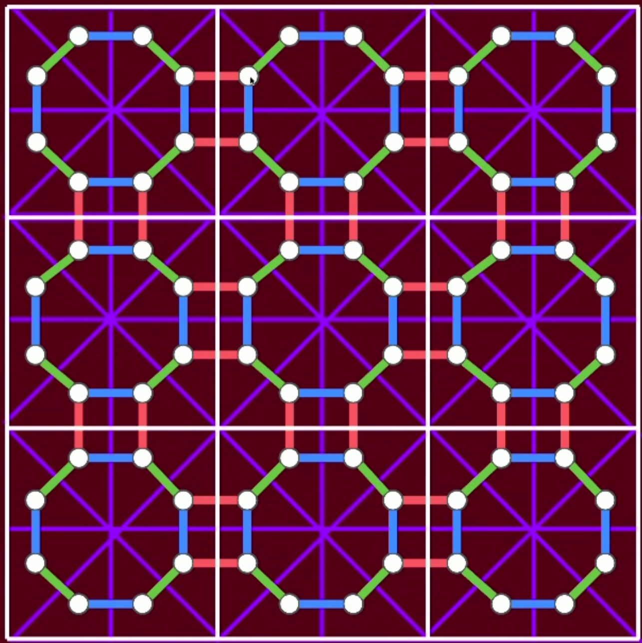

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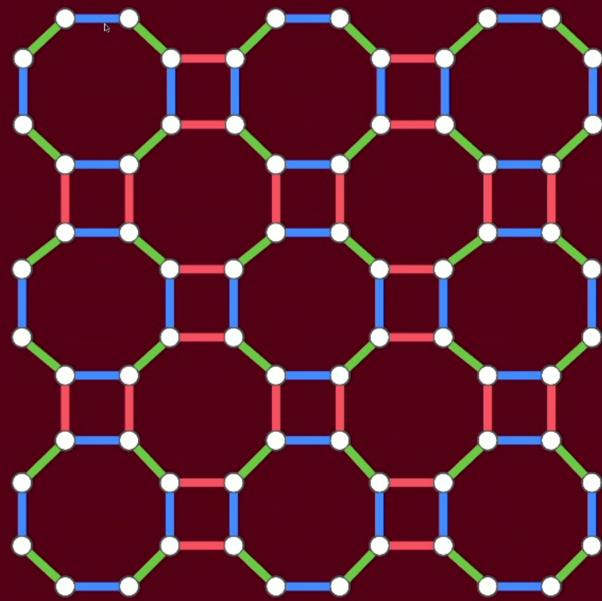
Draw an edge between two flags that only differs by one object. Colour the edge depending on the type of the changing object (where it belongs in the chain complex)

Associate a stabilizer to every maximal 2-coloured subgraph



C. Vuillot, N. Breuckmann, *Quantum Pin Codes*, 2019

Pin codes from chain complexes

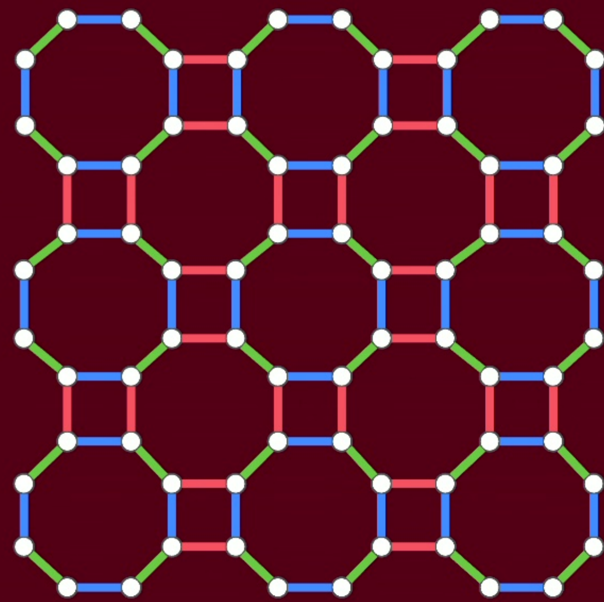
Different way to obtain a code from basically any chain complex

If the vertices and faces of the original chain complex have even weight, this defines a valid CSS code!

Observations:

1) If the original chain complex comes from a manifold, the generated code will be a colour code.

2) This procedure has been known since Bombin's first colour code paper, and is called fattening in this context.



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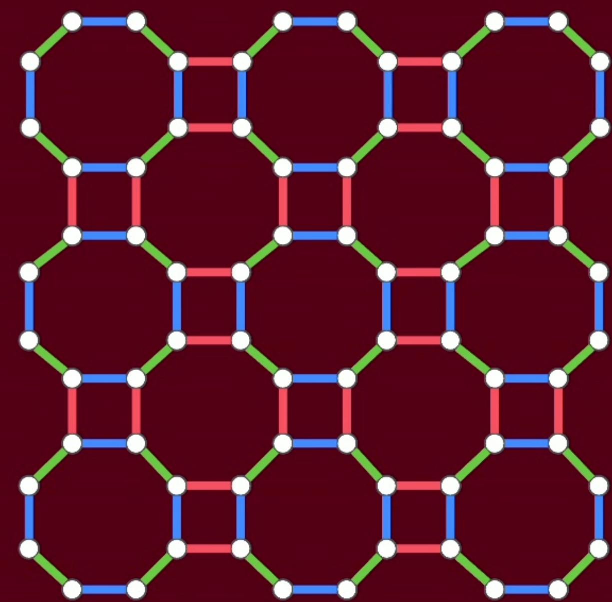


Observations:

1) If the original chain complex comes from a manifold, the generated code will be a colour code.

2) This procedure has been known since Bombin's first colour code paper, and is called fattening in this context.

3) In D-dimension, more flexibility in the choice of stabilizers: x & z-coloured maximal subgraphs, with x, z s.t.

$$x + z \geq D + 2$$

C. Vuillot, N. Breuckmann, *Quantum Pin Codes*, 2019

Pin codes from chain complexes

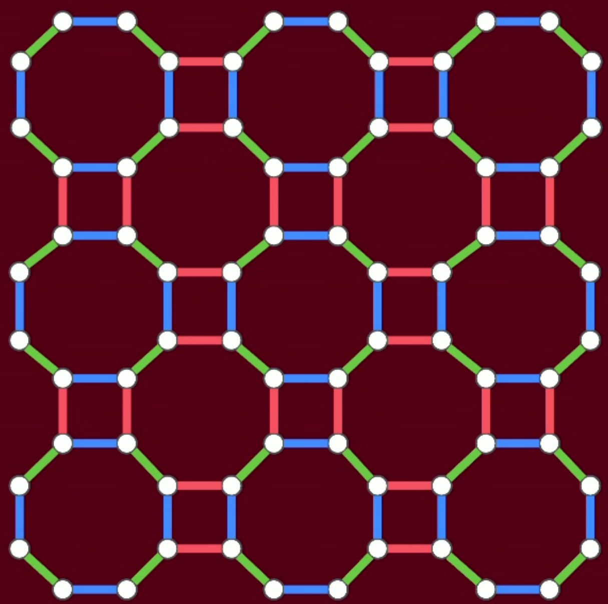
Different way to obtain a code from basically any chain complex

If the vertices and faces of the original chain complex have even weight, this defines a valid CSS code!

Observations:

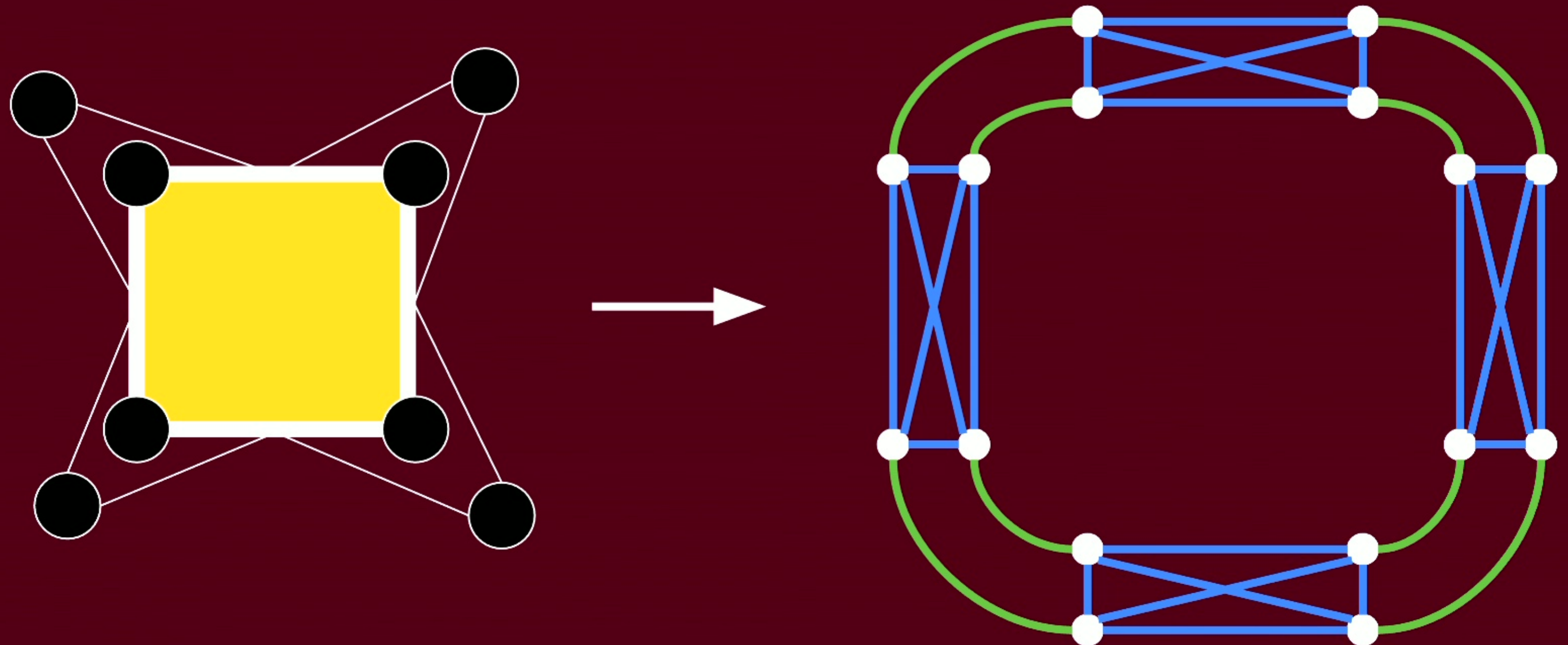
4) Pin codes can be generalized beyond chain complexes, to any coloured graph where all edge colours are represented at each vertex (e.g. constructions based on group theory).

5) The stabilizers of a pin code are D-orthogonal.
⇒ potential for transversal gates!



C. Vuillot, N. Breuckmann, *Quantum Pin Codes*, 2019

Pin codes beyond manifold



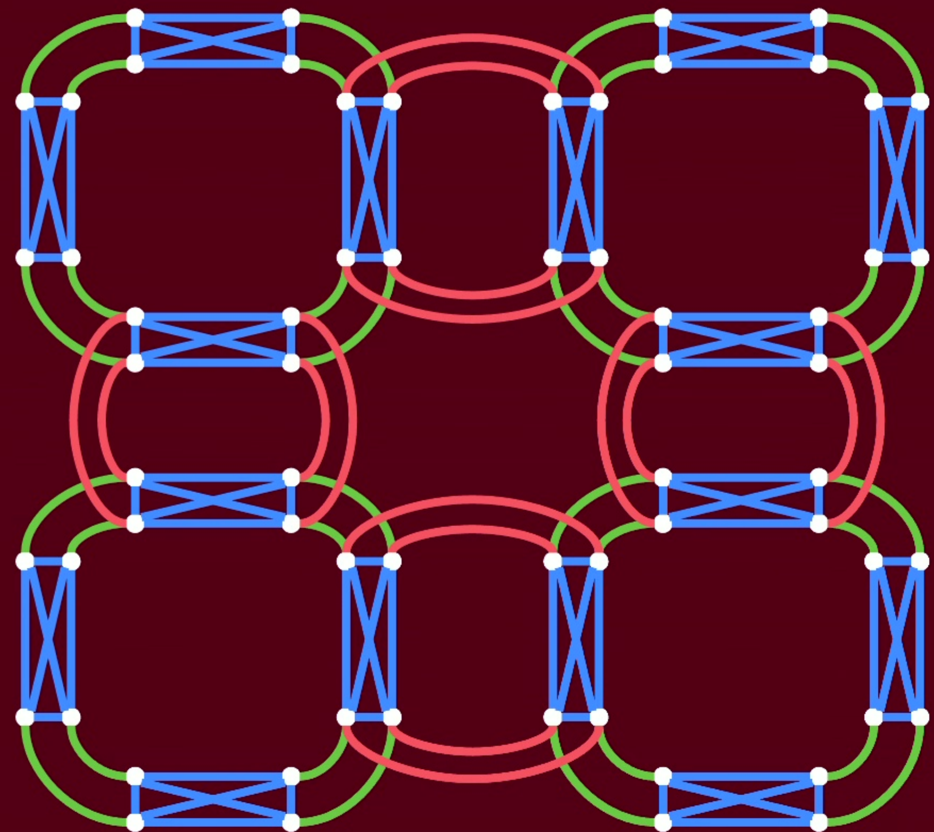
Pin codes beyond manifold

This defines a valid family of pin codes when using periodic boundaries

Question: what is the distance?

Answer: d=4 for all sizes!

All cycles made of two alternating colours are logicals of the code!



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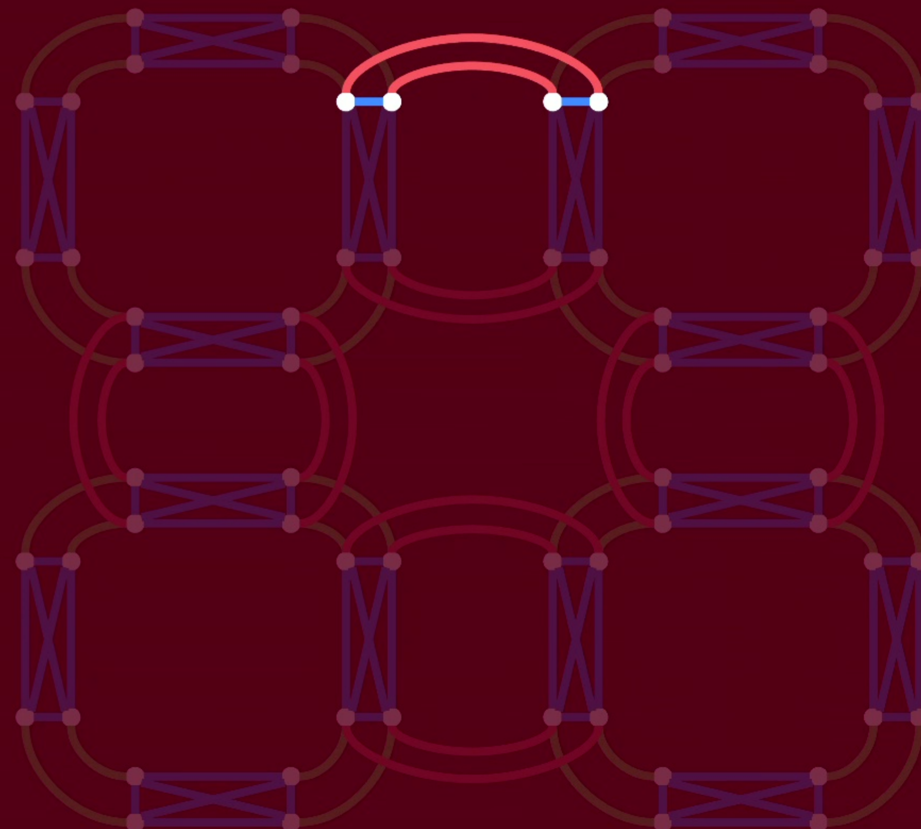

Pin codes beyond manifold

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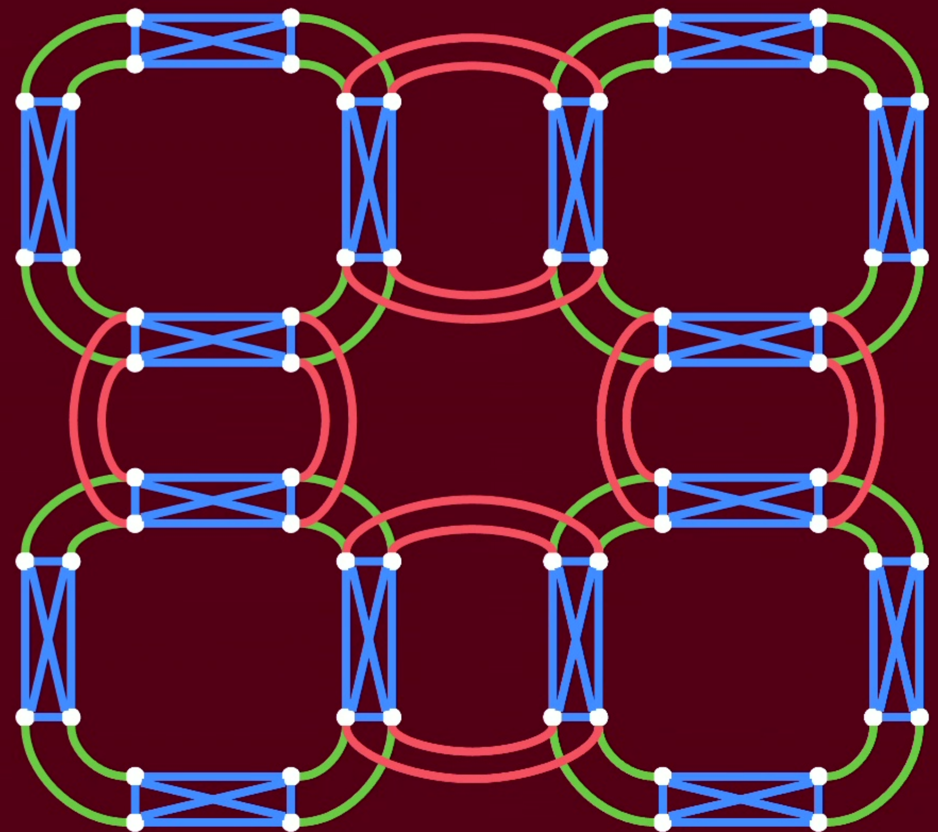
Answer: d=4 for all sizes!

All cycles made of two alternating colours are logicals of the code!

In the pin code paper, they experiment with many examples of code but can't go beyond d=4. This is the reason!

Question: how to solve this issue?



Rainbow codes: definitions

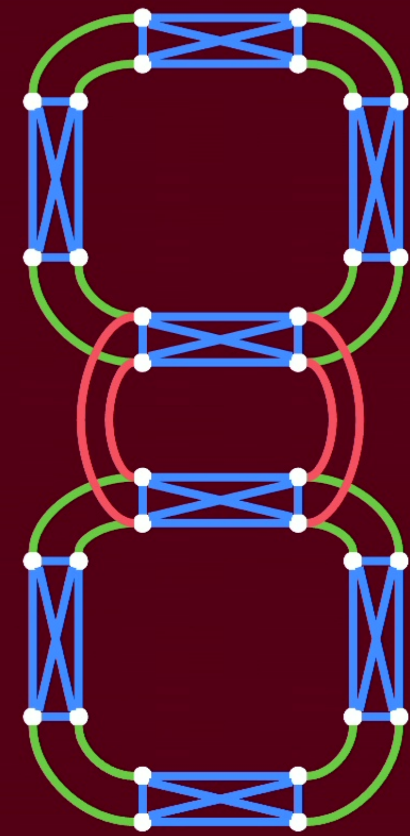
Let's consider a flag graph with D+1 colours.
Let C be a subset of colours (e.g. C={blue, green})

Definition (C-rainbow subgraph): subgraph where each vertex is connected to exactly one edge of every colour in C. By extension, a k-rainbow subgraph is a C-rainbow subgraph for |C|=k.

Example: a {blue, green}-rainbow subgraph is a cycle with alternating blue and green edges

Lemma: every maximally coloured subgraph has an even intersection with all C-rainbow subgraphs, where |C| < D+1

Consequence: we can define a code where all X stabilizers are 2-rainbow subgraphs and Z stabilizers 2-maximal subgraphs



Rainbow codes from hypergraph products

Hypergraph product codes

Why looking at hypergraph product codes?

1) They are simple, and a bridge towards more complicated LDPC codes
2) They have a CZ gate which can be interpreted as folding: direct link to colour codes!

Rainbow codes from hypergraph products

Hypergraph product codes

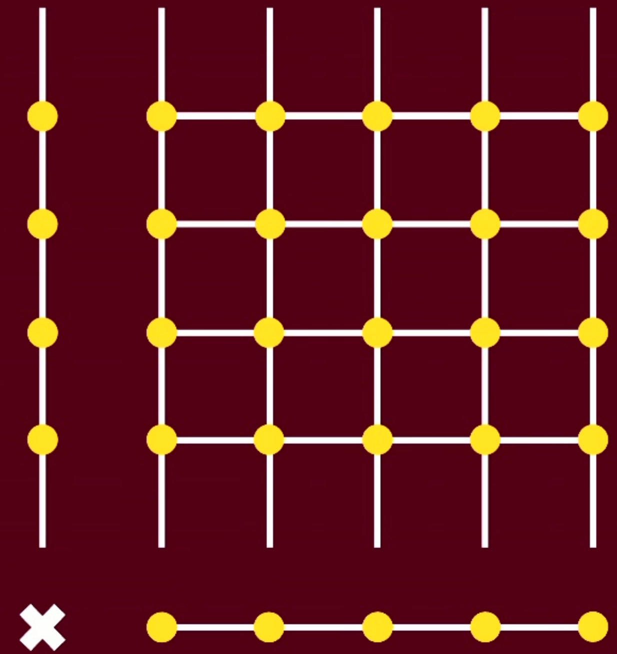
Idea: construct a quantum code from two classical codes

Example: the surface code is a HGP code of two repetition codes

In general, useful construction for several reasons:

1) The product of LDPC codes is LDPC
2) The product of $[[n_1, k_1, d_1]]$ and $[[n_2, k_2, d_2]]$ codes is an $[[n_1 n_2, k_1 k_2 + k_1^{\perp} k_2^{\perp}, \min(d_1, d_2)]]$ code

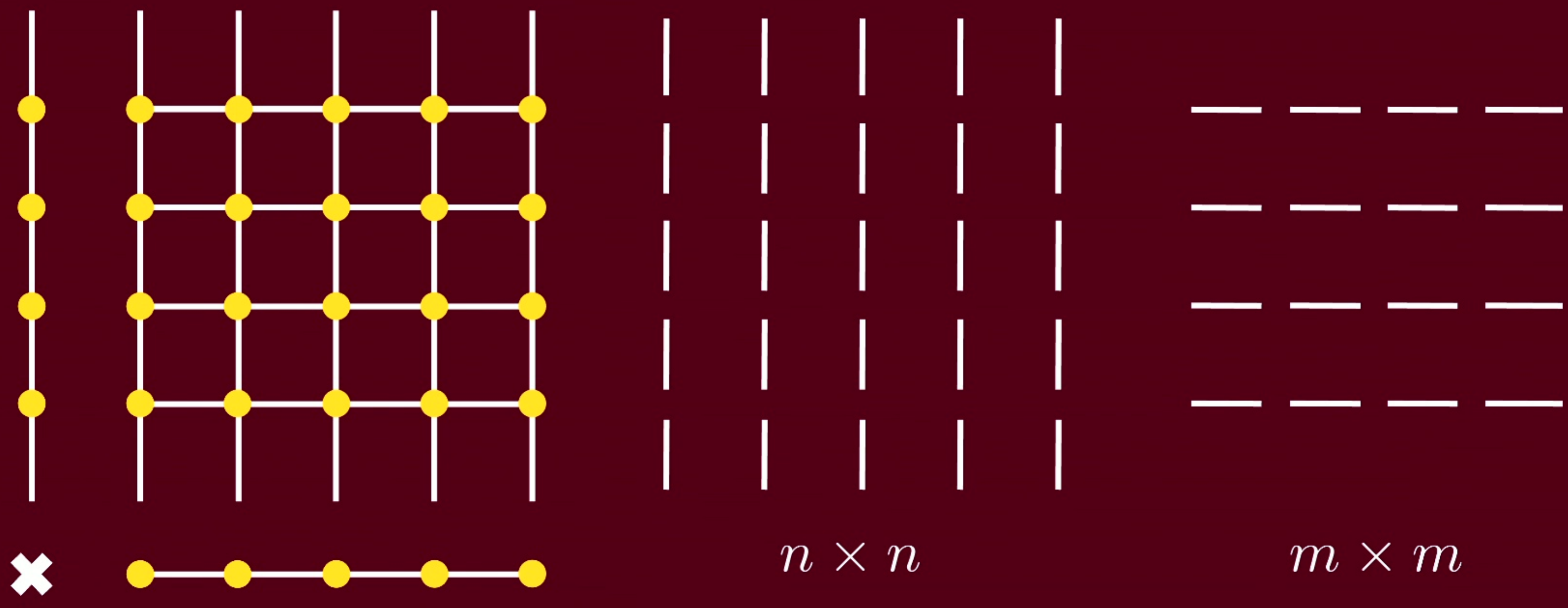
   ⇒ number of logical qubits often increases
3) Used in the construction of good LDPC codes



Rainbow codes from hypergraph products

Hypergraph product codes

Note: a hypergraph product can be represented in the following way:



Rainbow codes from hypergraph products

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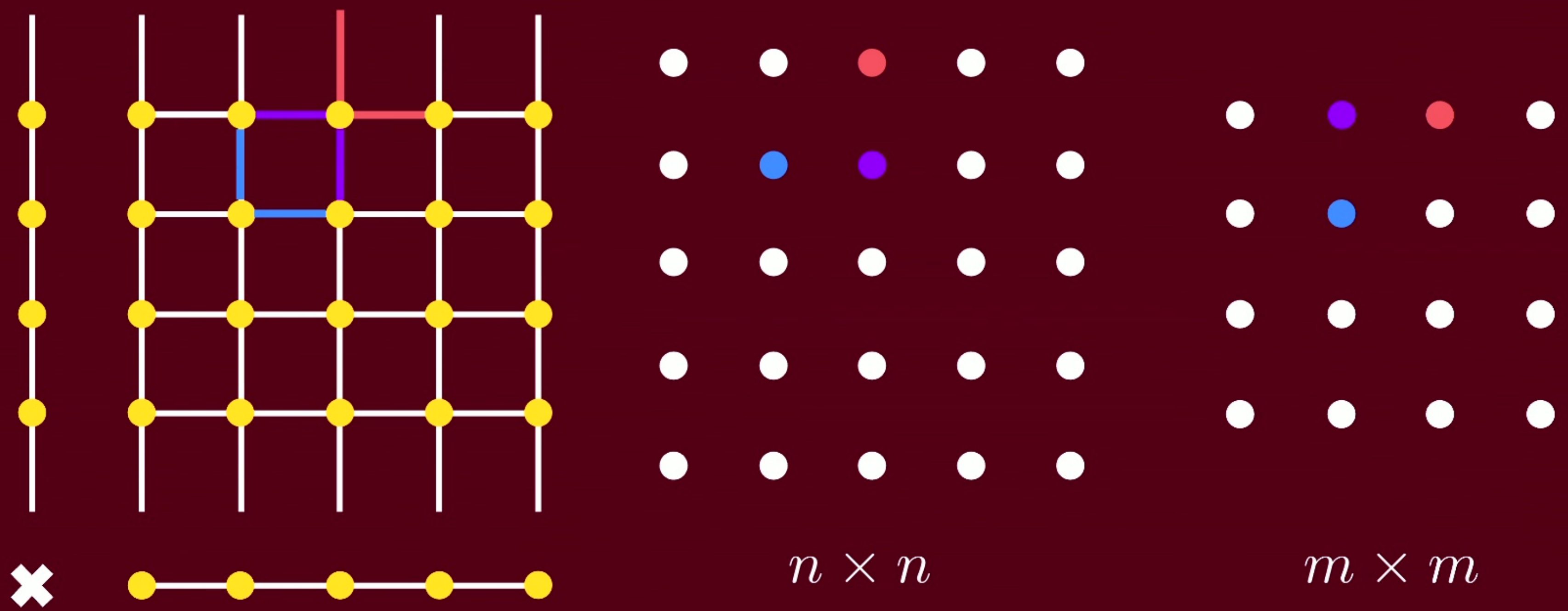


Rainbow codes from hypergraph products

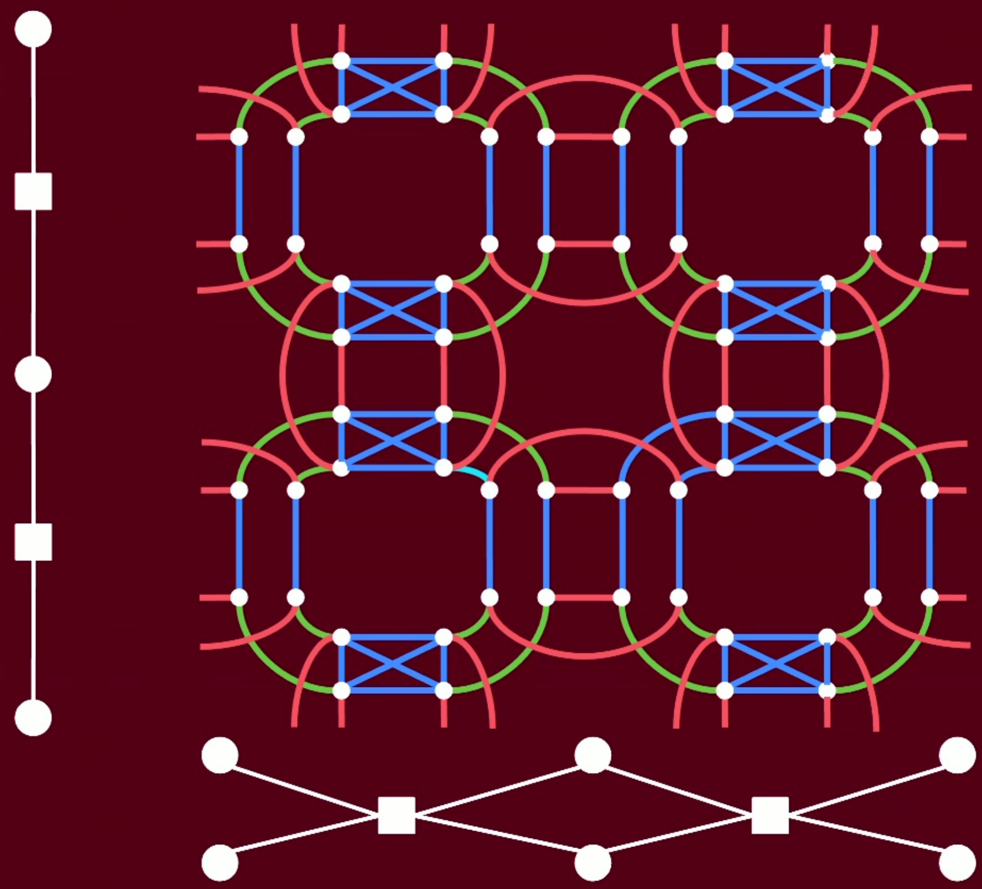
Hypergraph product codes

Note: a hypergraph product can be represented in the following way:

Consequence: the intersection of a “vertex” and a “face” of a hypergraph product must have exactly two “edges”.
⇒ the colour corresponding to “changing edge” must be be present exactly once per vertex of the flag graph

$n \times n$ $\quad$ $m \times m$

Rainbow codes from hypergraph products



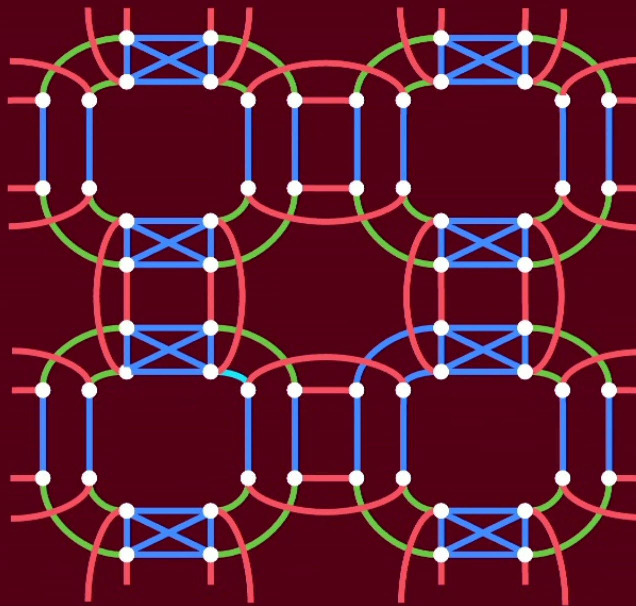
Rainbow codes from hypergraph products

Hypergraph product rainbow code, version 1 (“generic”):

- X stabilizers on maximal subgraphs
- Z stabilizers on rainbow subgraphs

Hypergraph product rainbow code, version 2 (“mixed”):

- In 2D: X and Z stabilizers on {1,2}, {2,3}-rainbow subgraph, and {1,3}-maximal subgraphs
- In 3D:
  - X stabilizers {1,2}, {2,3}, {2,4}, {3,4}-rainbow subgraphs & {1,4}-maximal subgraphs
  - Z stabilizers on {1,2,3}, {2,3,4}-rainbow subgraphs & {1,2,4}, {1,3,4}-maximal subgraphs



Convention:
color 0 = “changing vertex”
color 1 = “changing edge”
etc.

Rainbow codes from hypergraph products

Properties

Let ν_i the number of cycles of the i[th] tanner graph, and D the dimension of the HGP

	Generic version	Mixed version
#Logical qubits	$k = D\sum_i \nu_i$	$k = (D-1)\sum_i \nu_i + \sum_i \prod_{j\neq i} \nu_j$
#Logical qubits (3D)	$k = 3(\nu_1 + \nu_2 + \nu_3)$	$k = 2(\nu_1 + \nu_2 + \nu_3) + \nu_1\nu_2 + \nu_2\nu_3 + \nu_1\nu_3$
Distance	$d = \min_i \lvert\nu_i\rvert$	$d = (2\times)\min_i \lvert\nu_i\rvert$
Triorthogonality	Yes	Yes
Best family (best rate)	$[[n, n^{1/3}, \log(n)]]$	$[[n, n^{2/3}, \log(n)]]$

Rainbow codes from hypergraph products

Properties

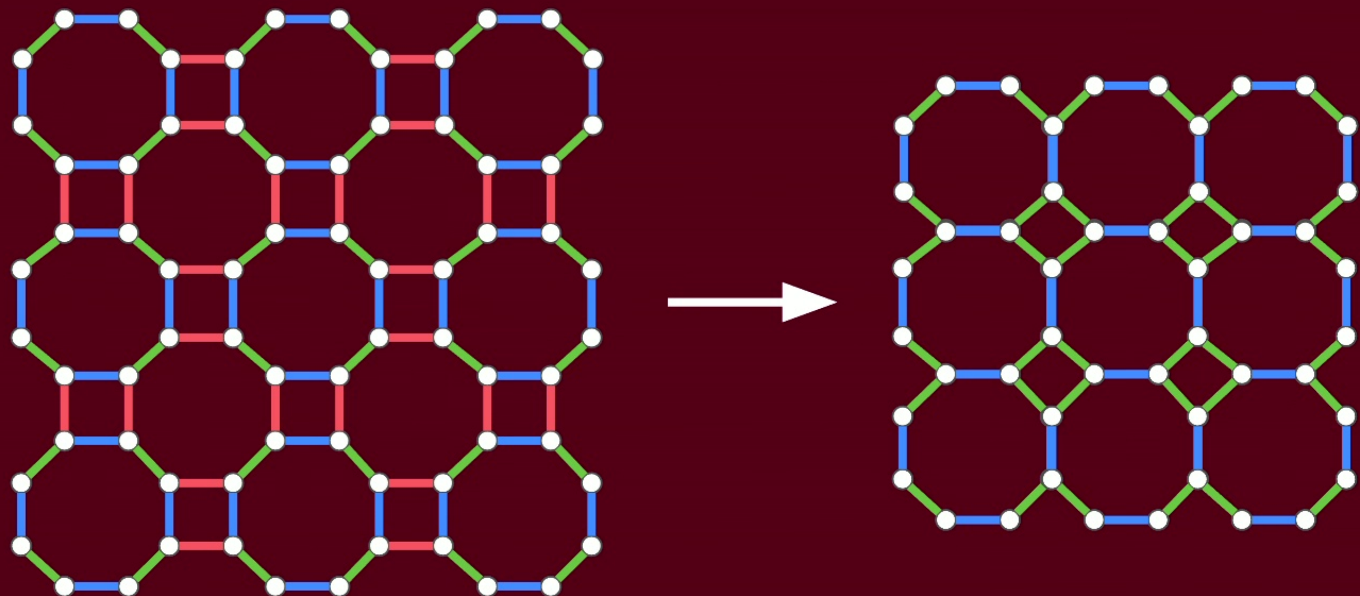
Let ν_i the number of cycles of the i[th] tanner graph, and D the dimension of the HGP

	Generic version	Mixed version
#Logical qubits	$k = D\sum_i \nu_i$	$k = (D-1)\sum_i \nu_i + \sum_i \prod_{j\neq i} \nu_j$
#Logical qubits (3D)	$k = 3(\nu_1 + \nu_2 + \nu_3)$	$k = 2(\nu_1 + \nu_2 + \nu_3) + \nu_1\nu_2 + \nu_2\nu_3 + \nu_1\nu_3$
Distance	$d = \min_i \lvert\nu_i\rvert$	$d = (2\times)\min_i \lvert\nu_i\rvert$
Triorthogonality	Yes	Yes
Best family (best rate)	$[[n, n^{1/3}, \log(n)]]$	$[[n, n^{2/3}, \log(n)]]$
Unfoldable?	Yes	?

Edge contraction of rainbow codes

Idea: by contracting edges of a certain colour, we often obtain a new valid CSS code

Example: we can contract all the blue edges of the 4.8.8. colour code



Discussion

Summary:

1) We found a new generalization of colour codes, that we can study analytically when starting from hypergraph codes, with better properties than their pin code cousins

2) They are the first product construction with full triorthogonality

3) However, their asymptotic parameters are far from ideal and further work is needed to improve them

Open questions:

1) What happens if we apply this construction to other product codes (lifted, balanced, etc.)?
2) Can we address logical qubits individually?
3) Is there a no-go theorem on transversal gates vs parameters for LDPC codes?