Title: Hypertoric 2-Categories O and Symplectic Duality

Speakers: Justin Hilburn

Series: Mathematical Physics

Date: April 25, 2024 - 11:00 AM

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Abstract: In this talk I will present an update on my work with Ben Gammage and Aaron Mazel-Gee on the 2-categories of boundary conditions in the A and B-twists. In particular I will explain how 2-categorical 3d mirror symmetry decategorifies to the Koszul duality of hypertoric categories O discovered by Braden-Licata-Proudfoot-Webster.

Zoom link

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Towards 2-categorical 3d mirror symmetry for non-abelian theories

Based on conversations with

Xin Jin
Ben Gammage
Sanath Devalapurkar

Recollections on 3d $\mathcal{N}=4$ theories

Higgs branch

$$G_H \curvearrowright M_H = T^*(N/G) \Longleftrightarrow$$

$$1 \to G \xrightarrow{i} \tilde{G} \xrightarrow{p} G_H \to 1$$

$$\tilde{G} \circlearrowright T^*N$$

Coulomb branch

$$\mathcal{M}_C = \mathrm{Spec}\, H^{BM}_\bullet(\mathrm{Maps}(\mathbb{B}, N/G))$$

usually singular

mass parameter $\mathbb{C}^\times \xrightarrow{m} T_H \subseteq G_H \Longrightarrow$ Line bundle $\mathcal{L}(m) \to \mathcal{M}_C$

$$\tilde{\mathcal{M}}_C(m) = \mathrm{Proj} \bigoplus_{k \in \mathbb{N}} \mathcal{L}(km) \to \mathcal{M}_C$$

partial resolutions

FI parameter $G \xrightarrow{\tau} \mathbb{C}^\times$

2-categorical 3d mirror symmetry

???? Has not been defined

2-categories

$$KRS(M_H) \cong \mathrm{Fuet}^{wr}(\widetilde{M}_C(m))$$

$L_H(m)$

$\downarrow K_{\mathbb{Q}}^{top}$

depends on holomorphic Liouville sector structure

$\Uparrow$

holomorphic Lagrangian $\widetilde{L}_H \subseteq \widetilde{M}_C$

1-categories

$$\mu KSh_{L_H(m)}(M_H) \cong \mathrm{Fuk}^{wr}(\widetilde{M}_C(m))$$

For sleaves $\mu KSh = \mu Sh$

Betti symplectic duality

Note From here turning on f is straight forward.

Ways of understanding $\mathrm{Fuk}^{wr}(\tilde{\mu}_c(n))$

1) Algebraically categorify $\mathrm{Fuk}^{wr}(\tilde{\mu}_c(n))$

<u>Ex</u> <u>Conj</u>

$$\begin{array}{ccc} \widetilde{\mathrm{Fuk}}^{wr}_{+}(T^*\mathbb{C}) & \simeq & 2\mathrm{Perv}(\mathbb{C},0) \\ \downarrow & & \downarrow \\ \mathrm{Fuk}^{wr}_{+}(T^*\mathbb{C}) & \simeq & \mathrm{Perv}(\mathbb{C},0) \end{array}$$

2) Categorify structural properties of $\mathrm{Fuk}^{wr}(\tilde{\mu}_c(n))$

- stop removal $+ \rightsquigarrow -$ or $|$
- microrestrictions $\tilde{\mu}_c(n) \overset{\text{open}}{\subseteq} X$
- sectorial gluing $\tilde{\mu}_c(n) = \bigcup_{i=1}^{n} U_i \Rightarrow \mathrm{Fuk}^{wr}(\mu_c(n)) \cong \operatorname{colim}_{I \subset 2^n} \mathrm{Fuk}(U_I)$

3) Compatibility with Langlands

- (2,2) Brane boxes

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Ways of understanding $\mathrm{Fuct}^{wr}(\tilde{\mu}_c(n))$

1) Algebraically categorify $\mathrm{Fuk}^{wr}(\tilde{\mu}_c(n))$

Ex Conj $\widetilde{\mathrm{Fuct}}^{wr}(T^*\mathbb{C})_{+} \cong 2\mathrm{Perv}(\mathbb{C},0) =: \{ v \underset{n}{\overset{\ell}{\rightleftarrows}} w \ / \ \text{twists invertible} \}$

$\downarrow \qquad\qquad \downarrow$

$\mathrm{Fuk}^{wr}(T^*\mathbb{C})_{+} \cong \mathrm{Perv}(\mathbb{C},0) \cong \{ v \underset{r}{\overset{\ell}{\rightleftarrows}} w \mid 1-\ell r,\ 1-r\ell \text{ invertible} \}$

2) Categorify structural properties of $\mathrm{Fuk}^{wr}(\tilde{\mu}_c(n))$

- stop removal $+ \rightsquigarrow -$ or $|$
- microrestrictions $\tilde{\mu}_c(n) \overset{\text{open}}{\subseteq} X$
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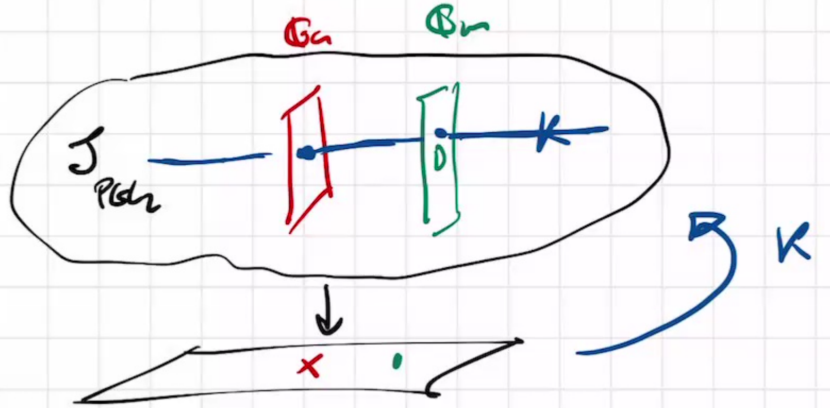
3) Compatibility with Langlands

- (2,2) Brane boxes

Pure gauge theory $\quad \mathcal{M}_H = T^*BG, \quad \mathcal{M}_C = J_{G^L}$

Thm (Jin) $\quad Fuk^{wr}(J_{G^L}) \cong Coh(T_{s.c.} /\!/ W)^{\pi_1 G}$

Ex $\quad G = SL_2$



Conj: $Fuk(J_{PGL_2}) \cong Rep(SL_2)\text{-Mod}$

$$End(K) \cong \mathcal{O}_T^{W} \cong K(BT)^W = K(BG)$$

Observation (H. + Gammage) When $G = G_{s.c.}$

$$Fuk^{wr}(J_{G^L}) \cong KShv(BG) =: \mu KShv(T^*BG)$$

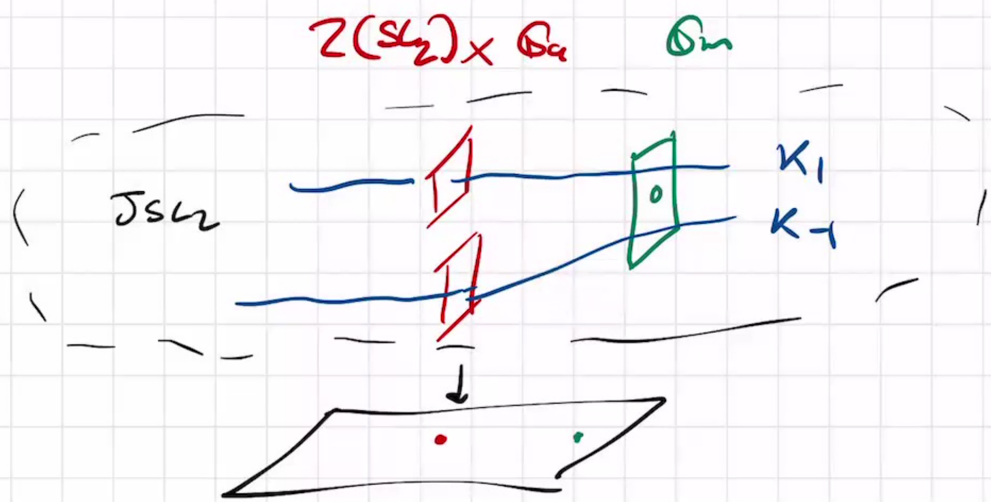
$$K \longmapsto \underline{K}_{BG} \longleftarrow End(\underline{K}_{BG}) = K(BG)$$

Ex $G = PGL_2$

$\mathbb{Z}/2\mathbb{Z} = Z(SL_2) = \pi_1(PGL_2)$

$\circlearrowright$

$\mathcal{O}_{T_{s.c}} \circlearrowleft W$



$\mathrm{Rep}(PGL_2)$
$\|$
$\mathrm{Rep}(SL_2)_{even} \oplus \mathrm{Rep}(SL_2)_{odd}$

$$\mathrm{End}(K_1 \oplus K_{-1}) = \begin{pmatrix} (\mathcal{O}_{T_{s.c}/\!/W})_{triv} & (\mathcal{O}_{T/\!/W})_{sgn} \\ (\mathcal{O}_{T_{s.c}/\!/W})_{sgn} & (\mathcal{O}_{T_{s.c}/\!/W})_{triv} \end{pmatrix} = K\left(\mathrm{End}_{\mathrm{Rep}(PGL_2)}(\mathrm{Rep}(SL_2))\right)$$

Recall $BSL_2 \to BPGL_2$ is $\mathbb{Z}/2\mathbb{Z}$ Gerbe then can be used to twist K-theory $K(BPGL_2)_{-1} := \mathrm{Rep}(SL_2)_{odd}$

Generalization (H. + Gaiotto)

$$FK^{tw}(\mathcal{J}_{SL_2}) \cong KSh(BPGL_2)$$

$$K_1, K_{-1} \longleftrightarrow \underline{K}_{BPGL_2}, \underline{K}_{-1}\, BPGL_2$$

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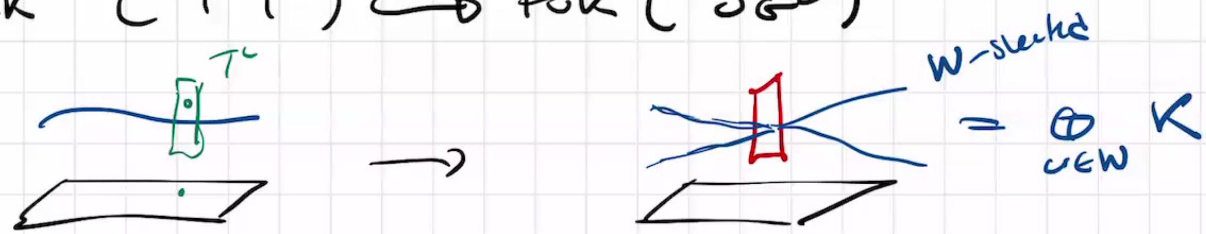
For simplicity assume $G^L = G^L_{ad}$, $G = G_{s.c}$

To define the Coulomb scalar structure on J_{G^L} Jin uses the Toda embedding $T^+T^L = (J_{PGL_2} - K) \hookrightarrow J_{PGL_2}$

Thm (Jin) $Fuk^{wr}(T^+T^L) \hookrightarrow Fuk(J_{G^L})$

$T^+T^L \overset{\sim}{\to} TT^L \to J_G$

$T^+BT \leftarrow BB \times_{BG} TG \to T^+BG$



$$\cong \qquad\qquad\qquad \cong$$

$$Coh(T) \xrightarrow{p_*} Coh(T//W)$$

$$\cong \qquad\qquad\qquad \cong$$

$$KSh(BT) \xrightarrow{PInd_T^G} KSh(BG)$$

$$q^+ \nwarrow KSh(BB) \nearrow p_*$$

Observation (H., Gammage)

↑

dual to Teleman

Conj

$$N^T(BB) \longrightarrow T^*(BT \times BG)$$

Can I recover $T^*T^\vee \hookrightarrow J_G^\vee$?

$$\begin{array}{ccc} & \longrightarrow & BB_K \\ \downarrow \ \lrcorner & & \downarrow \\ BT_G \times BG_G & \longrightarrow & BT_K \times BG_K \end{array}$$

Conj

$$N^\vee(BB^\vee) \longrightarrow T^*(BT \times BG)$$

Can I recover $T^*T^\vee \hookrightarrow J_{G^\vee}$?

$$\underbrace{G_O\backslash G_K/U_K T_O}_{FR^{\infty/2}} = T_O \times G_O \backslash T_K \times G_K / B_K \longrightarrow BB_K$$

$$\begin{array}{ccc} T_O \times G_O \backslash T_K \times G_K / B_K & \longrightarrow & BB_K \\ \downarrow & \lrcorner & \downarrow \\ BT_O \times BG_O & \longrightarrow & BT_K \times BG_K \end{array}$$

$$\mathcal{O}_{T^*T^\vee} \ni 1$$

$$\begin{array}{c} \mathcal{O}_{T^*T^\vee} \\ \| \\ H_0(T_O\backslash T_K/T_O) \end{array} \circlearrowright H_0(G_O\backslash G_K/U_K T_O) \circlearrowleft \begin{array}{c} H_0(G_O\backslash G_K/G_O) \\ \| \\ \mathcal{O}_{J_{G^\vee}} \end{array}$$

$$\mathcal{O}_{T^*T^\vee} \cong H_0(G_O\backslash G_K/U_K T_O) \longleftarrow$$

$$\psi \quad 1$$

Conj $N^T(BB) \longrightarrow T^*(BT \times BG)$

Can I recover $T^*T^L \xrightarrow{i} J_{G^L}$?

$$G_O\backslash G_K/U_K T_O = T_O \times G_O \backslash T_K \times G_K / B_K \longrightarrow BB_K$$

$U_K T_O$: $Fl^{\infty/2}$

$$\begin{array}{ccc} T_O \times G_O \backslash T_K \times G_K / B_K & \longrightarrow & BB_K \\ \downarrow & \lrcorner & \downarrow \\ BT_O \times BG_O & \longrightarrow & BT_K \times BG_K \end{array}$$

$$\mathcal{O}_{T^*T^L} \ni 1$$

$$\|$$

$$H_0(T_O\backslash T_K/T_O) \circlearrowright H_0(G_O\backslash G_K/U_K T_O) \circlearrowleft H_0(G_O\backslash G_K/G_O)$$

Conj $\mathcal{O}_{T^*T^L} \cong H_0(G_O\backslash G_K/U_K T_O) \xleftarrow{i^*} \mathcal{O}_{J_{G^L}}$

$1 \in \mathcal{O}_{T^*T^L}$, $\quad \mathcal{O}_{J_{G^L}} \ni m^{\Sigma \omega_i}$

Invert this element $\longrightarrow$ $m^{\Sigma \omega_i}$

Attracting correspondences and sectional covers of Coulomb branches

The mass parameter gives rise to an attracting set Lagrangian

$$\mathcal{M}_H^{\mathbb{G}_m} \longleftarrow \mathcal{M}_H^{\mathbb{G}_m,+} \longrightarrow \mathcal{M}_H$$

Defined in the following way

Notation $X \xrightarrow{f} Y$ spaces over $B\mathbb{G}_m$, $K: B\mathbb{G}_m \to B\mathbb{G}_m$

$$\begin{array}{ccc} \mathrm{Maps}(X,Y)_K & \longrightarrow & \mathrm{Maps}(Y, B\mathbb{G}_m) \\ \downarrow & & \downarrow f^+ \\ \{X \to B\mathbb{G}_m \xrightarrow{K} B\mathbb{G}_m\} & \longrightarrow & \mathrm{Maps}(X, B\mathbb{G}_m) \end{array}$$

Some spaces over $B\mathbb{G}_m$ are

$\mathcal{M}_H/\mathbb{G}_m$ $\qquad \mathbb{G} \qquad \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \hookleftarrow 0/\mathbb{G}_m$

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By summing over $\mathrm{Maps}(-, M_H/G)_K$ for all $K \in \mathbb{Z}$

get

$$\underbrace{\coprod_K \mathrm{Maps}(BG_m, M_H/G)_K}_{M_H^{G_m,0}} \longleftarrow \underbrace{\coprod_K \mathrm{Maps}(\mathbb{A}^1/G_m, M_H/G)_K}_{M_H^{G_m,+}} \longrightarrow M_H$$

As a component this has

$BT \leftarrow BB \rightarrow BG$

Ex $\mathbb{C}^\times \curvearrowright BG$ trivial

$$\coprod_{\gamma: G_m \to G} BG^{\gamma,0} \longleftarrow \coprod_{\gamma: G_m \to G} BG^{\gamma,+} \longrightarrow BG$$

$$\coprod BL \longleftarrow \coprod BP \longrightarrow BG$$

Conjecture Components of

$$M_H^{\mathbb{G}_m, 0} \longleftarrow M_H^{\mathbb{G}, +} \longrightarrow M_H$$

are dual to open subsets

$$D_\gamma \hookrightarrow \widetilde{M}_C(m) \longrightarrow M_C$$

$$\begin{array}{ccc} \| & & \| \\ \{ s_\gamma \neq 0 \} & & \mathrm{Proj} \bigoplus_{k \geq 0} L(km) \\ & & \in \\ & & s_\gamma \end{array}$$

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$$H_0(G_O \backslash G_K / G_O) \circlearrowright H_G(pt)$$

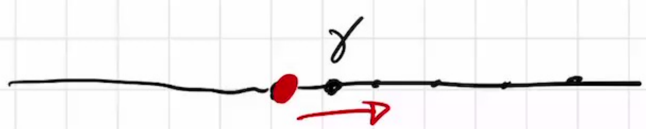
$$\|$$

$$\mathbb{Z} \in (\pi_1 G)^\vee \circlearrowright M_C$$

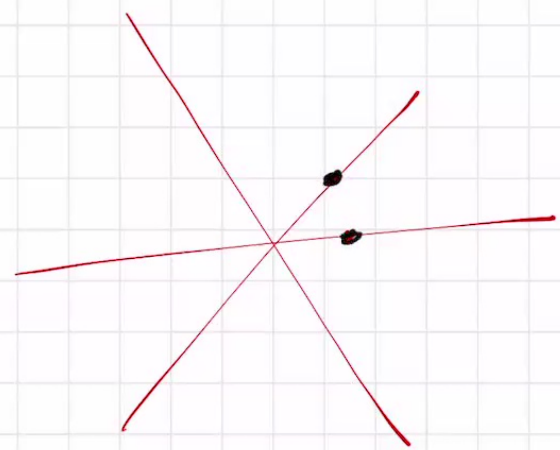
alter

G/PGL_2

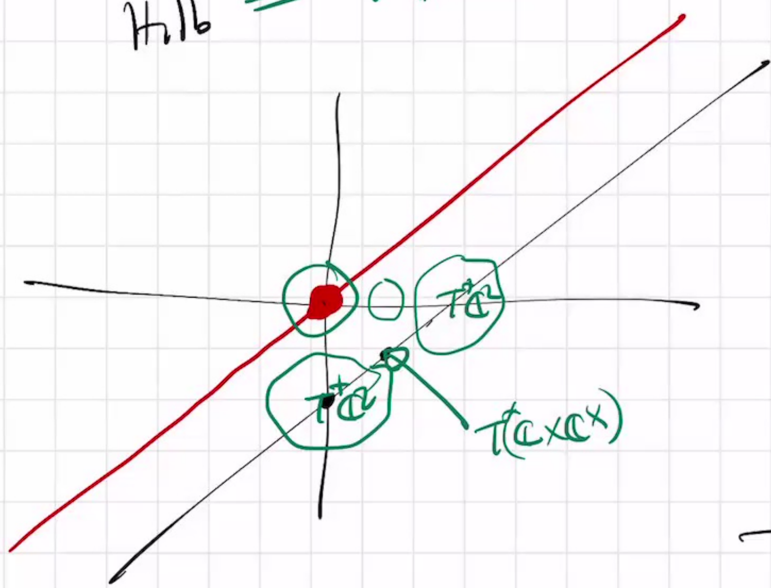
t



$$D(s_\gamma) = T^\dagger T$$



$\mathrm{Hilb} = T^*\mathbb{P}^1 \times T^*$



$\bigoplus \mathcal{O}(k)$

$k \geq 1$

$\in$ torsor

$p^{-1}(1)$

$T \longrightarrow \tilde{T} \xrightarrow{\rho} \mathbb{C}^n$

hard symplectic geometry of setting of Coulomb branch structure

$\mathrm{Fuk}^{cw}(D_\gamma) \hookrightarrow \mathrm{Fuk}^{cw}(\tilde{\mathcal{M}}_{C(\gamma)})$

dual to parabolic

$\mathrm{KSh}(M^{\mathrm{dom},o}(\gamma)) \longrightarrow \mathrm{KSh}(M)$