Title: Mathematical Physics Lecture

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$M^3 = \partial N^4$

A gauge field on M

extend to N

$$\int_{M^3} CS(A) = \int_{N^4} tr(F(A) \wedge F(A))$$

Let's write $\mathrm{tr}\, F(A) \wedge F(A)$ as a total derivative

$$F(A) = dA + \tfrac{1}{2}[A, A]$$

$$\mathrm{tr}\, F(A)^2 = \mathrm{tr}\; dA \wedge dA + \mathrm{tr}\,[A, A]\, dA + \tfrac{1}{4}\mathrm{tr}\,([A, A][A, A])$$

$$\mathrm{tr}\,[A A][A A] = \mathrm{tr}(A\,[A\,[A A]])$$

$$= \mathrm{tr}(A_i [A_j [A_k A_l]])\, \varepsilon^{ijkl} = 0 \text{ by Jacobi}$$

$$\operatorname{tr} F(A)^2 = \operatorname{tr}(dA\,dA + dA[AA])$$

$$= d \operatorname{tr}\left(A\,dA + \tfrac{1}{3} A[AA]\right)$$

$$= d\, CS(A)$$

We're interested in 3d theory
with Lagrangian

$$\frac{k}{2\pi}\int_{M^3} CS(A)$$

Let's check this is gauge invariant

2 steps

1) Small gauge trans $\delta A = d\chi + [A, \chi]$

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1) Small gauge trans $\qquad \delta A = d\chi + [A,\chi]$

2) Large gauge trans

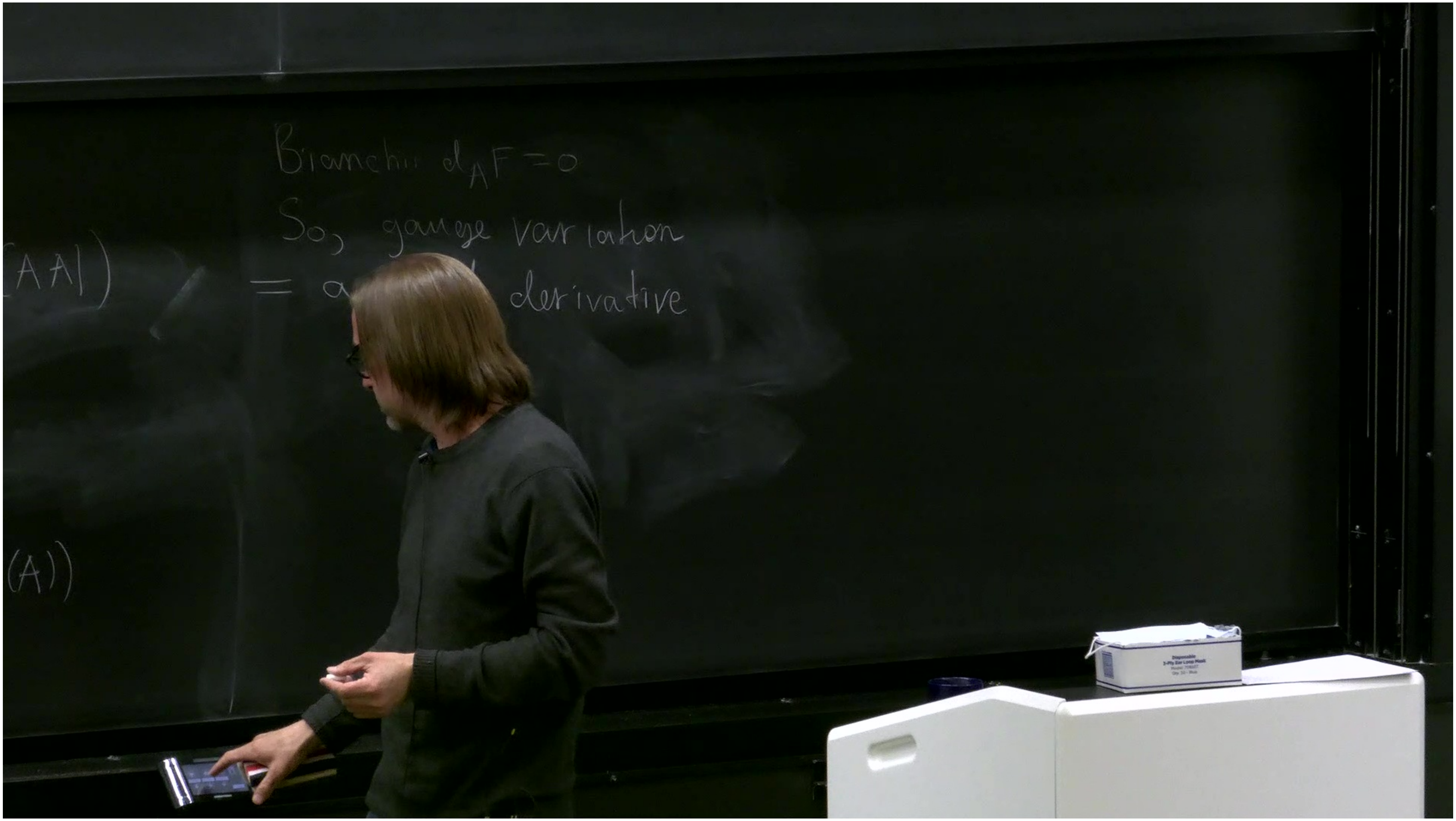
1)

$$CS(A+\delta A) = tr(\delta A\, dA + A\, d\delta A + \delta A[A A])$$

$$= 2tr(\delta A\, F(A) + d\, tr(A\, \delta A)$$

$$(\Rightarrow \text{EOM are } F(A)=0)$$

$$CS(A+d\chi+[A,\chi]) = d\, tr(A(d\chi+[A,\chi]) + tr(d_A \chi F(A))$$


Bianchi: $d_A F = 0$
So, gauge variation
derivative

What about large gauge transformations?

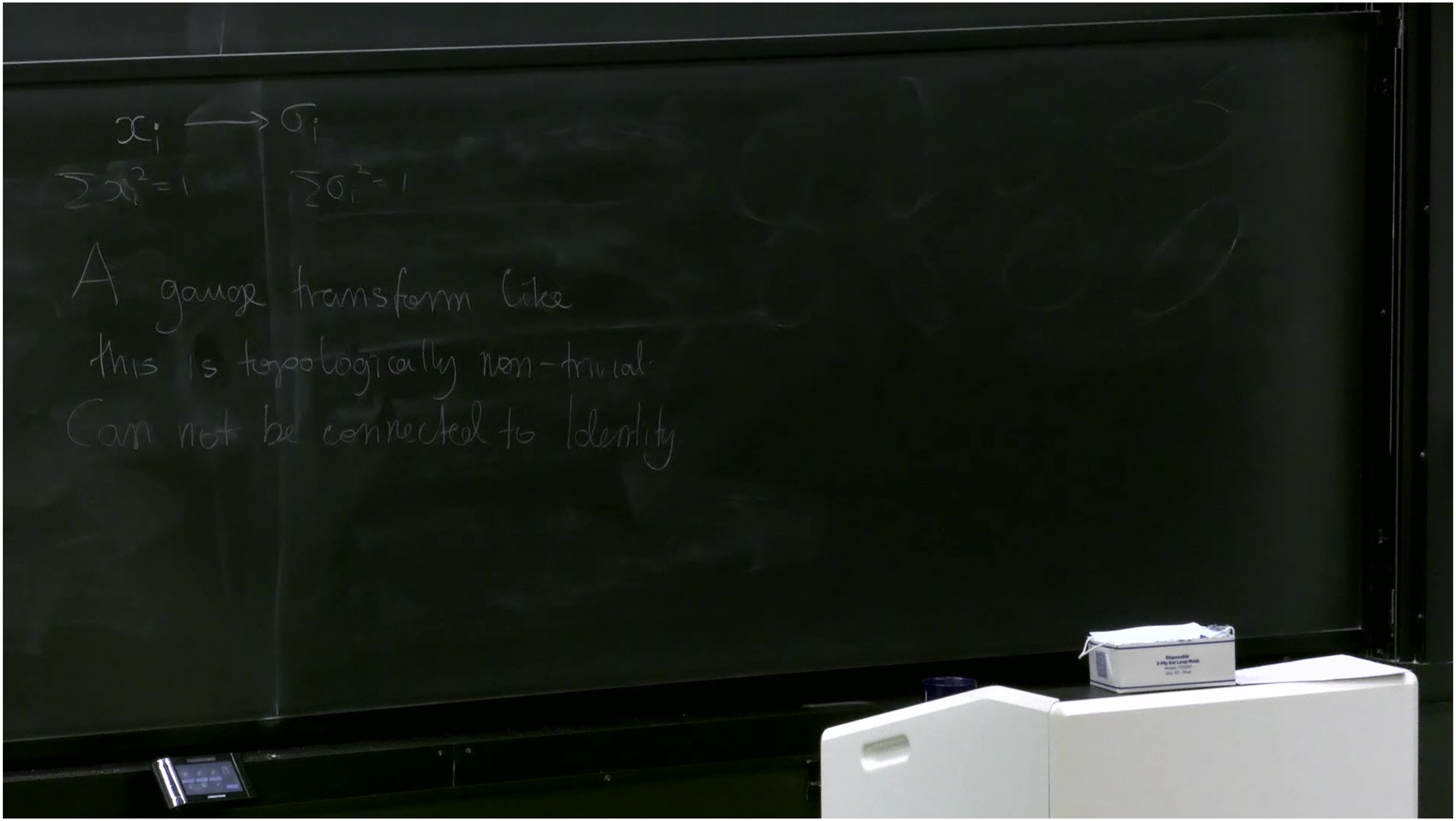
Ex $M^3 = S^3$

Gauge group is $SU(2) = 3$

Gauge transformation

$S^3 \longrightarrow SU(2)$ which is

an isomorphism


$x_i \longrightarrow \sigma_i$
$\sum x_i^2 = 1$ $\sum \sigma_i^2 = 1$
A gauge transform like
this is topologically non-trivial.
Can not be connected to Identity.

If $\sigma : S^3 \longrightarrow SU(2)$

is my gauge transformation

Trivial connection $A = 0 \rightsquigarrow \sigma^{-1} d\sigma$

We need to look at

$CS(\sigma^{-1} d\sigma)$

$$\mathrm{tr}\,[A\,A][A\,A] = \mathrm{tr}(A\,[A\,[A\,A]])$$

$$= \mathrm{tr}(A_i[A_j\,[A_k\,A_l]])\,\varepsilon^{ijkl} = 0 \text{ by Jacobi}$$

$$CS(\sigma^{-1}d\sigma) = \mathrm{tr}\left(\sigma^{-1}d\sigma\; d(\sigma^{-1}d\sigma) + \tfrac{1}{3}\sigma^{-1}d\sigma\,[\sigma^{-1}d\sigma, \sigma^{-1}d\sigma]\right)$$

$$d\sigma^{-1} = -\sigma^{-1}(d\sigma)\sigma^{-1}$$

$$CS(\sigma^{-1}d\sigma) \propto \mathrm{tr}((\sigma^{-1}d\sigma)^3)$$

On the manifold $SU(2) = S^3$, there is a volume form which is $\mathrm{tr}((X^{-1}dX)^3)$

$X \in SU(2)$

Conclude

$$\int_{S^3} CS(\sigma^{-1} d\sigma) \propto \int_{S^3} \sigma^* d\,\mathrm{Vol}_{SU(2)}$$

This is the 3d version of winding.

which is $tr((X^{-1}dX)^3)$
$X \in SU(2)$

is an integer measures how many times we wrap S^3
$(\times \pi \ldots)$

Conclusion

$e^{\frac{ik}{2\pi}\int CS(A)}$ is gauge invariant

even under large gauge transformations

Example

$G = U(1)$

$M^3 = S^1 \times \Sigma$

θ coord. on S^1

If my gauge transform is

$f(\theta) = e^{2\pi i \theta}$

$\delta A = 2\pi i \, d\theta$

Example

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If my gauge transform is

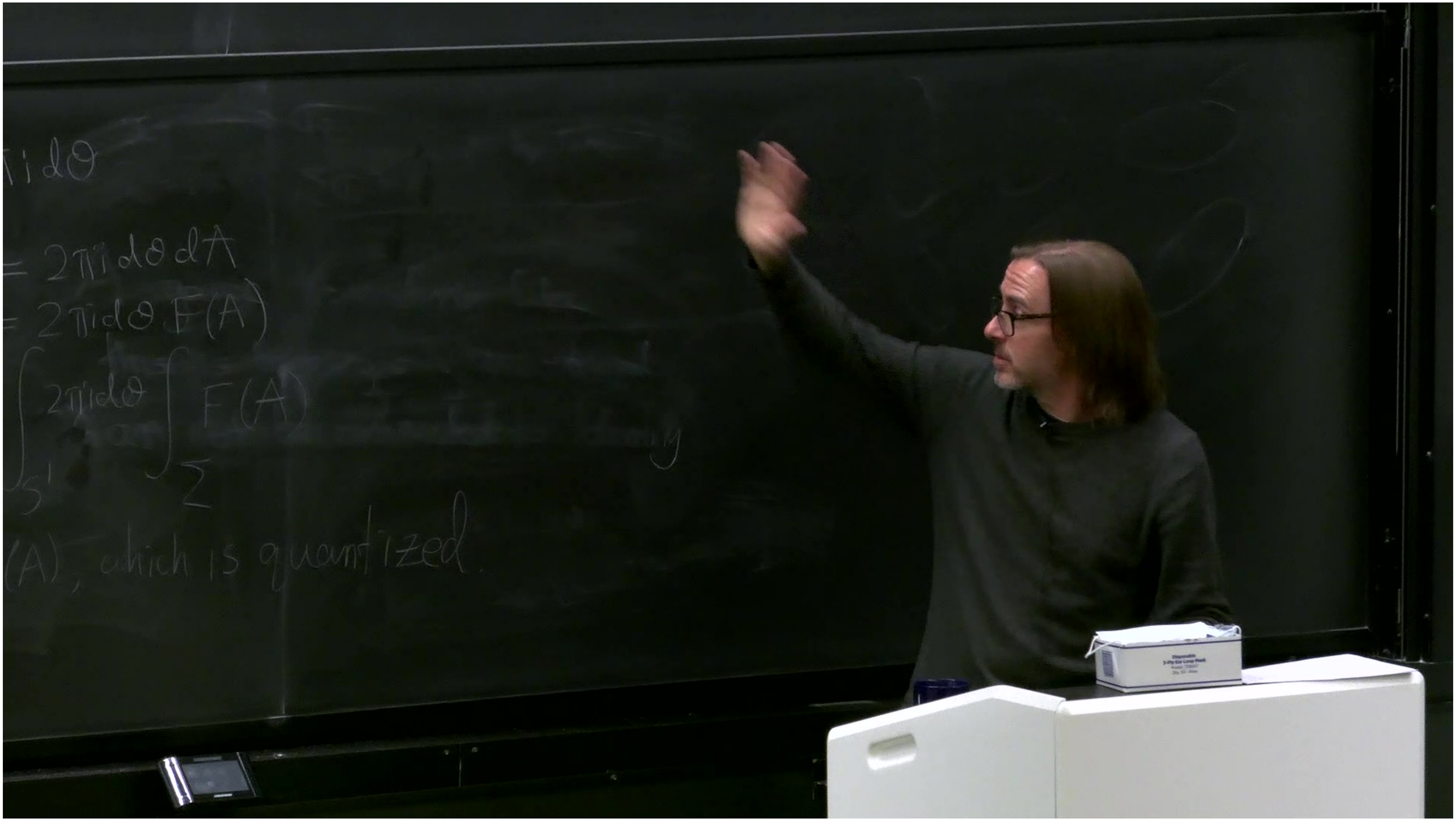
$$f(\theta) = e^{2\pi i \theta}$$

$$\delta A = 2\pi i \, d\theta$$

$$\delta(A\,dA) = 2\pi i \, d\theta \, dA$$
$$= 2\pi i \, d\theta \, F(A)$$

$$\delta \int CS(A) = \int_{S^1} 2\pi i \, d\theta \int_{\Sigma} F(A)$$

$\propto \int_{\Sigma} F(A)$, which is quant


$\pi i d\theta$
$= 2\pi i \, d\theta \, dA$
$= 2\pi i \, d\theta \, F(A)$
$\int_{S^1} 2\pi i \, d\theta \int_{\Sigma} F(A)$
(A), which is quantized.

Other terms which are quantized

M symplectic manifold

$\gamma : \mathbb{R} \longrightarrow M$

Extend γ to a map

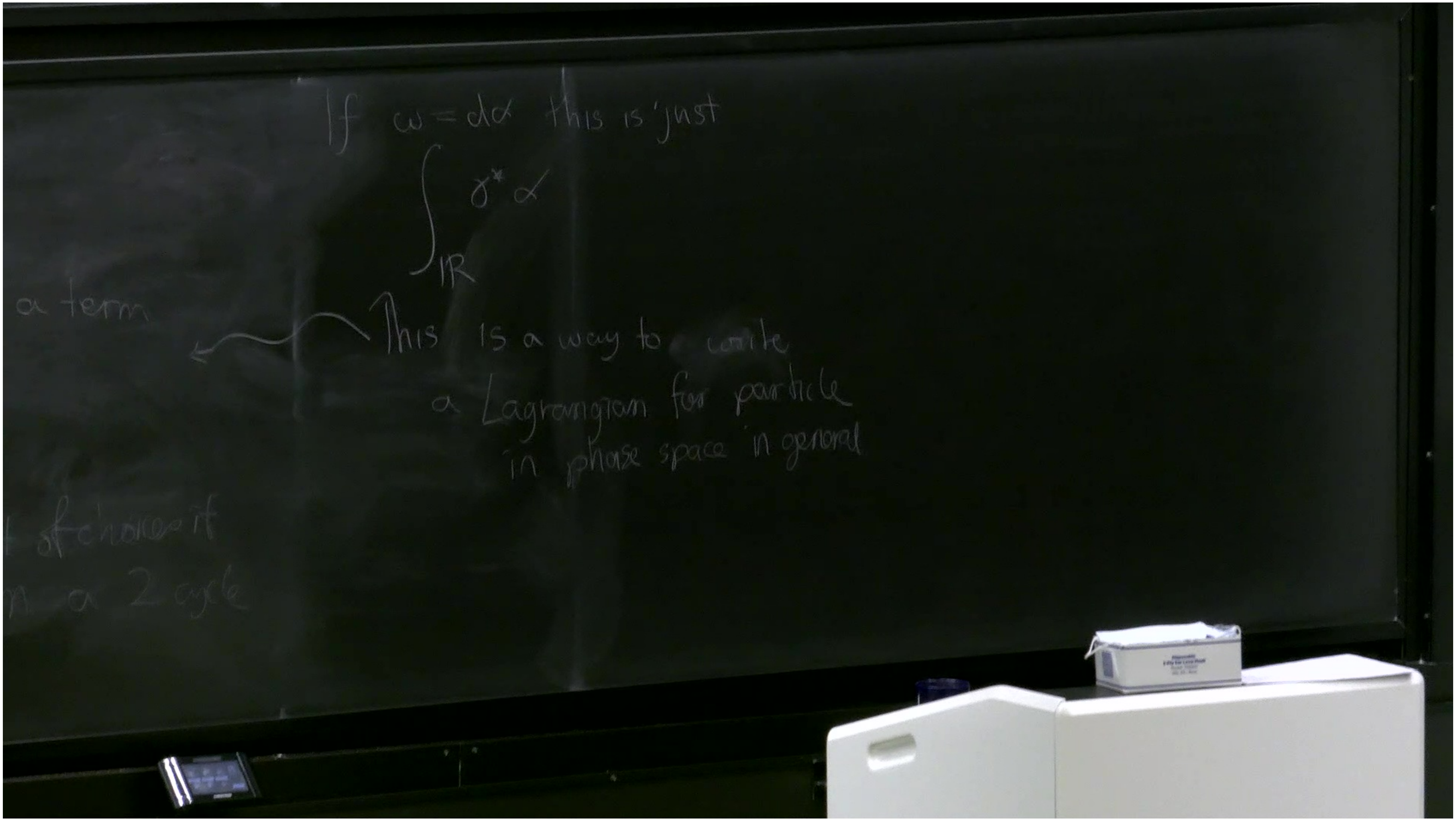
$\tilde{\gamma} \; \mathbb{R}\times\mathbb{R}_{\geq 0} \longrightarrow M$

We can have a term

$$\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \tilde{\gamma}^{*}\omega$$

This is independent of choice if

$\int_{\Sigma} \omega \in \mathbb{Z}$, $\Sigma \subseteq M$ a 2 cycle


If $\omega = d\alpha$ this is 'just
$\int_{\mathbb{R}} \gamma^* \alpha$
a term
This is a way to write
a Lagrangian for particle
in phase space in general
of choices if
a 2 cycle

If N^d is a d-manifold

X is a manifold with

a $d+1$-form $\Omega \in \Omega^{d+1}(X)$

$d\Omega = 0$, $\int_{\substack{d+1 \\ \text{manifold}}} \Omega \in \mathbb{Z}$

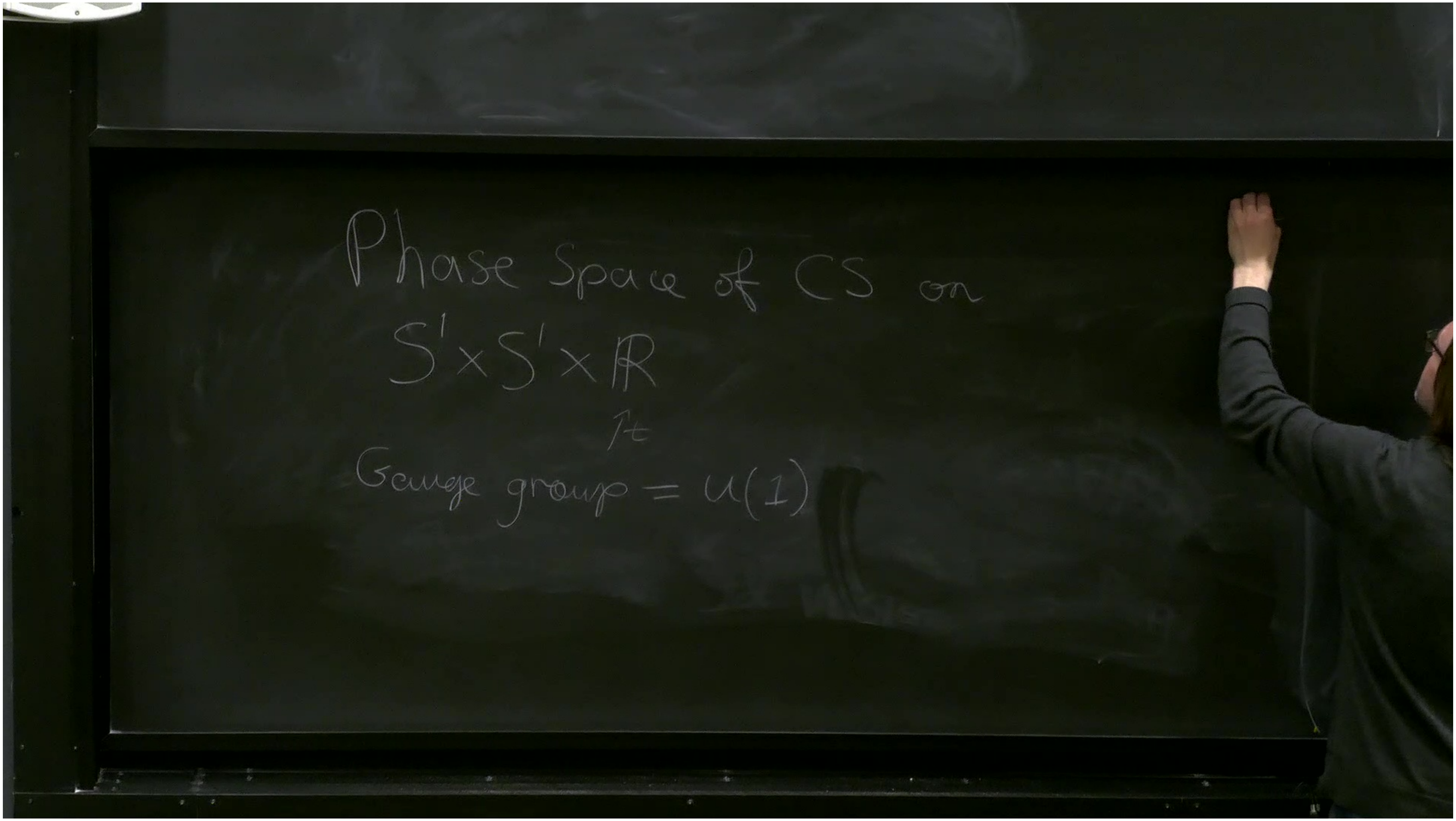
Then if

Then if $\sigma : N^d \to X$

is a field, can have a term

$$\int_{M^{d+1}} \sigma^* \Omega \qquad \partial M^{d+1} = N$$

This is the Wess-Zumino term


Phase Space of CS on
$S^1 \times S^1 \times \mathbb{R}$
t
Gauge group $= U(1)$

Equations of motion tell us we have

a flat $U(1)$ bundle A, $F(A) = 0$

$$A = A_1 \, d\theta_1 + A_2 \, d\theta_2 + A_t \, dt$$

Satisfying $\partial_2 A_1 = \partial_1 A_2$

$$\partial_i A_t = \partial_t A_i$$

up to gauge tr

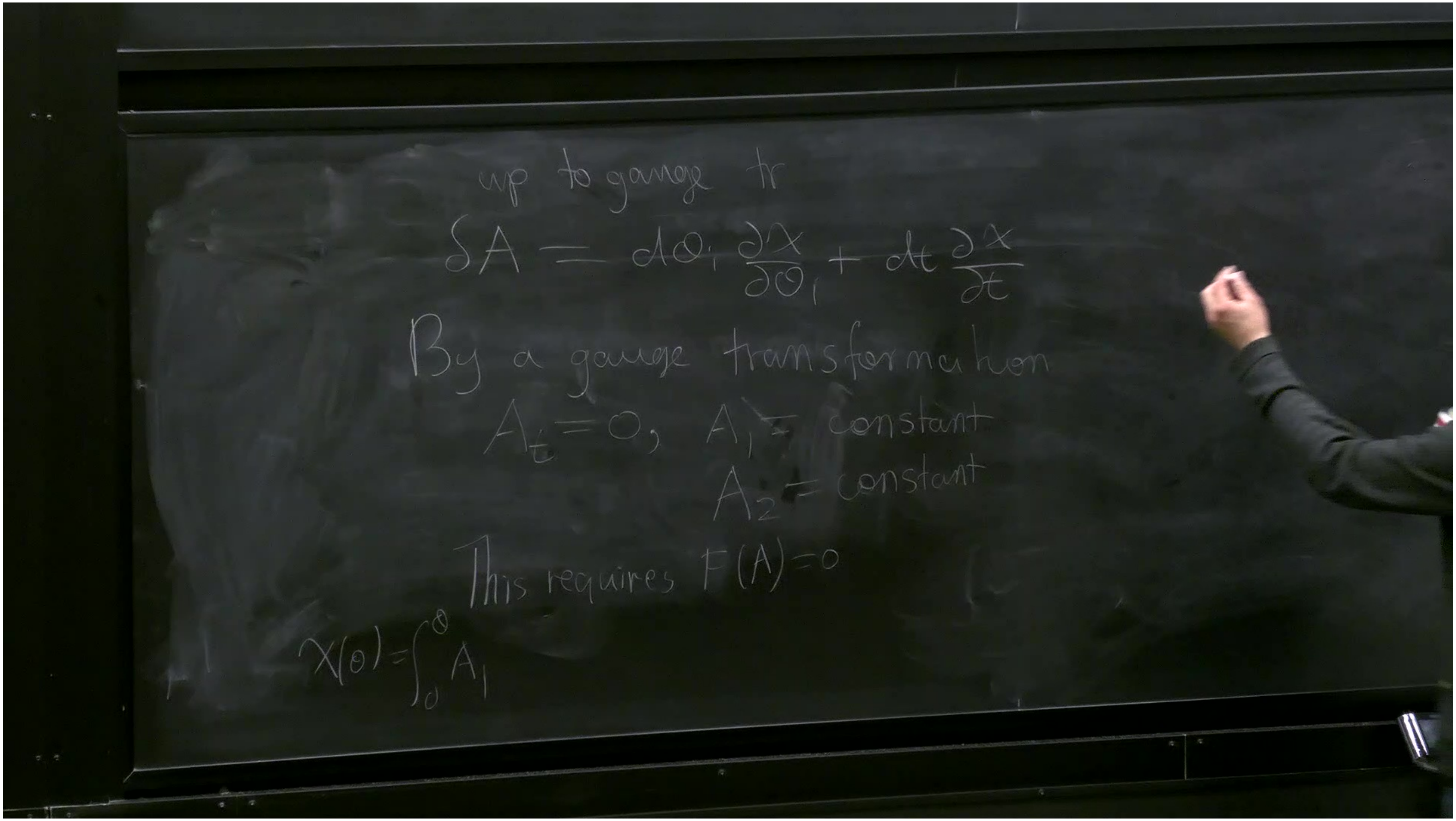
$$\delta A = d\theta_i \frac{\partial x}{\partial \theta_i} + dt \frac{\partial x}{\partial t}$$

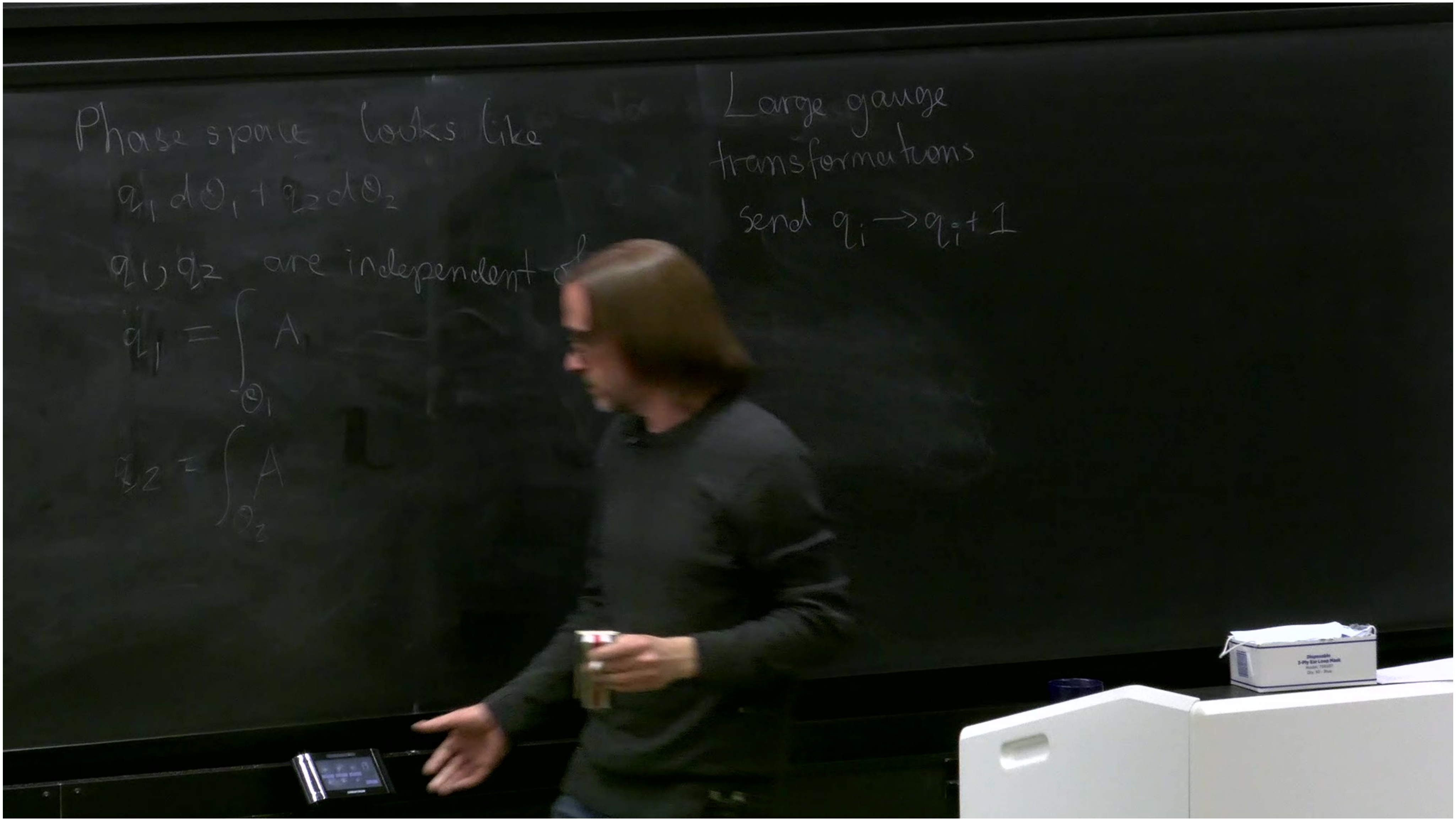
By a gauge transformation

$$A_t = 0, \quad A_1 = \text{constant}$$

$$A_2 = \text{constant}$$

This requires $F(A) = 0$


up to gauge tr
$\delta A = d\theta_1 \frac{\partial \chi}{\partial \theta_1} + dt \frac{\partial \chi}{\partial t}$
By a gauge transformation
$A_t = 0, \; A_1 = \text{constant}$
$A_2 = \text{constant}$
This requires $F(A) = 0$
$\chi(\theta) = \int_0^\theta A_1$

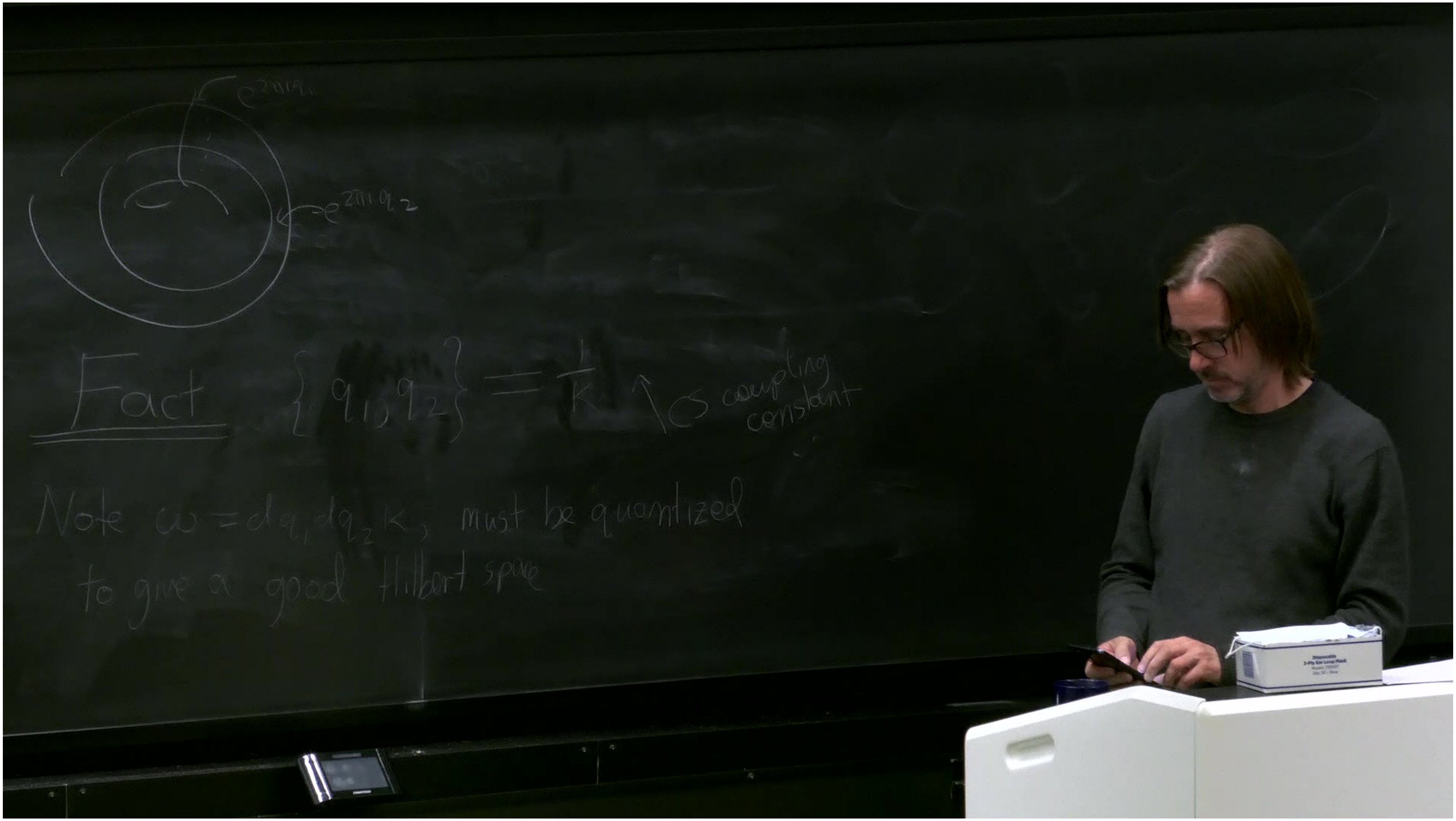

Phase space looks like
$q_1 d\Theta_1 + q_2 d\Theta_2$
q_1, q_2 are independent of
$q_1 = \int_{\Theta_1} A$
$q_2 = \int_{\Theta_2} A$
Large gauge
transformations
send $q_i \to q_i + 1$

Phase space is

$$S^1 \times S^1$$

$$e^{2\pi i q_1} = \text{A Wilson line in CS wraps } \theta_1 \text{ cycle}$$

$$e^{2\pi i q_2} = \text{'' } \theta_2 \text{ cycle}$$


$e^{2\pi i q_1}$
$e^{2\pi i q_2}$
Fact $\{q_1, q_2\} = \frac{1}{k}$ is coupling constant
Note $\omega = dq_1 dq_2 k$, must be quantized
to give a good Hilbert space