Title: The Ising Model on S^2 - The Affine Conjecture

Speakers: Richard Brower

Series: Quantum Matter

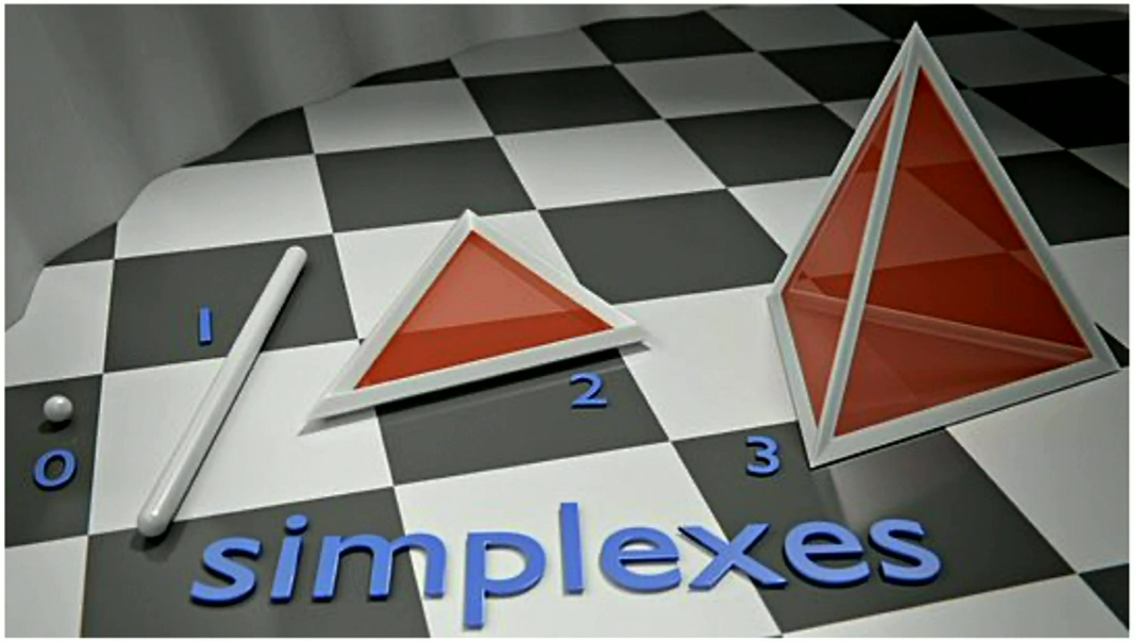
Date: April 19, 2024 - 11:00 AM

URL: https://pirsa.org/24040102

Abstract: A formulation of the 2-dimensional Ising model on a triangulated Riemann sphere is proposed that converges to the exact conformal field theory (CFT) in the continuum limit. The solution is based on reconciling Regge's simplicial geometry for the Einstein Hilbert action with an Affine map to quantum correlators on the tangent plane. Numerical tests of the 2d Ising sphere and radial quantized phi4 theory on $R x S^2$ are presented. Extending the method to more general fields theories on curved manifolds is discussed.

Zoom link

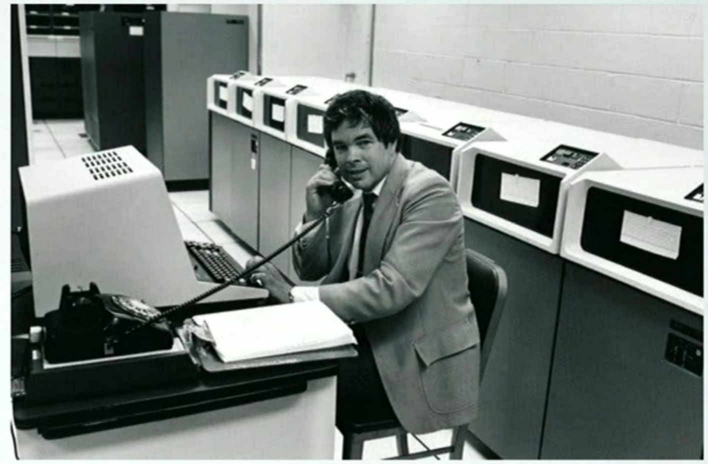
Ising Model on the Riemann Sphere
The Affine Conjecture



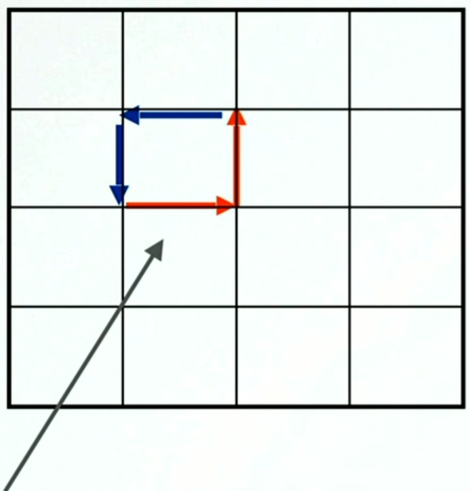
Perimeter, April 19, 2024

Richard Brower, Boston University

FIRST 50 YEARS: WILSON'S LATTICE QCD*



$$Z_{wilson} = \int_{Haar} dU \, \mathbf{e}^{\frac{6}{g^2} \sum_{\Box} Tr[U_{\Box} + U_{\Box}^{\dagger}]}$$



* K.G. Wilson, Phys. Rev. D 10 (1974), 2445.

$$U^{ij}(x, x+\mu) = e^{iagA_{\mu}^{ij}(x)}$$

$$i, j = 1, 2, 3$$

SU(3) Gauge Transport on each link.
Exact per site gauge invariance

$$U_{\Box_{\mu\nu}}(x) = [U(x, x+\mu)U(x+\mu, x+\mu+\nu)][U(x, x+\nu)U(x+\nu, x+\nu+\mu)]^{\dagger}$$
$$\simeq 1 + a^2 iF_{\mu\nu} - (a^4/2)F_{\mu\nu}^2 + \cdots$$

GOAL

- CONSTRUCT LATTICE FIELD THEORIES ON CURVED MANIFOLDS THAT GIVE EXACT RESULTS IN THE CONTINUUM LIMIT

- PRINCIPLE MAXIMALLY SYMMETRIC SPACES

- Conformal Field Theories are more easily studied on Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces (Gauge/Gravity Duality)

$\mathbb{S}^d$ $\mathbb{R} \times S^{d-1}$ $\mathbb{A}d\mathbb{S}^{d+1}$

Ok, BE REALISTIC TO GET GOING!

> The art of doing mathematics consists finding that special case which contains all the germs of generality.

David Hilbert
Mathematician, Physicist, Philosopher

Author of Geometry and the Imagination

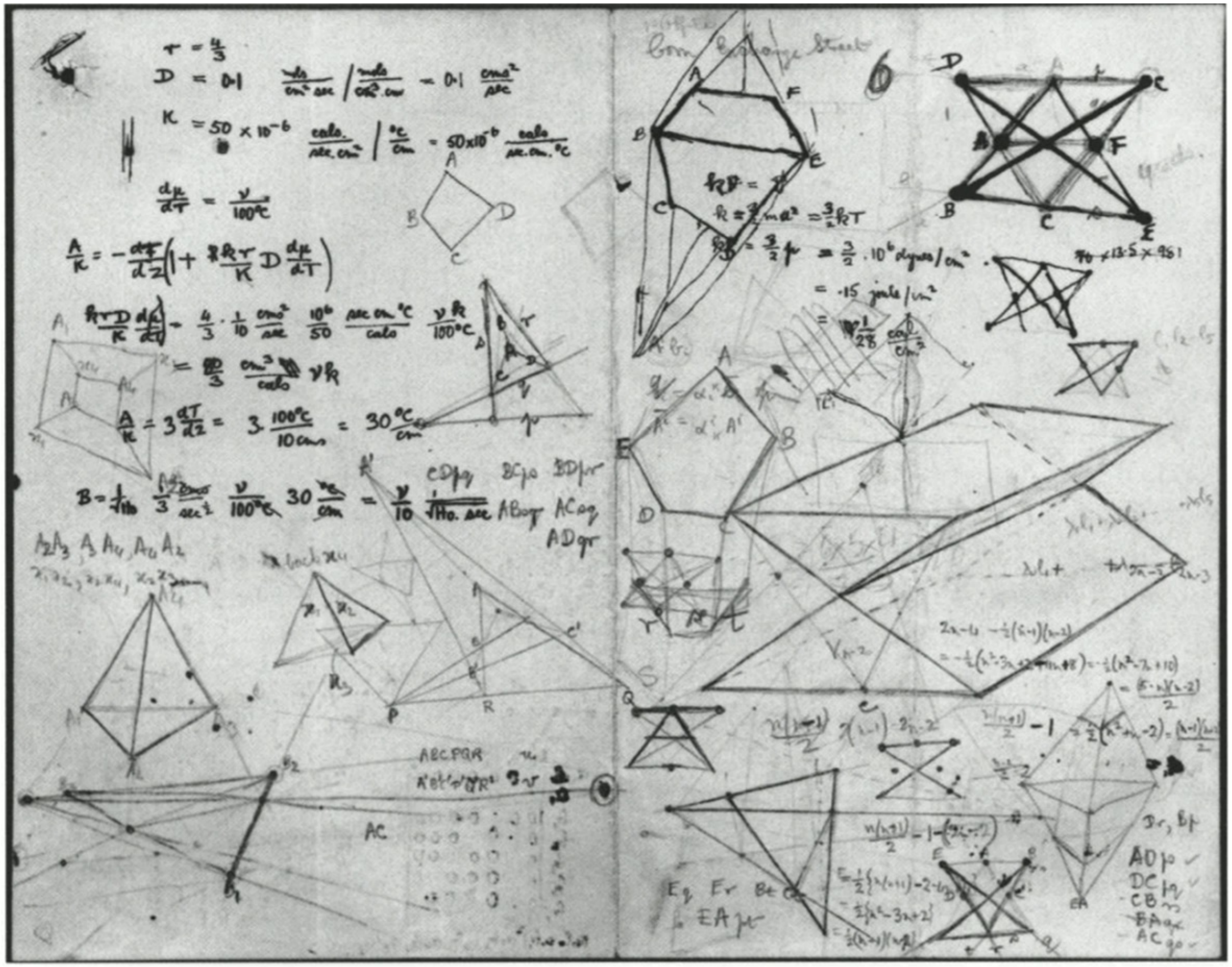
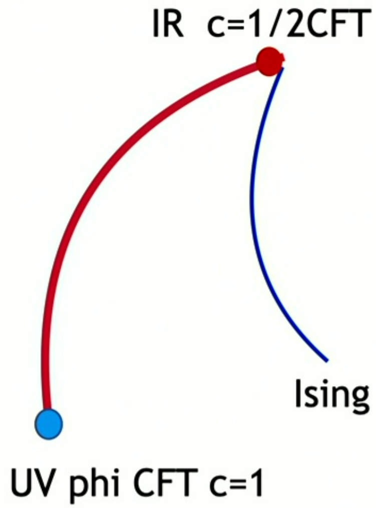
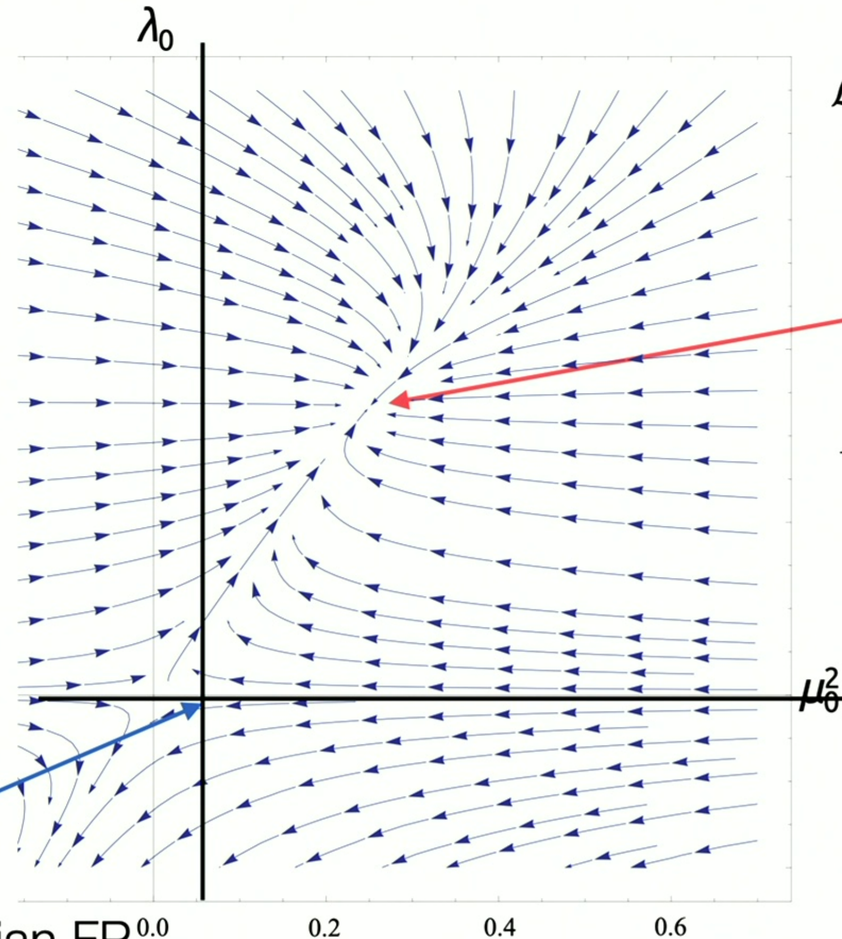


FIGURE 1. Paul Dirac, Geometrical Sketches, in the Paul A. M. Dirac Papers, Florida State University, Tallahassee, Florida; hereafter PDP. By permission of the Florida State University Libraries.

Scalar Phi4/Ising Model





$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

$$H_{Ising} = -\sum_{\langle i,\rangle j} K s_i s_j = \frac{K}{2}\sum_{\langle i,\rangle j}(s_i - s_j)^2$$

Magic of
UNIVERSALITY:
Long Distance (IR)
Microscopic physics
(UV)

First step: Construct the Classical Simplicial Action

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x\sqrt{g(x)}\,[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

$g_{\mu\nu}(x)$

Quantum field $\phi(x)$

Regge Calc Geometry

Finite Element Method

Classical Simplicial Action

$$S_{FEM} = \frac{1}{2}[\sum_{y\in\langle x,y\rangle} K_{xy}(\phi_x - \phi_y)^2 + \sqrt{g_x}[\xi\ Ric\ \phi_x^2 + m_0^2\phi_x^2 + \lambda_0\phi_{t,x}^4]]$$

BACK TO THE FUTURE

1974 WILSON "Confinement of Quarks" LATTICE QCD

1985 CARDY "Universal amp in Finite Size Scaling"

1961 REGGE "General Relativity without Coordinates"

1984 T D LEE et al " Lattice Gravity Near the Continuum"

202? Ising Solution and "The Affine Conjecture?"

Classical Field Geometry

$$S = S_{EH} + S_M = \int dx \sqrt{g}\left[\frac{1}{2\kappa} R + \mathcal{L}_M\right]$$

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \implies \quad R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x)$$

Quantum Field Geometry

$$\frac{\delta \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle}{\delta g^{\mu\nu}(x)} = \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) T_{\mu\nu}(x)\rangle$$

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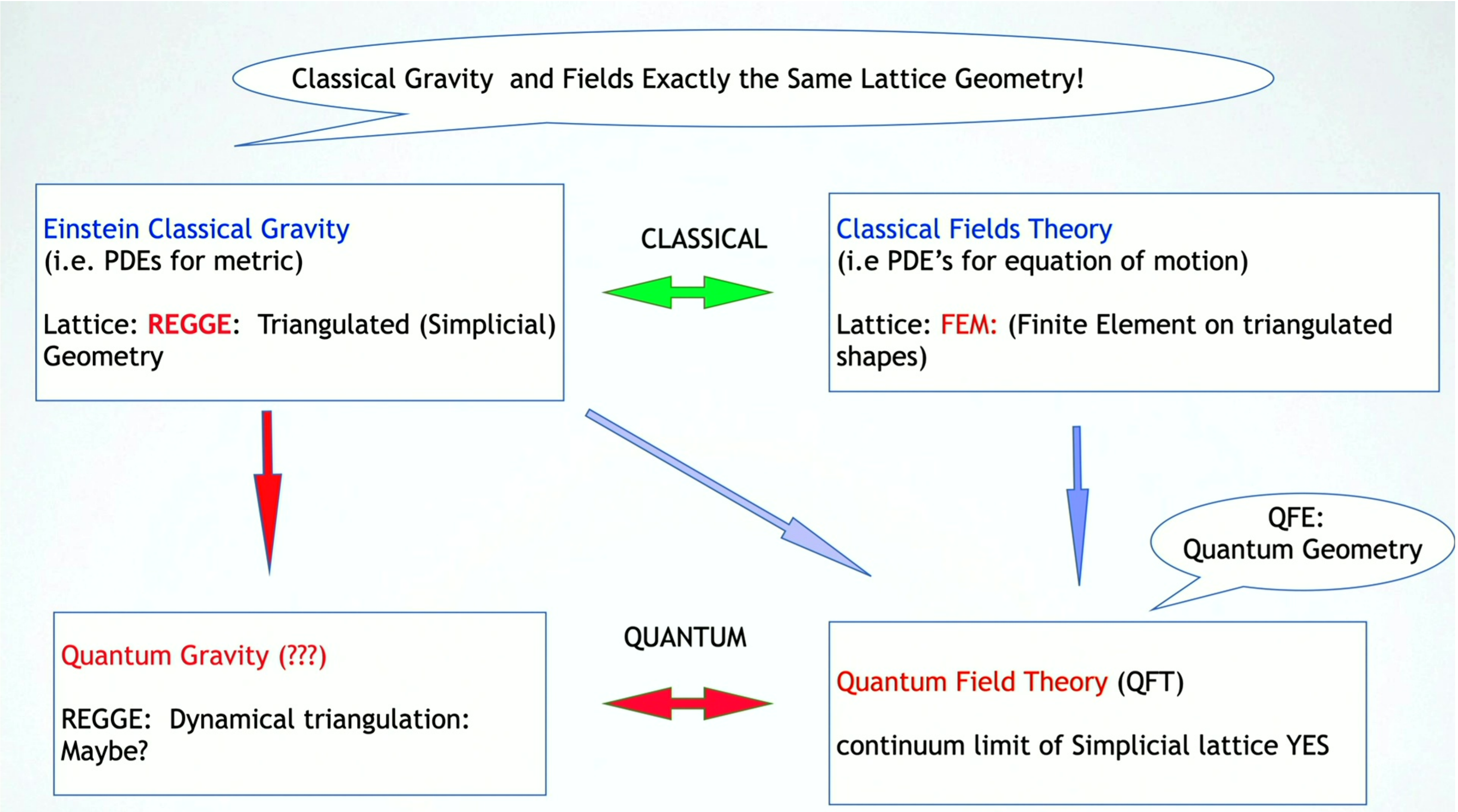
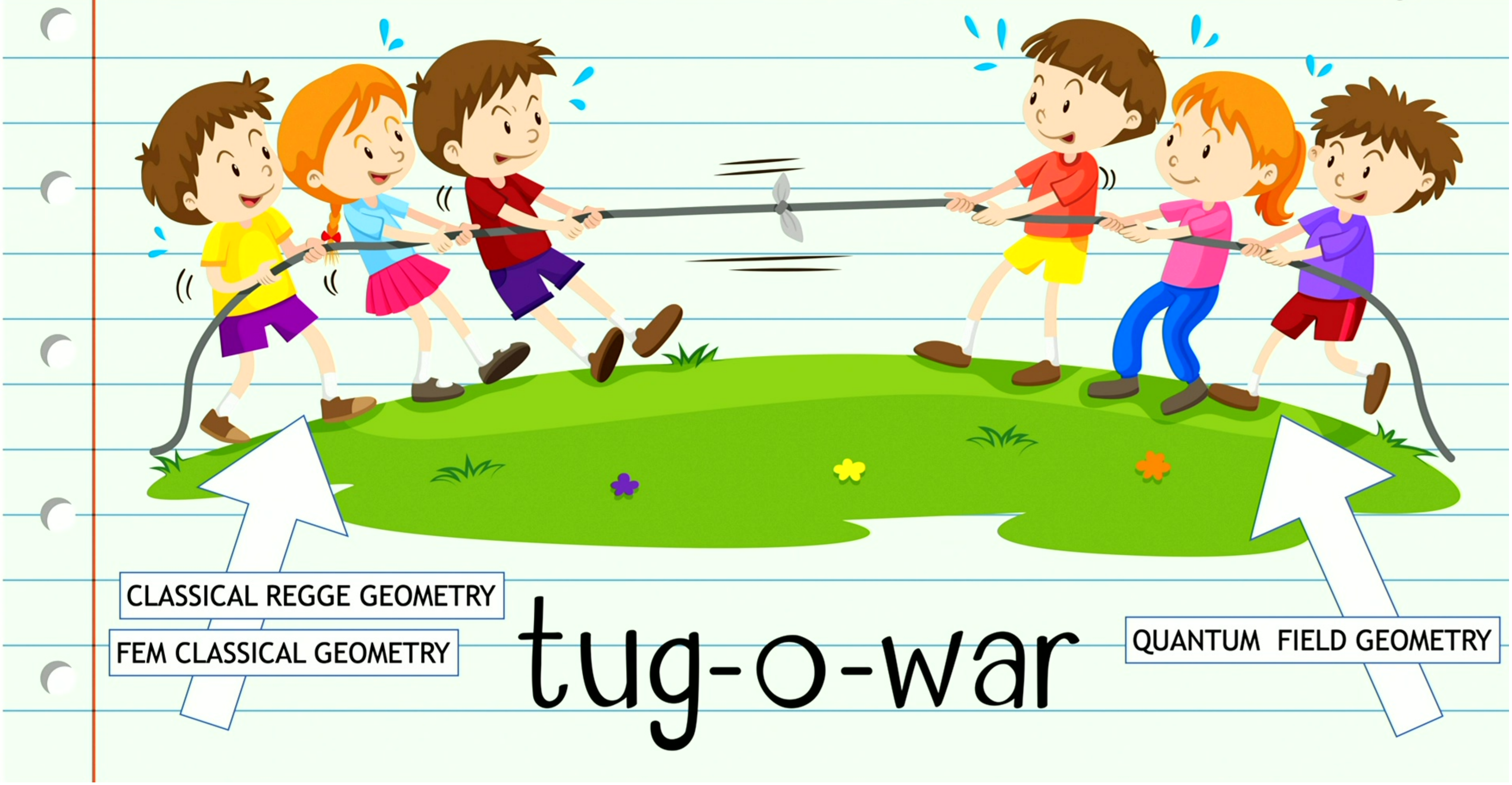
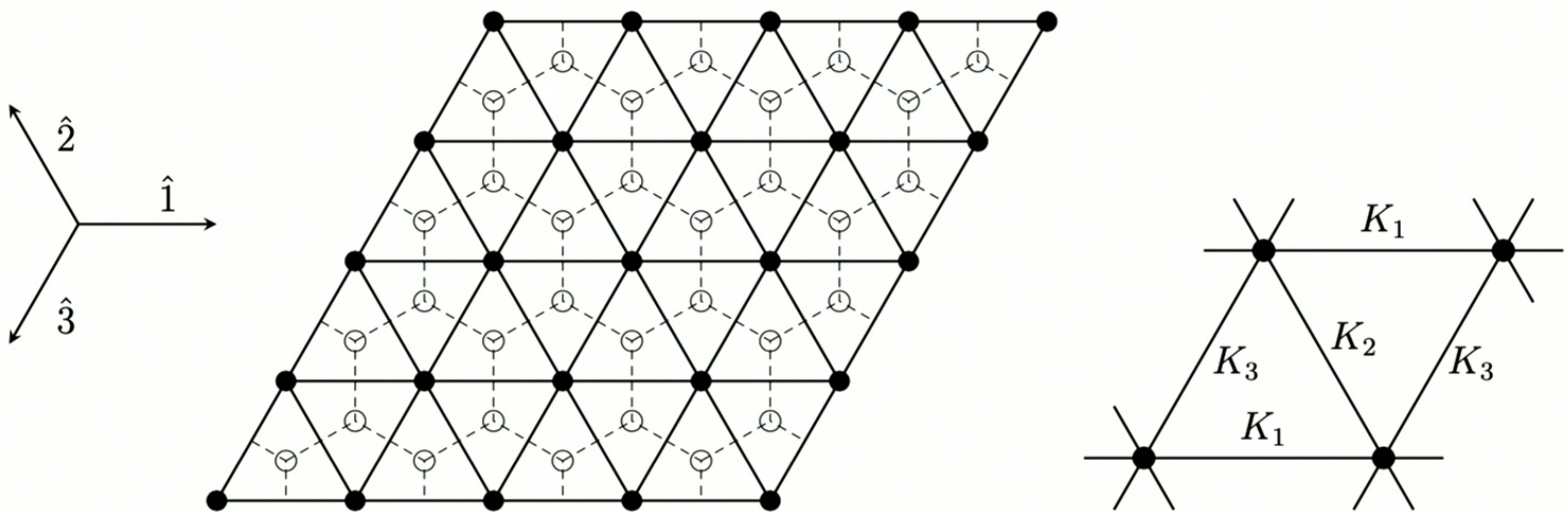

Classical Gravity and Fields Exactly the Same Lattice Geometry!
Einstein Classical Gravity
(i.e. PDEs for metric)
Lattice: REGGE: Triangulated (Simplicial) Geometry
CLASSICAL
Classical Fields Theory
(i.e PDE's for equation of motion)
Lattice: FEM: (Finite Element on triangulated shapes)
QFE:
Quantum Geometry
Quantum Gravity (???)
REGGE: Dynamical triangulation: Maybe?
QUANTUM
Quantum Field Theory (QFT)
continuum limit of Simplicial lattice YES

The Problem of Classic vs Quantum Geometry



Ising Model on the Affine Plane



$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{\,K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}} \,,$$

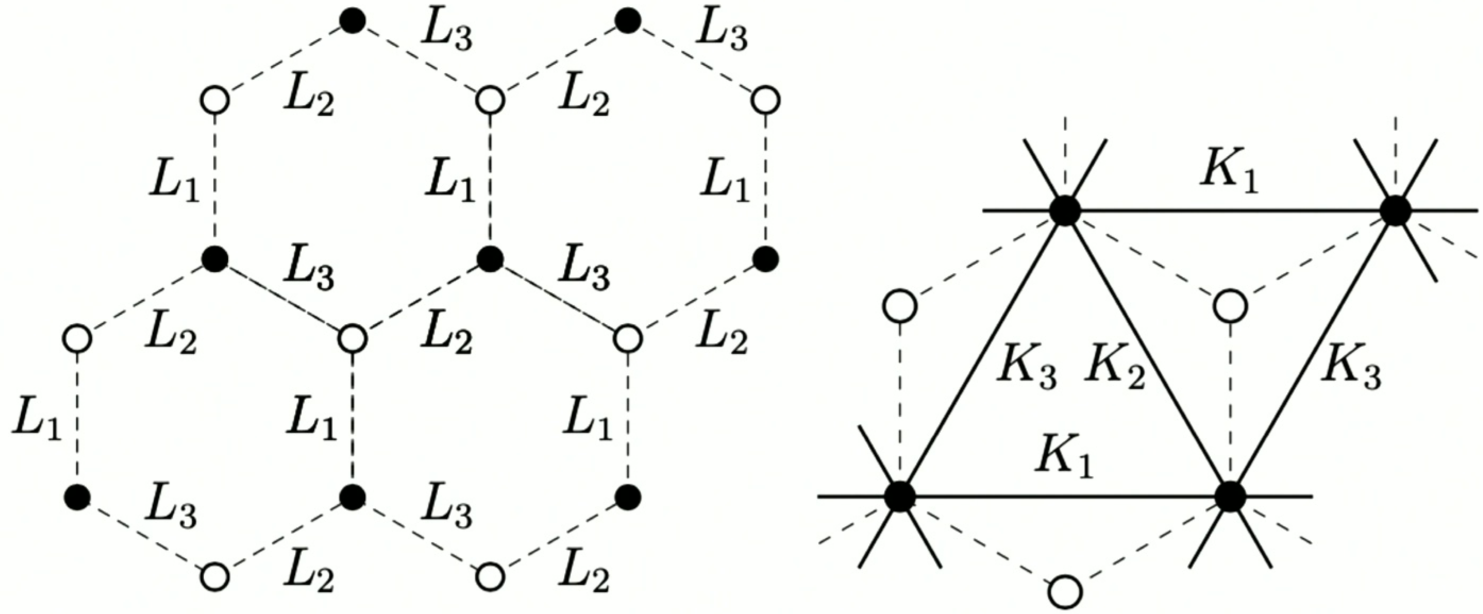
Affine Parameters:
2d Affine transformation takes circle to ellipse:

$$\langle \phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

- d = 2 Poincare 1 rotation 2 translation
- New Affine plus 1 major/minor + 1 orientation + 1 scaling
- General Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex - local metric

25

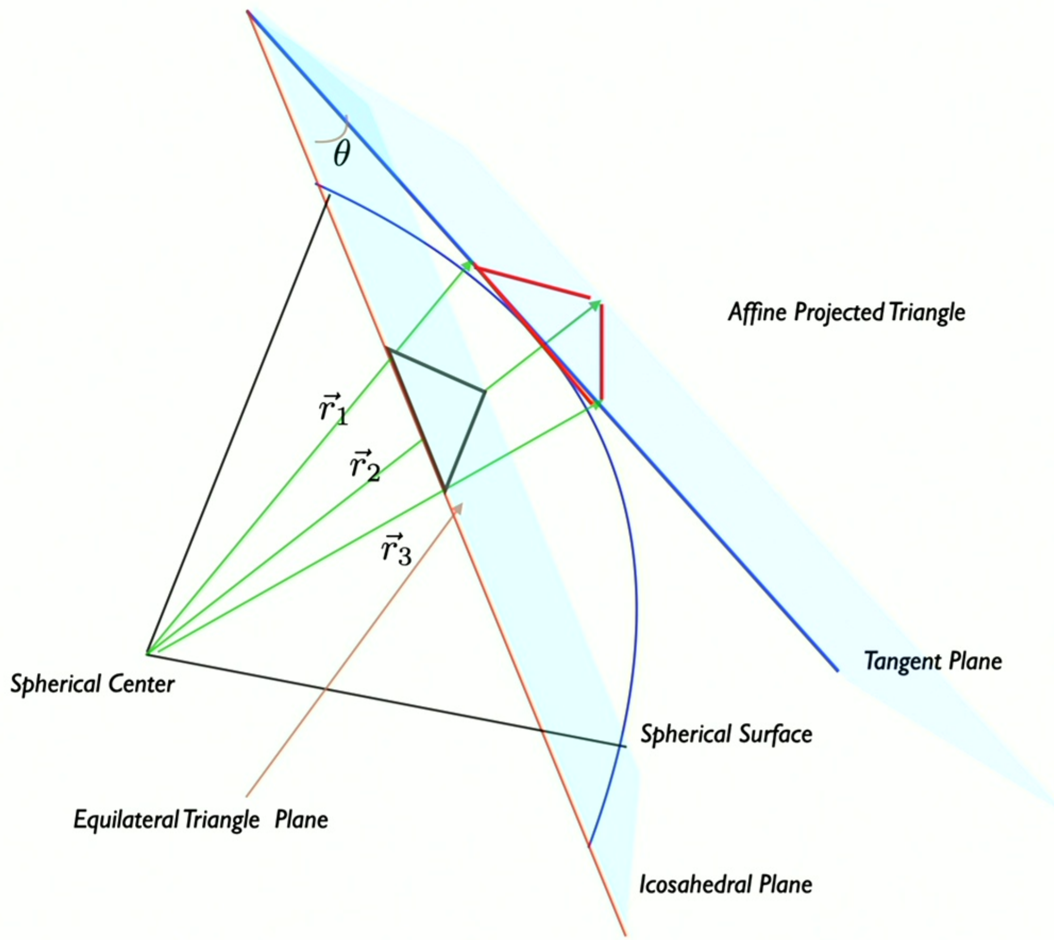
Step I : Star Triangle ID: Hex to Triangle Map



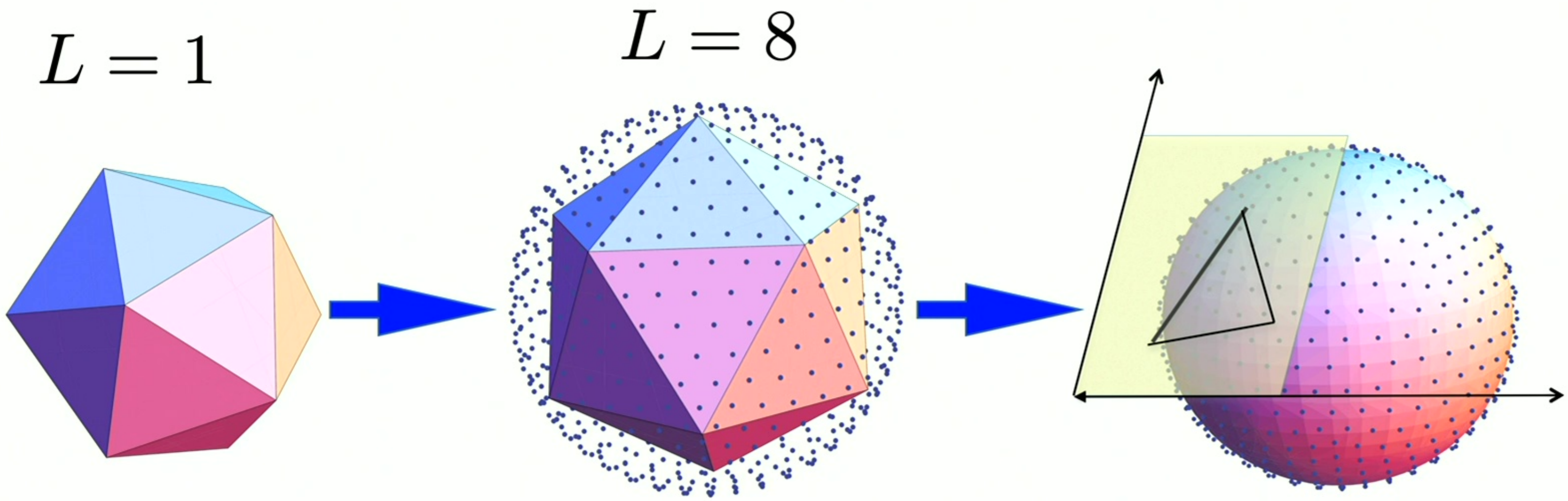
$$h \sinh(2K_1)\sinh(2L_1) = h \sinh(2K_2)\sinh(2L_2) = h \sinh(2K_3)\sinh(2L_3) = 1$$

$$h(K_1, K_2, K_3) = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4\sqrt{(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)}} \qquad \text{with } v_i = \tanh(K_i)$$

To O(a^2) the tangent plane is an Affine lattice on each tangent plane.



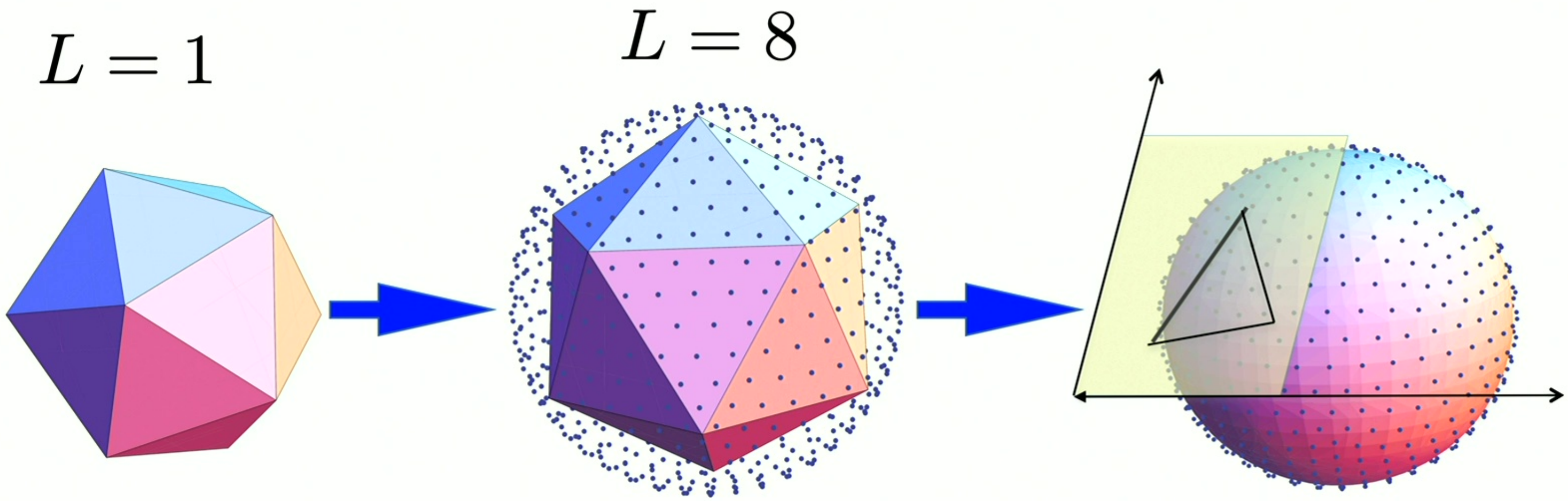
First Attempt (with good results): Classical FEM with UV counter term



Start with maximum regular Tessellation: preserve Icosahedral group upon refinement

l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

First Attempt (with good results): Classical FEM with UV counter term

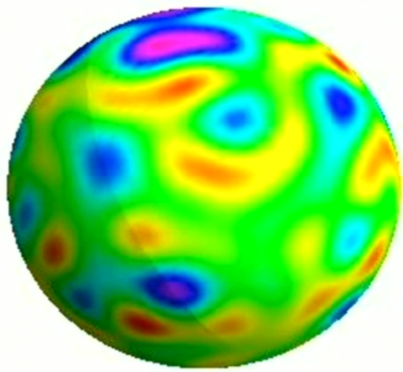


Start with maximum regular Tessellation: preserve Icosahedral group upon refinement

l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

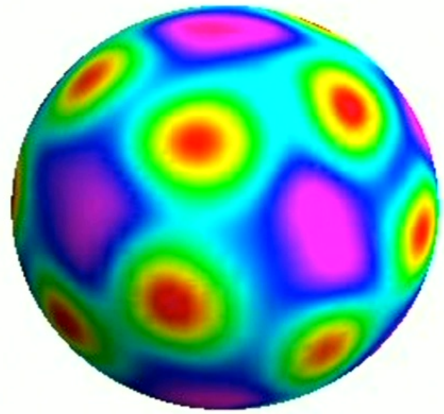
Now add $\lambda\phi^4$ term: What happens to FEM?

$\phi^2(x)$

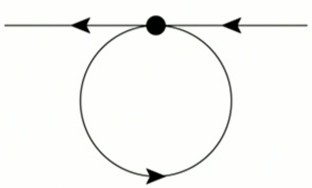


one configuration

$\langle \phi^2(x) \rangle$



Average of config.



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$

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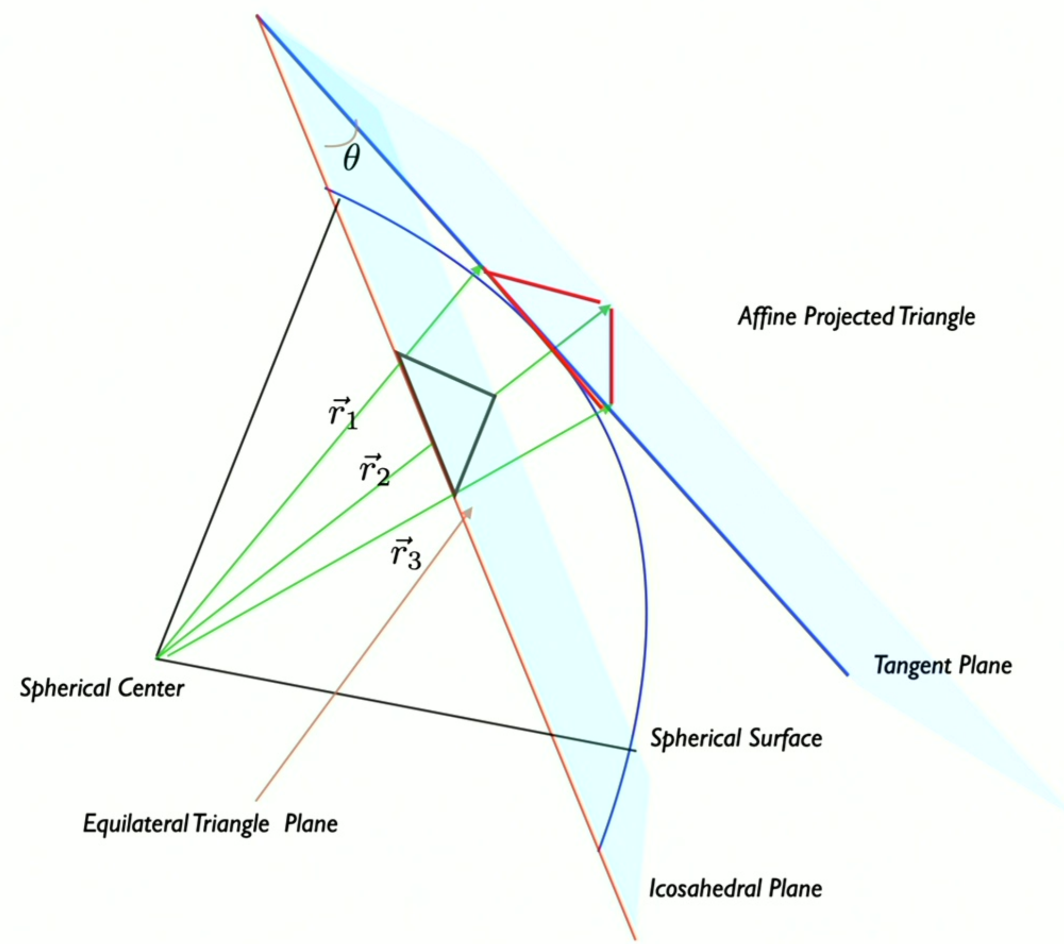
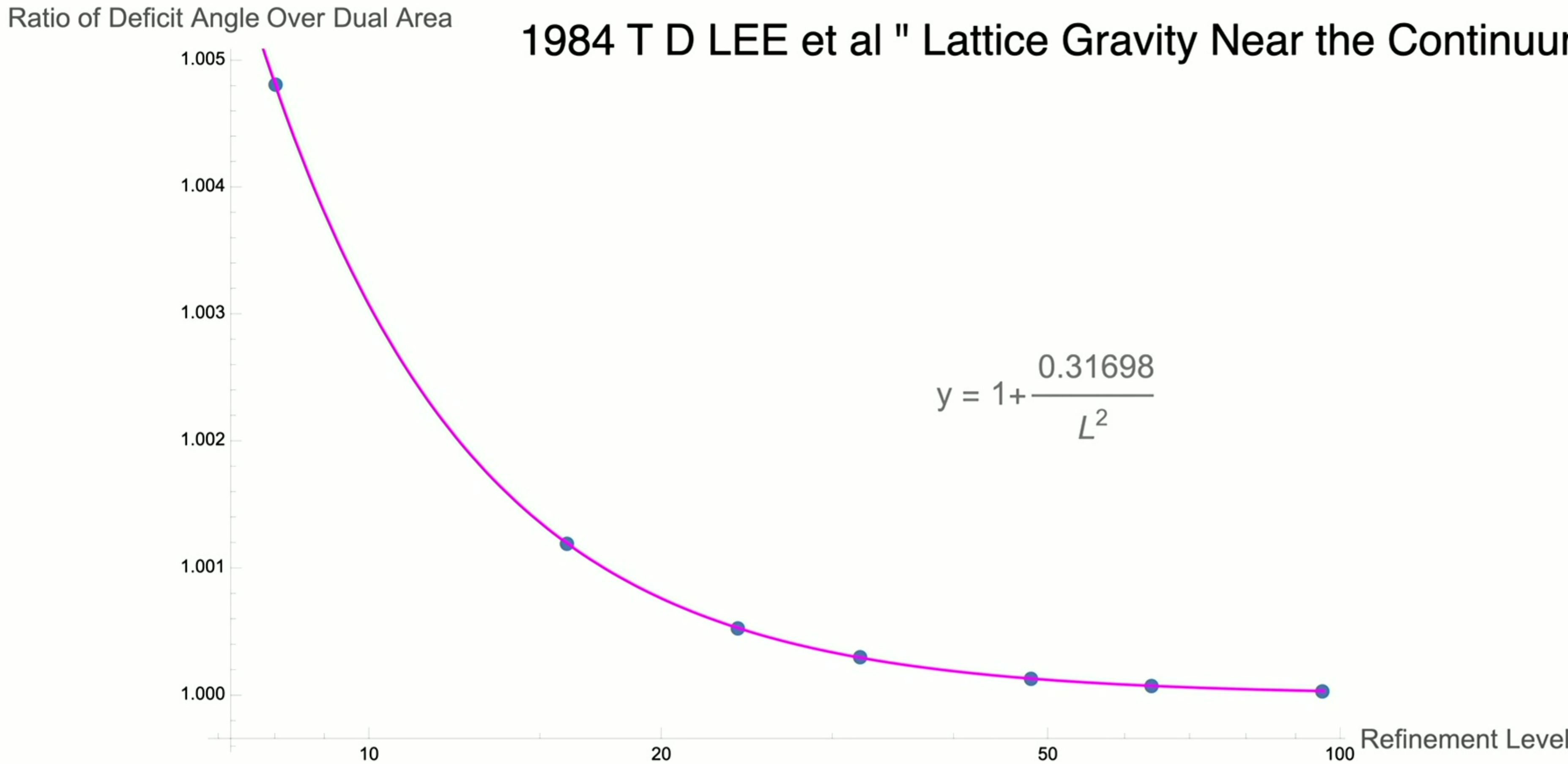


Smooth Scalar Curvature Theorem

1984 T D LEE et al " Lattice Gravity Near the Continuum"

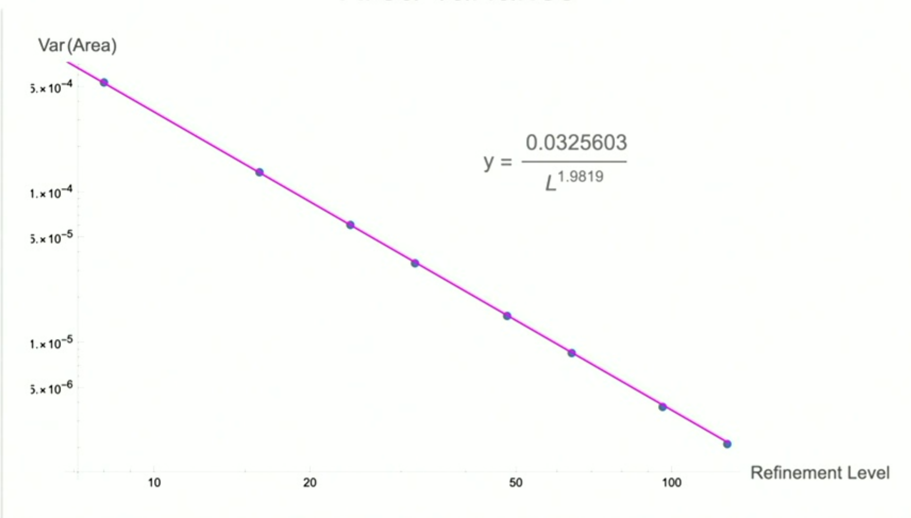


Area Optimization to smooth scalar curvature

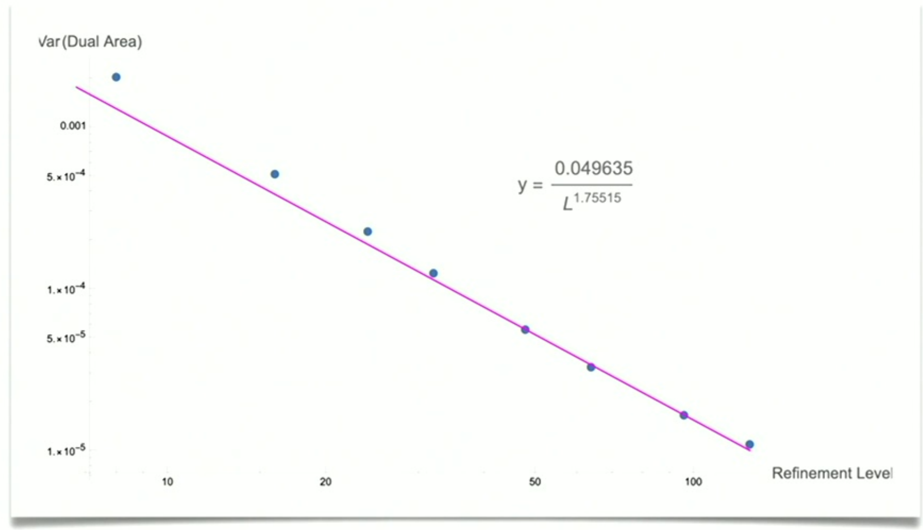
$$S(\ell_{ij}) = N^{-1} \sum_{\triangle} A_{\triangle}^2(l_{ij})$$

$$\text{dof:} \quad 2N = 4 + 20L^3$$

Area Variance



Dual Area Variance



$$4A(a,b,c)^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$= a^2 b^2 c^2 / R_{\triangle}^2$$

$$a^2 = \ell_{12}^2 = |\vec{r}_1 - \vec{r}_2|^2 = 2 - 2\vec{r}_1 \cdot \vec{r}_2$$

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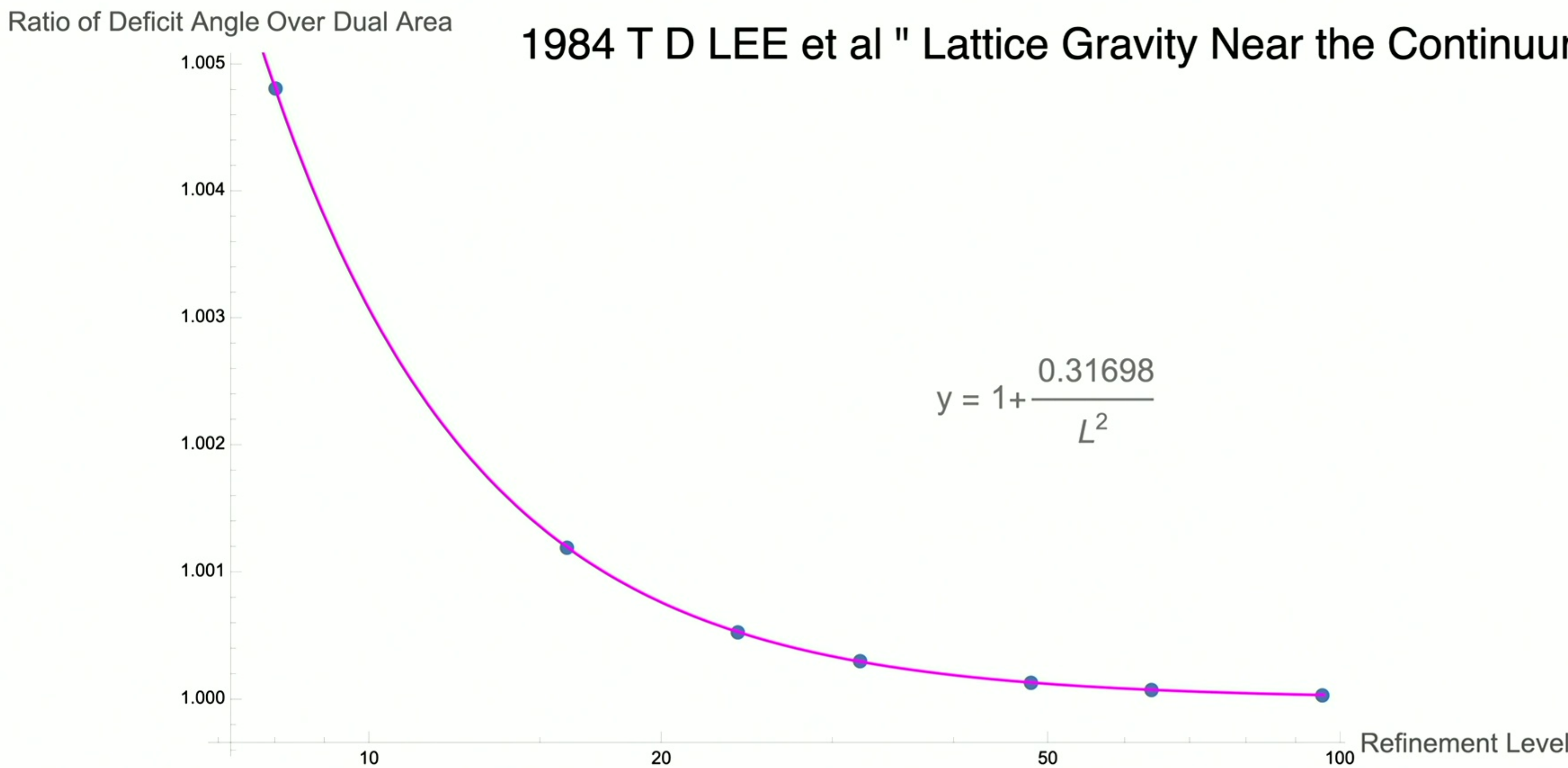
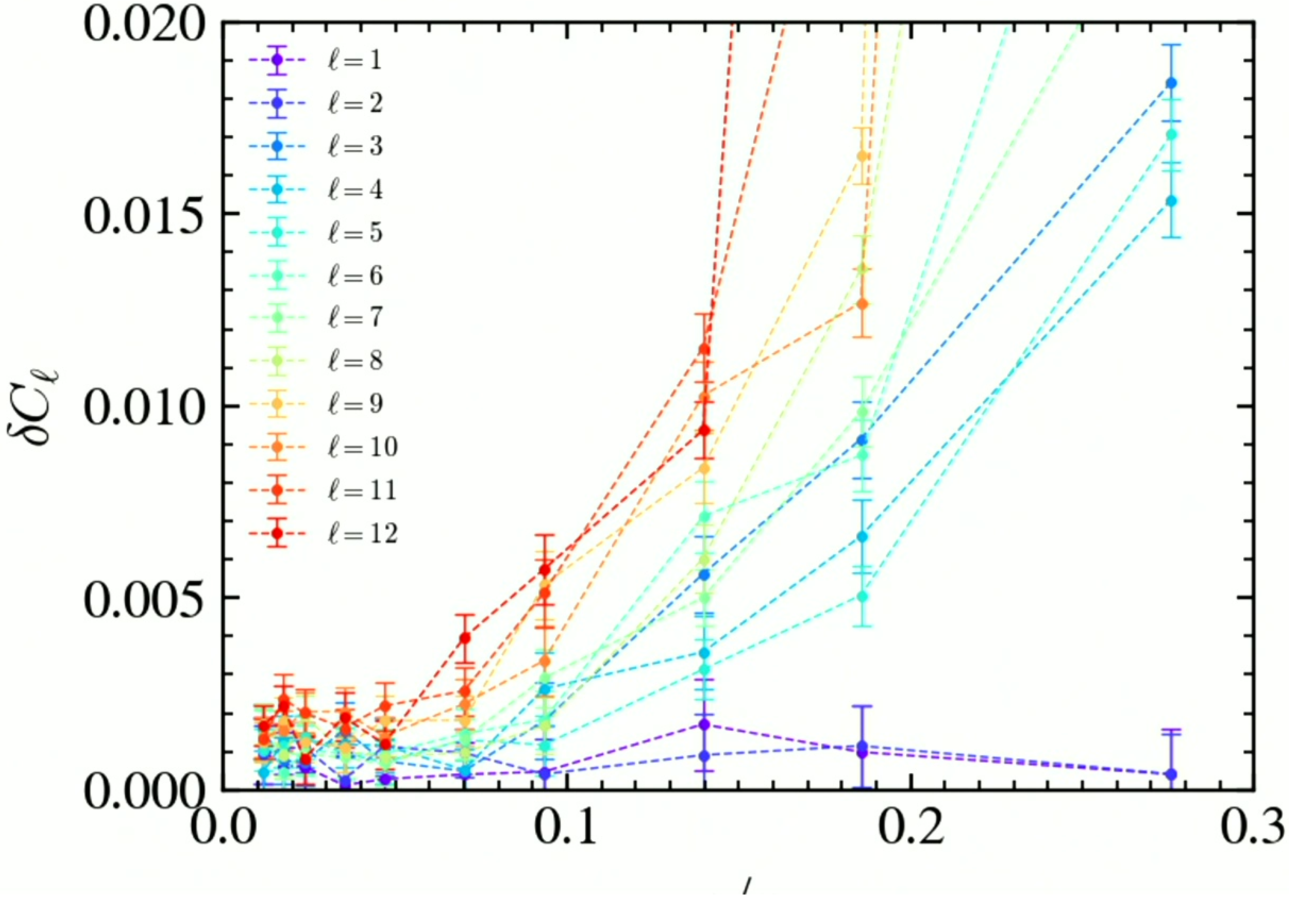
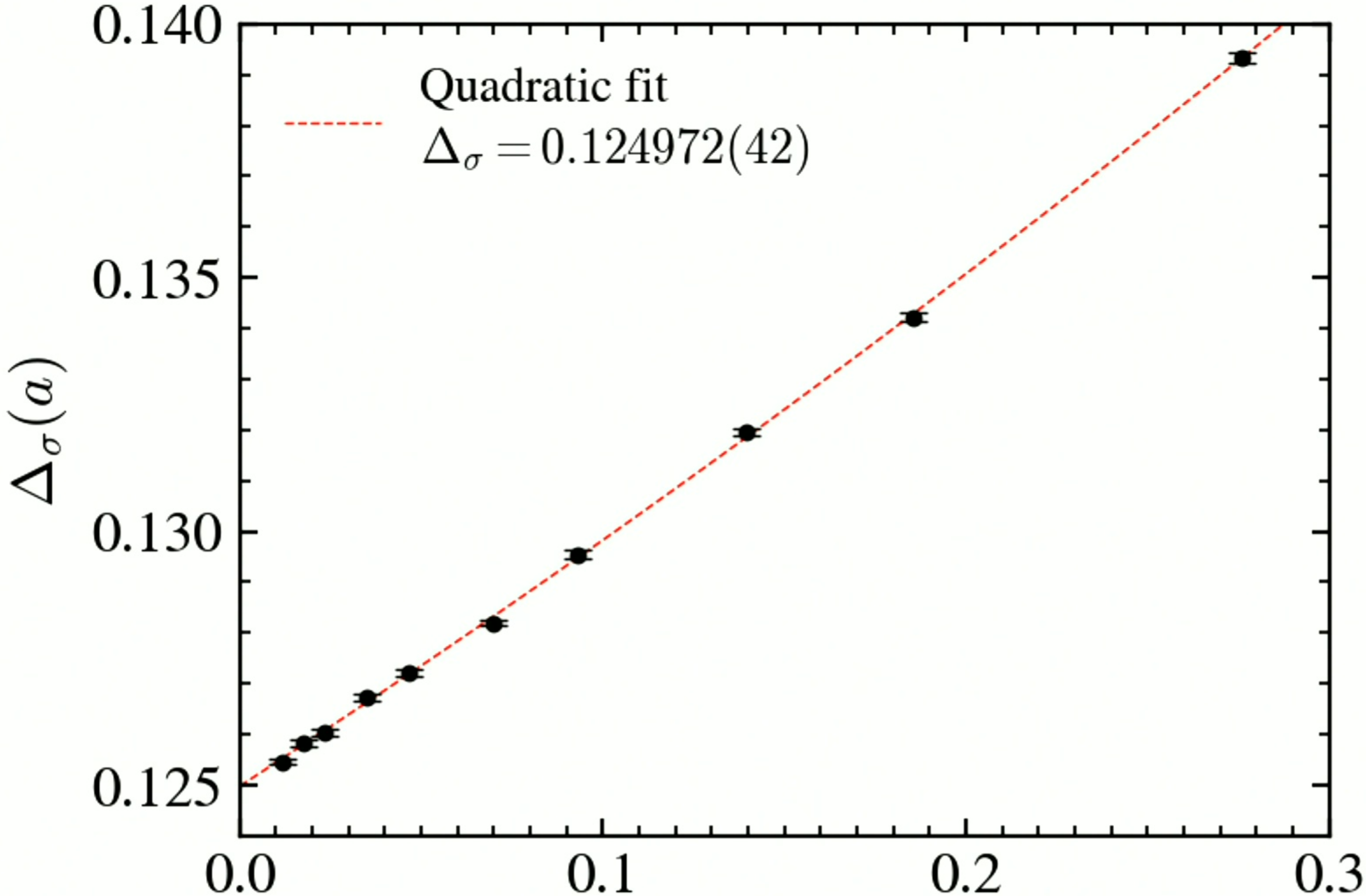
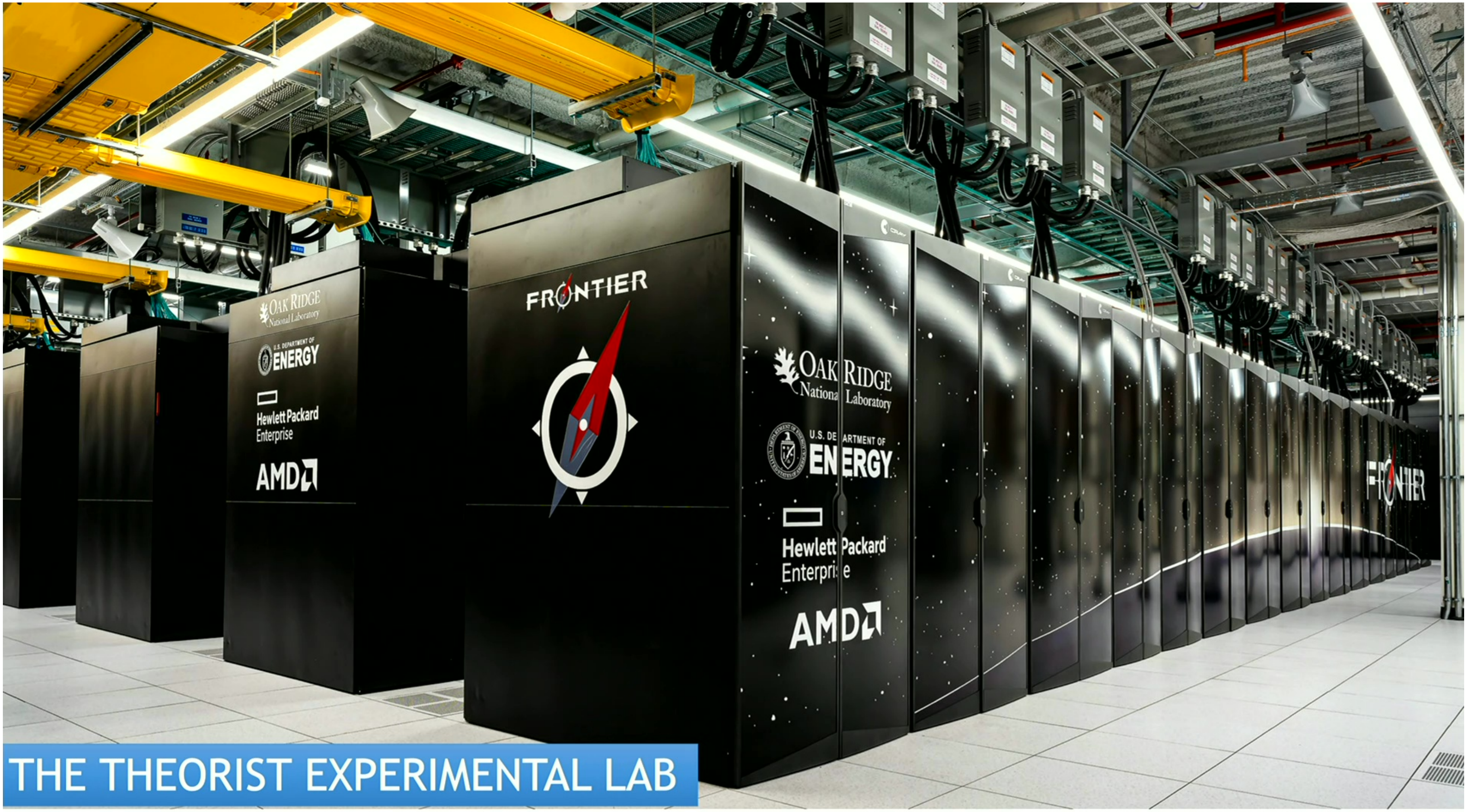




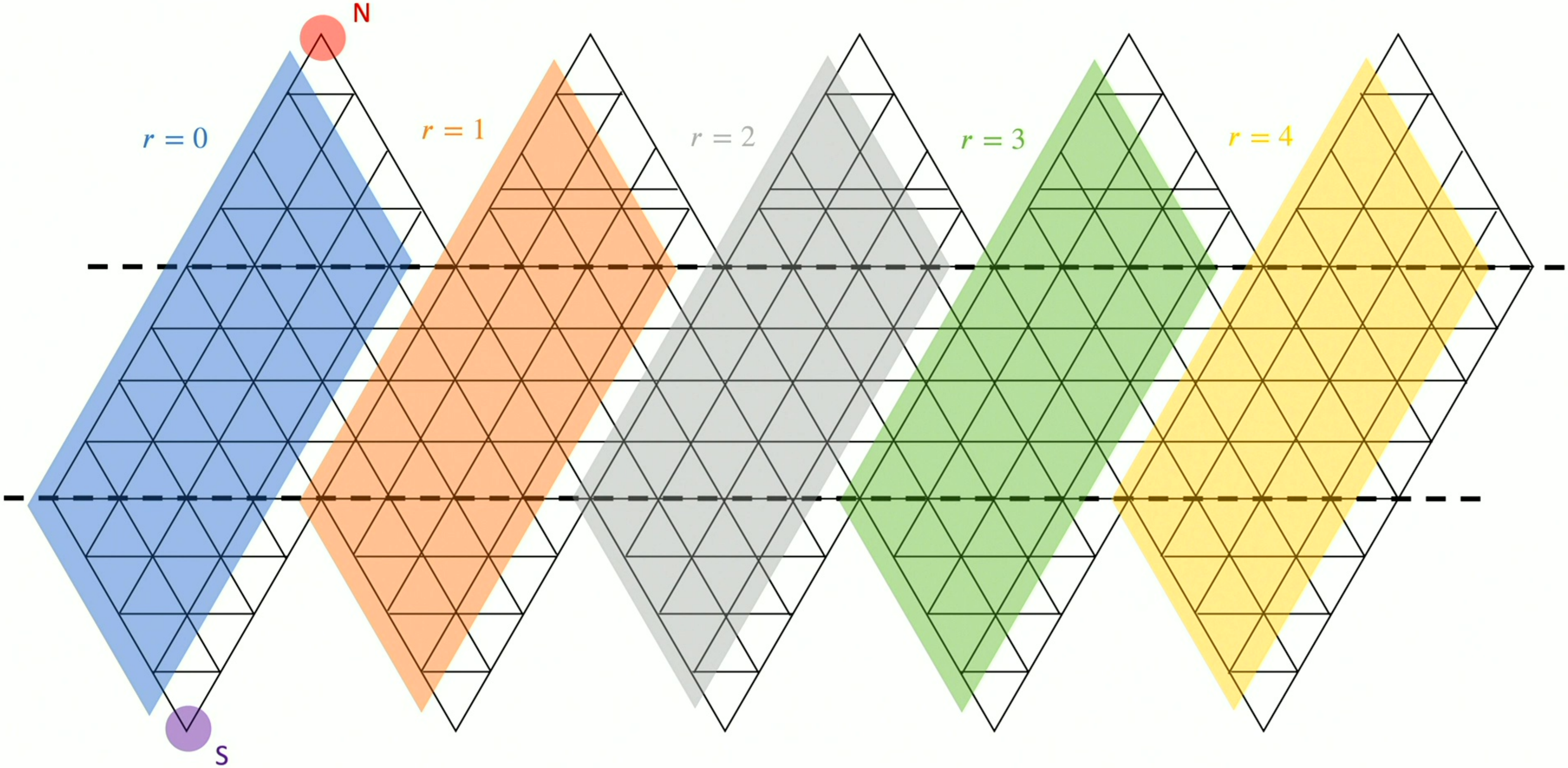
0.020
0.015
0.010
0.005
0.000
0.0
0.1
0.2
0.3
δC_ℓ
$\ell = 1$
$\ell = 2$
$\ell = 3$
$\ell = 4$
$\ell = 5$
$\ell = 6$
$\ell = 7$
$\ell = 8$
$\ell = 9$
$\ell = 10$
$\ell = 11$
$\ell = 12$



0.140
0.135
0.130
0.125
$\Delta_\sigma(a)$
Quadratic fit
$\Delta_\sigma = 0.124972(42)$
0.0
0.1
0.2
0.3


FRONTIER
OAK RIDGE National Laboratory
U.S. DEPARTMENT OF ENERGY
Hewlett Packard Enterprise
AMD
THE THEORIST EXPERIMENTAL LAB

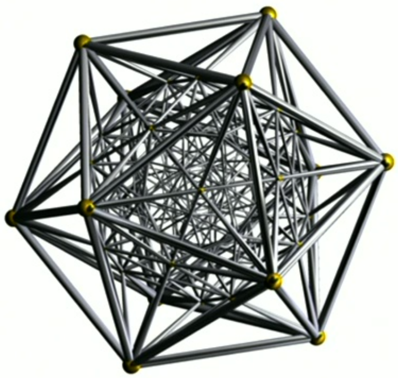
Aiming to implement MPI, we divide the lattice into 5 patches + two pole points:



Uncolored points have an identical point somewhere else.

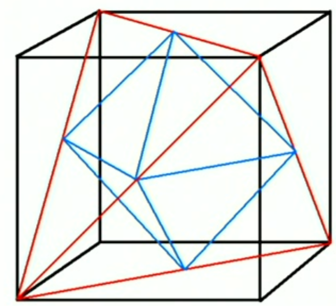
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: "Square of the icosahedron" −Symmetries 1440= 120 * 120 the 120 copies of icosahedron

$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.