Title: Resurgence and non-perturbative effects in topological string theory

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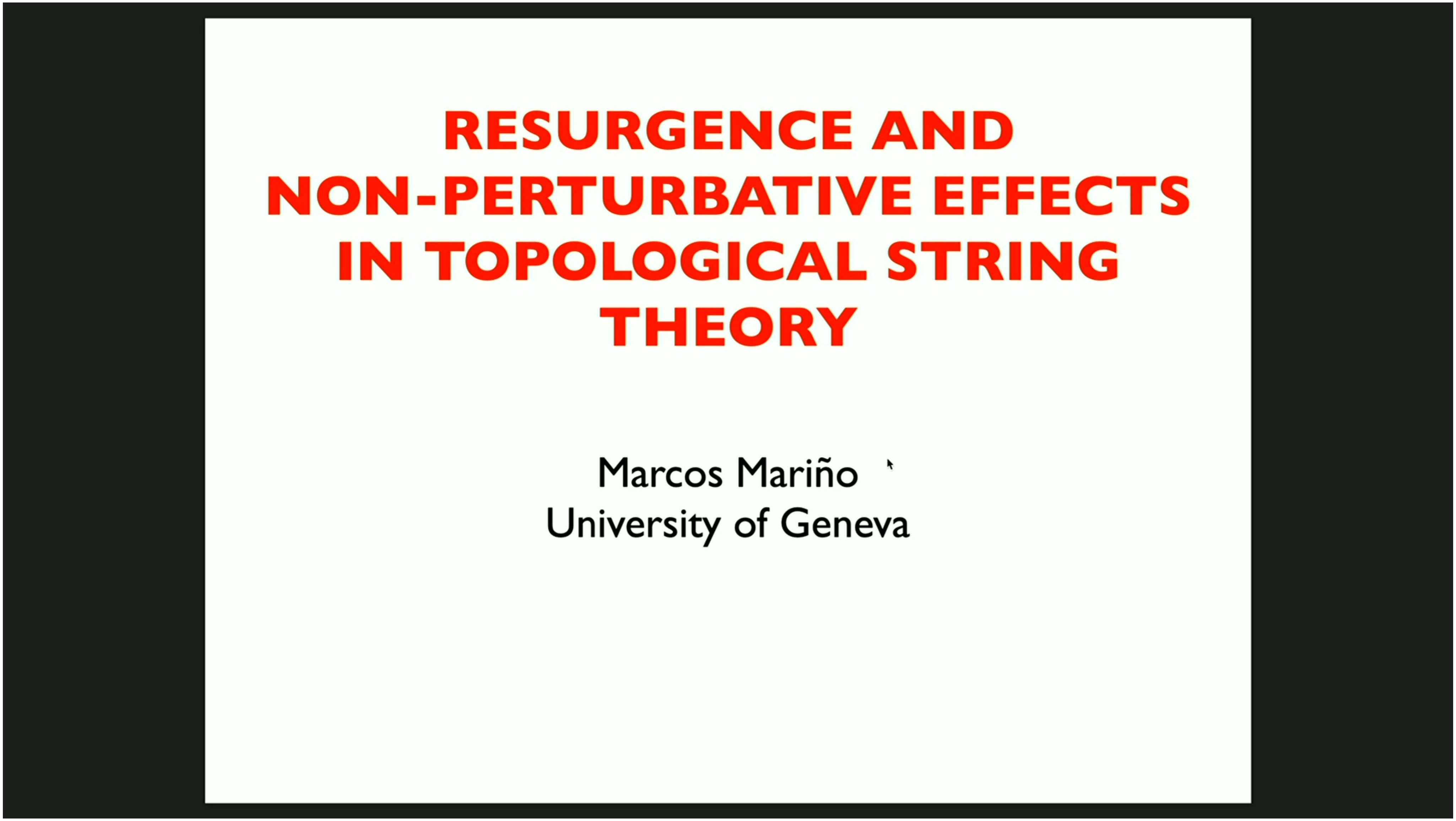
Abstract: Topological strings are well understood in perturbation theory, but their non-perturbative structure has been the subject of much work and speculation. A useful approach to this problem is to unveil the non-perturbative sectors of the theory by looking at the large order behavior of the perturbative series. This approach can be formulated in a mathematically rigorous way by using the theory of resurgence. In this talk I review the basic ideas behind this approach and I show that it can be successfully applied to topological string theory on arbitrary Calabi-Yau threefolds. Thsee methods lead, not only to explicit non-perturbative topological string amplitudes, but also to a conjecture relating the "invariants" of the theory of resurgence (also known as Stokes constants) to BPS invariants. Physically, this conjecture implies that the large order behavior of the topological string perturbative series contains information about the spectrum of stable D-brane sectors.

Zoom link


RESURGENCE AND NON-PERTURBATIVE EFFECTS IN TOPOLOGICAL STRING THEORY
Marcos Mariño
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In most quantum theories, we can use perturbation theory to calculate observables as power series in a "small" coupling z:

$$\varphi(z) = \sum_{n \geq 0} a_n z^n$$

Unfortunately, these series are typically factorially divergent

[Dyson]

$$a_n \sim n!$$

It is therefore nontrivial to extract, say, numerical predictions from these series!

String theory is no exception!

String Perturbation Theory Diverges

David J. Gross and Vipul Periwal
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 11 April 1988)

It has been recognized for a long time that the factorial divergence of perturbative series is a signal of "non-perturbative effects" that have to be taken into account, of the form

$$e^{-A/z}$$

Bender and Wu made a quantitative connection between the factorial growth of perturbation theory and non-perturbative effects

Large-Order Behavior of Perturbation Theory

Carl M. Bender*
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

Tai Tsun Wu†
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138,‡ and Deutsches Elektronen-Synchrotron, Hamburg, Germany
(Received 28 June 1971)

This idea was developed in quantum field theory to give some of our best insights on non-perturbative physics (instantons, renormalons, non-perturbative condensates…)

see my talk on Friday

In string theory, the factorial divergence of the genus expansion has been linked to non-perturbative effects due to D-branes [Polchinski, Shenker…]. So far, it has been difficult to make this picture concrete, except in non-critical strings [Martinec, Alexandrov-Kazakov-Kutasov, …]

In this talk I will describe how this picture is realized in topological string theory on Calabi-Yau (CY) threefolds, and leads to a beautiful mathematical story/conjecture.

To make the picture precise I will need some ingredients from Ecalle's theory of resurgence, which gives a mathematical framework for the insights of Dyson, Bender-Wu, and others

From wild series to analytic functions

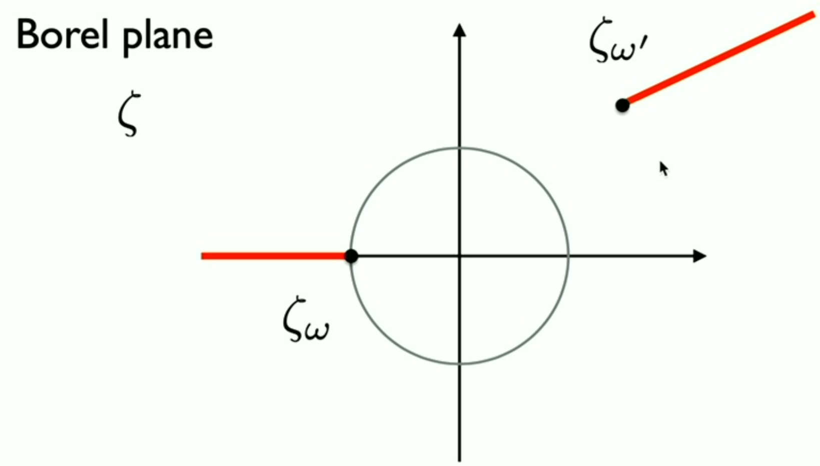
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \qquad a_n \sim n!$$

These are sometimes called Gevrey-1 series. The **Borel transform** is a deceptively simple way of transforming these series into "nice" functions

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. We now demand that it can be "endlessly analytically continued" to the complex plane, displaying a set of **singularities** (poles, branch cuts)



Main reason for looking at this: **the singularities of the Borel transform contain information about the non-perturbative sectors of the theory**

Concretely, the local expansion of the Borel transform at the singularities leads to **new formal power series**. A typical example is a logarithmic singularity at $\zeta = \zeta_\omega$

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega\, \widehat{\varphi}_\omega(\zeta - \zeta_\omega)\frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \text{regular}$$

The function $\widehat{\varphi}_\omega(\xi)$ is typically analytic at the origin

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0} \widehat{a}_{n,\omega}\xi^n$$

but we can think about it as the Borel transform of a **new, factorially divergent power series** associated to the singularity:

$$\varphi_\omega(z) = \sum_{n\geq 0} a_{n,\omega} z^n \qquad a_{n,\omega} = n!\,\widehat{a}_{n,\omega}$$

The constant S_ω is called a **Stokes constant**. Its value depends on the normalization of $\varphi_\omega(z)$

singularity ζ_ω → new power series $\varphi_\omega(z)$

inverse Borel transform of the local expansion

singularity ζ_ω → Stokes constant S_ω

With this, we build up a **non-perturbative amplitude**

$$\Phi_\omega(z) = \mathsf{S}_\omega \, \mathrm{e}^{-\zeta_\omega / z} \varphi_\omega(z)$$

This is an example of a **trans-series**, involving exponentially small terms

In physics, trans-series can be sometimes interpreted as **instanton corrections**, i.e. expansions around non-trivial saddle-points of the path integral.

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z)$$

instanton action

quantum fluctuations around the instanton

Resurgence and asymptotics

The "largest" exponentially small trans-series, associated to the Borel singularity closese to the origin

$$\Phi_{\mathcal{A}} = \mathrm{e}^{-\mathcal{A}/g_s}(c_0 + c_1 g_s + \cdots)$$

turns out to determine the asymptotic behavior of the perturbative series (Darboux" theorem)

$$a_n \sim \frac{\mathcal{A}^{-n}}{2\pi}\Gamma(n)\left(c_0 + \frac{c_1 \mathcal{A}}{n-1} + \cdots\right)$$

The trans-series **"resurges"** in the perturbative series

Resurgent structures

Given a factorially divergent perturbative series, we can collect the set of all formal power series and Stokes constants associated to the singularities of its Borel transform. I will call this collection its **resurgent structure**. It encodes in a mathematically precise way **all the non-perturbative information which can be obtained from perturbation theory**.

Understanding the resurgent structure of quantum theories has been an ongoing pursuit for many years [Voros, Zinn-Justin, Brezin, Parisi, 't Hooft, …]. We expect that topological fields and strings will provide workable and interesting examples of these structures

Topological string theory

Let *M* be a Calabi-Yau (CY) threefold. At each genus *g* one can consider the topological string free energy $F_g(X)$, which depends on the moduli of the CY, given by "flat coordinates" *X* (I will often consider one-modulus CYs for simplicity)

At large *X* ("large radius") this has an expansion encoding Gromov-Witten invariants of *M*, which "count" holomorphic curves of genus *g* and degree *d*:

$$F_g(X) = \sum_d N_{g,d} \, \mathrm{e}^{-dX}$$

I recall that in the mirror manifold $M^\star$ one can calculate **periods** by integrating the holomorphic 3-form over a symplectic basis of 3-cycles. This determines both X and the genus zero free energy [Candelas-de la Ossa-Green-Parkes]

$$X^I = \int_{\alpha^I} \Omega \qquad \mathcal{F}_I = \int_{\beta_I} \Omega = \frac{\partial F_0}{\partial X^I}$$

String perturbation theory tells us that the **total free energy** is given by a genus expansion in a small parameter, a.k.a. the string coupling constant

$$F(X, g_s) = \sum_{g \geq 0} F_g(X) g_s^{2g-2}$$

As I mentioned, general arguments [Gross-Periwal, Shenker] indicate that this series grows doubly-factorially, at fixed X

$$F_g(X) \sim (2g)!, \qquad g \gg 1$$

What is the resurgent structure associated to this series?

This is a difficult problem. Note that in this case the resurgent structure depends on the moduli of the CY manifold, parametrized by X.

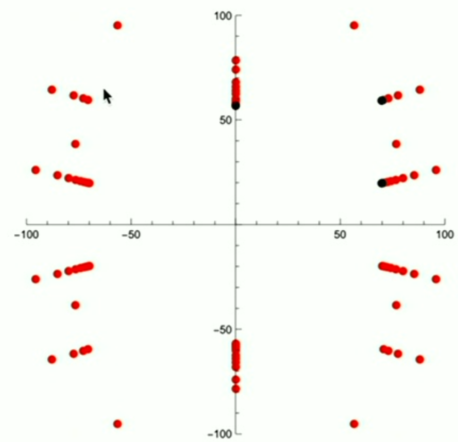
I will present a conjecture on this resurgent structure: it is governed by the spectrum of D-branes on the CY

Borel plane and CY periods

Conjecture: the Borel singularities for the perturbative series of TS free energies are **integral periods** of the mirror CY

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

Determining which integral periods are actual singularities is more difficult and it sometimes relies on numerical calculations



Trans-series for topological strings

How do we determine the trans-series associated to the singularities?

The **holomorphic anomaly equations** of BCOV make it possible to obtain the perturbative series. One can then try to solve them with a "trans-series ansatz" [Couso-Edelstein-Schiappa-Vonk] as in Ecalle's theory of ODEs

$$F = \sum_{g \geq 0} F_g(X) g_s^{2g-2} + \mathrm{e}^{-\mathcal{A}/g_s} \sum_{n \geq 0} F_n^{(1)}(X) g_s^{n-1} + \cdots$$

perturbative series   instanton correction

In recent work [Gu-M.M, Gu-Kashani-Poor-Klemm-M.M.] we obtained an all-orders, **exact solution** for the trans-series, for **any** CY (compact or not)

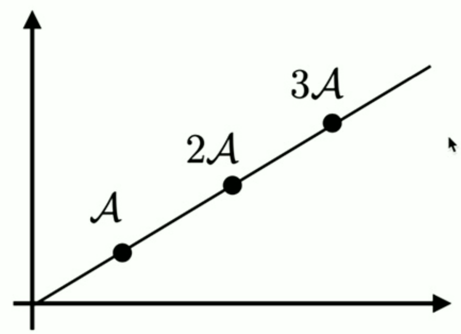
$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

$$\Phi_{\mathcal{A}} = \mathsf{S}_{\mathcal{A}} \left(1 + g_s c^J \partial_J F(X^I - g_s c^I)\right) \mathrm{e}^{F(X^I - g_s c^I) - F(X^I)}$$

This is a universal formula for the "one-instanton amplitude" associated to $\mathcal{A}$. It can be written in terms of perturbative data only, and it suggests that the the CY periods X^I are **quantized** in units of the string coupling constant, as in large *N* dualities

Multi-instantons and Stokes discontinuities

Given an instanton action $\mathcal{A}$ we have Borel singularities at positive integer multiples $\ell\mathcal{A}$ encoding **multi-instanton** amplitudes



This is similar to the multi-covering phenomenon for worldsheet instantons

The natural quantity to encode all the multi-instanton amplitudes is the **Stokes discontinuity** across the ray, which has a rigorous definition in Ecalle's theory. It can be written more compactly in terms of the dual topological string partition function

$$\tau(X,\lambda) = \sum_{k\in\mathbb{Z}} \lambda^k Z(X+kg_s) \qquad Z = \mathrm{e}^F$$

and one finds [Iwaki-M.M]

$$\mathfrak{S}\tau(X,\lambda) = \mathrm{e}^{\mathsf{S}_{\mathcal{A}}\mathrm{Li}_2(\lambda)}\tau\left(X - cg_s\mathsf{S}_{\mathcal{A}}\log(1-\lambda),\lambda\right)$$

D-branes on CYs

We expect that non-perturbative effects are due to D-branes. The spectrum of BPS D-branes on a CY has been the subject of intense study and can be formalized in the theory of Donaldson-Thomas (DT) invariants.

These BPS D-branes are labelled by a charge vector identical to the one appearing in the instanton action

$$\gamma = (c_I, d^I)$$

In fact, the corresponding central charge can be identified with the action

$$Z_\gamma = \mathcal{A} = c_I X^I + d^I \mathcal{F}_I$$

A conjecture

The connection between resurgent structure and non-perturbative effects suggests the following conjecture

1) **Spectrum**: the Borel singularities of the topological string free energy are given by the central charges of stable BPS D-branes

2) **Invariants**: Stokes constants are given by DT invariants

$$\mathsf{S}_{\mathcal{A}} = \frac{\Omega(\gamma)}{2\pi}$$

See my talk at Strings2019 and recent work by myself and Alexandrov, Alim, Grassi, Gu, Hao, Hollands, Kashani-Poor, Klemm, Neitzke, Pioline, Schwick, Teschner, Tulli

Therefore, the factorial divergence of the topological string free energy encodes conjecturally all the information about the spectrum of stable D-branes in the CY!

There is a similar story in complex Chern-Simons theory, where Stokes constants of perturbative series turn out to be related to the **DGG index** [Garoufalidis-Gu-M.M.-Wheeler].

Evidence for the conjecture

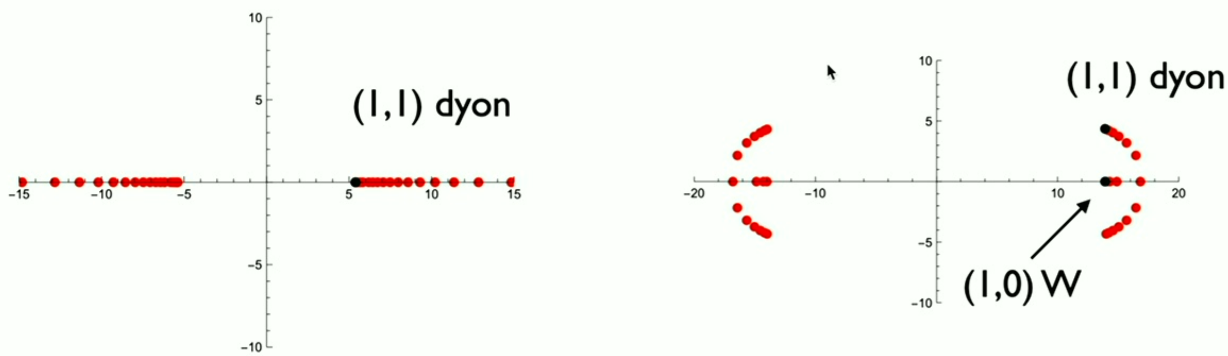
1) There is structural evidence for the conjecture coming from the Stokes discontinuity formula, which is identical to formulae appearing in the DT setting [Alexandrov-Pioline, Coman-Longhi-Teschner, …]

2) There is a derivation for the resolved conifold [Schiappa-Pasquetti, Alim et al.] and more generally for D2-D0 BPS states [GKKM]

3) Direct (numerical) evidence for some CY manifolds

N=2 SW theories

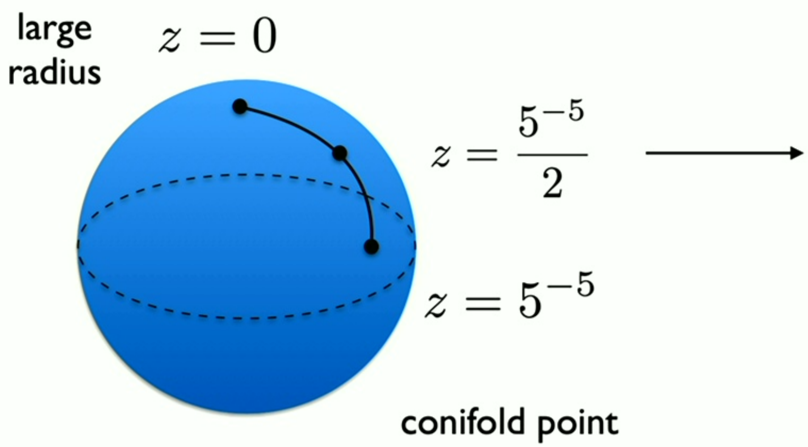
Given an *N*=2 SW theory, one can construct a topological string genus expansion directly from the SW curve, by geometric engineering or topological recursion. Direct calculations of the resurgent structure display the expected structure of BPS spectrum



Wall-crossing in *N*=2, *SU(2)* SYM

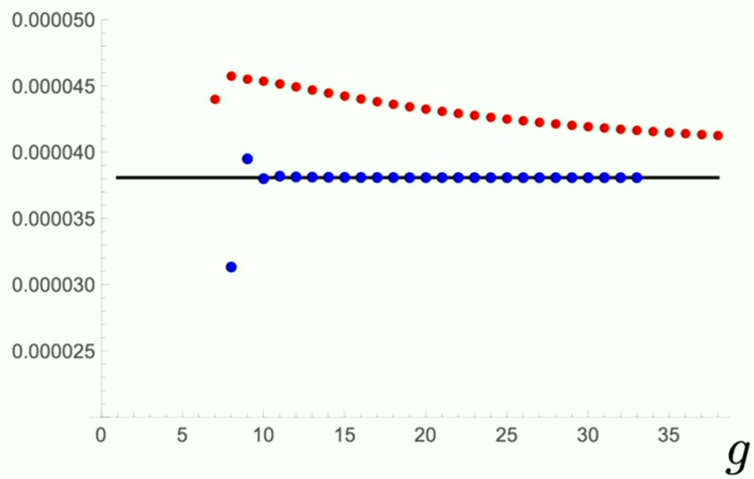
A test of our results for the quintic CY

The closest singularity of the Borel transform determines the asymptotics of perturbation theory, and we can test the one-instanton amplitude against perturbative data, in e.g. the famous quintic CY



$$\mathcal{A} = \mathcal{F}_0$$

vanishing period at the conifold point =stable D6 brane



red dots: sequence $\frac{\mathcal{A}^{2g-1}}{\Gamma(2g-1)} F_g$

blue dots: Richardson acceleration

black line: prediction from our one-instanton formula

Conclusions and outlook

The theory of resurgence gives a precise mathematical framework to understand non-perturbative sectors, which can be applied successfully to (topological) string theory.

To do this, we have developed an "instanton calculus" for the Kodaira-Spencer theory of BCOV. We have found universal **exact** solutions for multi-instanton amplitudes, and conjectured that the resurgent structure encodes the spectrum of D-brane BPS states.

Our results determine the instanton physics of other systems governed by the HAE, like large N matrix models, and give a new view on the BPS spectra of N=2 theories

Does resurgence provide new, efficient methods to calculate BPS spectra?

Can we deduce the multi-instanton amplitudes from a spacetime approach for D-branes?

What is the relation to other proposals for non-perturbative definitions of the topological string, like the one in [Grassi-Hatsuda-M.M.]?