Title: Hamiltonian formulation of the second-order self-force in the small mass ratio approximation

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Series: Strong Gravity

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Abstract: The two body problem in general relativity is of great theoretical and observational interest, and can be studied in the post-Newtonian, post-Minkowskian and small mass ratio approximations, as well as with effective one body and fully numerical techniques. An issue that arises is whether the motion can be decomposed into dissipative and conservative sectors for which the conservative sector admits a Hamiltonian description. This has been established to various orders in the post-Newtonian and post-Minkowskian approximations. In this talk, I will go over recent work where we showed that in the small mass ratio approximation, the motion of a (spinning) point particle under the conservative piece of the first-order self force is Hamiltonian in any stationary spacetime. After this, I describe two issues that arise when attempting to extend these results to subleading order in the mass ratio, namely infrared divergences and ambiguities in the conservative/dissipative splittings. I suggest resolutions of these issues and successfully derive a subleading Hamiltonian conservative sector for the scalar self force, as a toy model for the gravitational case.

Zoom link

Progress towards a Hamiltonian description of second-order self-force

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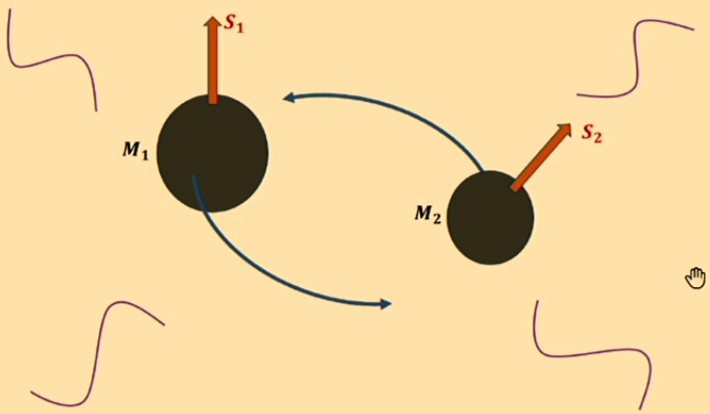
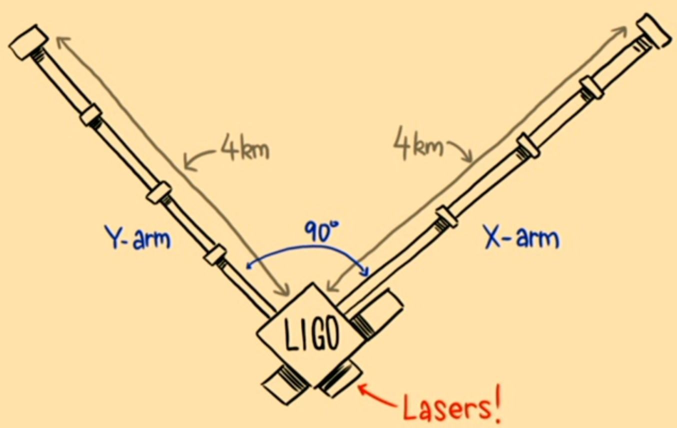
Binary Systems



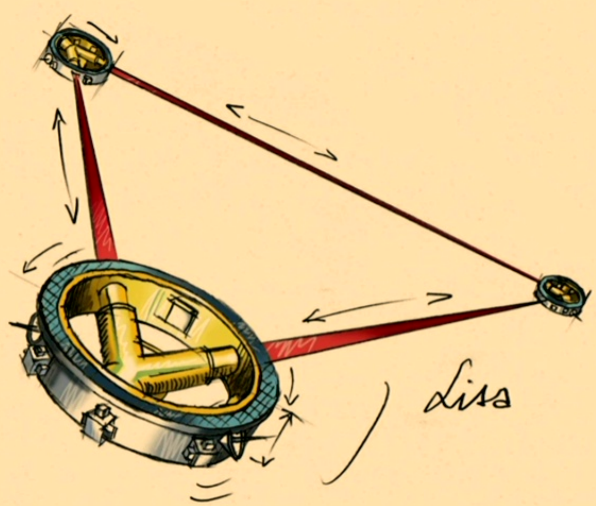
- Compact objects with masses between $10 M_\odot$ to $10^6 M_\odot$ orbiting each other.
- Potentially spinning around different axis.
- Complex, eccentric orbits.
- Primary sources of gravitational waves.

Laser Interferometer Gravitational-Wave Observatory (LIGO)



- Frequency sensitivity at $10\text{Hz} < f < 10^4\text{Hz}$.
- Comparable-mass inspirals of neutron stars or black holes between $1M_\odot$ and $10M_\odot$.
- Typical signals last seconds.

Laser Interferometer Space Antenna (LISA)

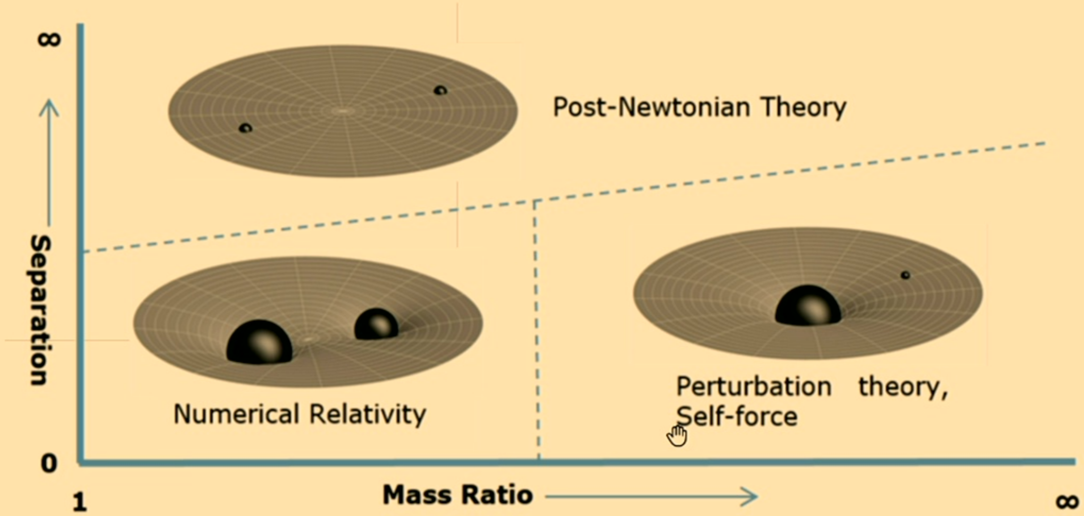


- Will launch in (hopefully) 10 years.
- Frequency sensitivity at 10^{-4}Hz$< f < 10^{-1}$Hz
- Typical sources have masses $\sim 10^6 M_\odot$:
  - Comparable-mass inspirals of supermassive black holes.
  - Extreme-mass ratio Inspirals of stellar-mass objects into supermassive black holes.
- Typical signals can last months.

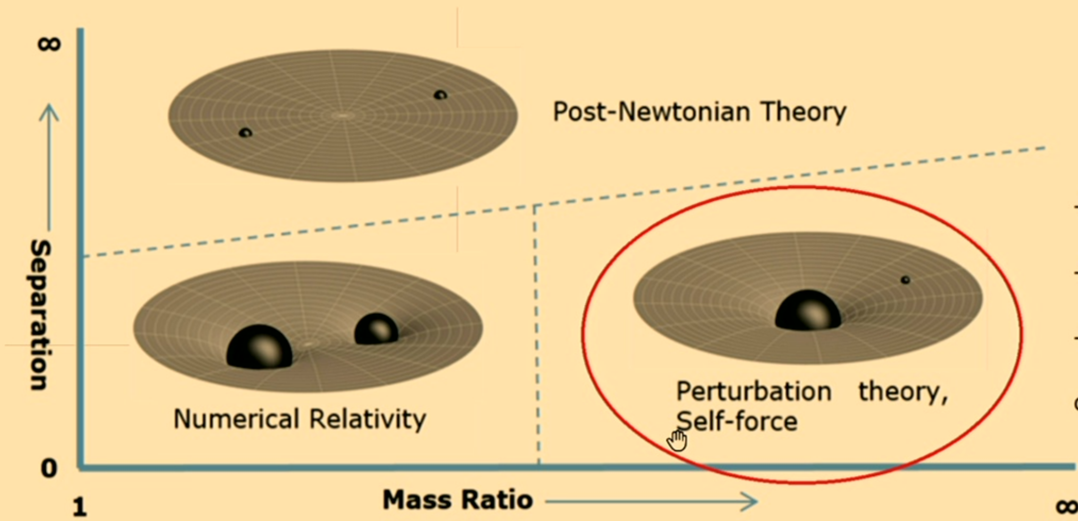
Extreme mass-ratio inspirals (EMRI's)

- Separation of scales: The primary has mass M and the secondary μ.
  - Orbital timescale $T_o = M$
  - Radiation-reaction timescale $T_r = \frac{M^2}{\mu}$
- Goes through $\frac{M}{\mu} \sim 10^5$ cycles before merging.
- Unparalleled probe of strong-gravity region around the primary.
- Longer signals will require higher accuracy for the waveforms.

Binary systems in different regimes



Binary systems in different regimes



-Models EMRI's.

-Models IMRI's.

-Useful for crosschecks with other approximations.

Conservative and Dissipative sectors of the dynamics?

What are their effects?

- The dissipative piece drives the inspiral.
- The conservative piece modifies orbital features such as the ISCO frequency, periastron advance, etc.

Conservative and Dissipative sectors of the dynamics?

Why is it useful?

- Usually we only need one piece of the dynamics to calculate waveforms.
- The conservative sector can be casted as a Hamiltonian system which greatly simplifies calculations.
- Hamiltonian formulation allows to explore integrability.

Crash course on Hamiltonian mechanics

- Phase space coordinates:

$$Q^A = (x^i, p_i) \tag{1}$$

- Non-degenerate symplectic form:

$$\Omega = dp_i \wedge dx^i \tag{2}$$

- Hamiltonian function $H(Q)$ that determines flow on phase space:

$$\frac{dQ^A}{d\lambda} = \Omega^{AB} \frac{\partial H}{\partial Q^B} \tag{3}$$

Action principle

The equations of motion can be derived from an action principle

$$S[x,p] = \int p_i dx^i - \int H_0(x,p)dt \tag{4}$$

varying respect to x^i and p_i.

Integrable systems

A N-dimensional Hamiltonian system is integrable when:

- There exists N conserved quantities (including the Hamiltonian).
- which all commute with each other.

Perturbations on Hamiltonian systems

An arbitrary perturbation looks like

$$\frac{dQ^A}{d\lambda} = \Omega^{AB}\frac{\partial H}{\partial Q^B} + F^A \quad (5)$$

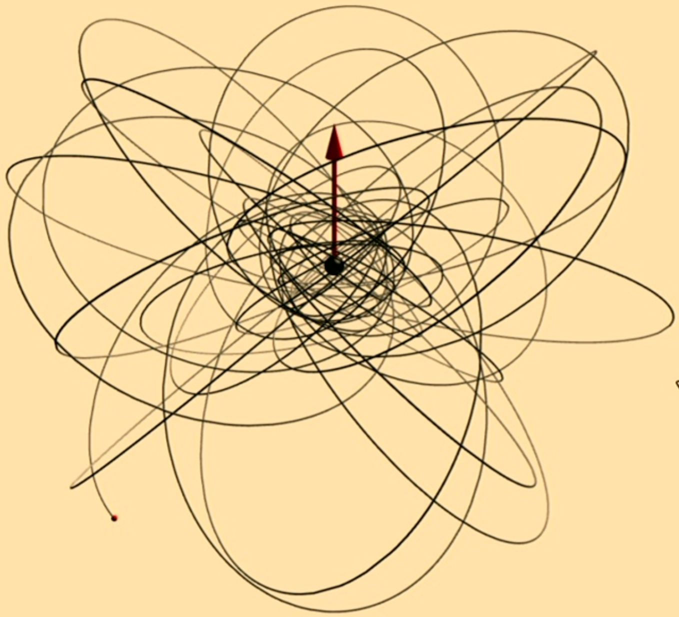
When can this be recasted as a Hamiltonian system?

$$\Omega^{AB}\frac{\partial H}{\partial Q^B} + F^A = \tilde{\Omega}^{AB}\frac{\partial \tilde{H}}{\partial Q^B} \quad (6)$$

Some results on Hamiltonian perturbations

Approximation method	**Small parameter**	**Zeroth Order**	**Is conservative piece Hamiltonian?**
Post-Newtonian	$\frac{1}{c^2}$	Newtonian gravity	Up to 4PN 1805.07240
Post-Minkowskian	G	Special relativity	Up to 3PM 1901.04424
Small mass ratio	$\frac{m}{M}$	Geodesic motion	Linear in m/M 2205.01667 (Flanagan and FMB)

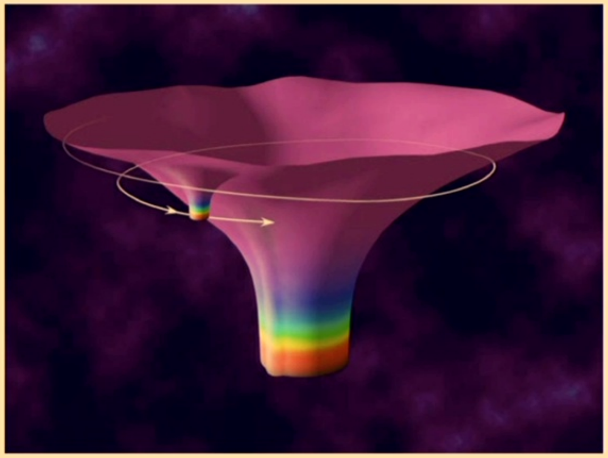
Geodesic motion in Kerr is Hamiltonian



- It has three conserved quantities (E, J, C).
- It is also integrable.

What happens when we perturb this with the self-force?

Dynamics of binaries in the small mass-ratio regime



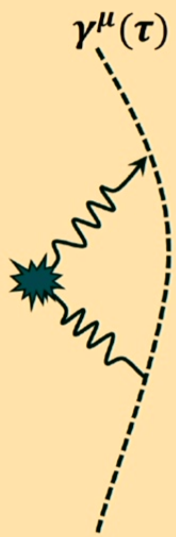
- The primary sources the background metric $g_{(0)}$.
- The secondary moves on an effective metric $\tilde{g} = g_{(0)} + h$.
- Geodesic motion in the effective metric is equivalent to forced motion with respect to the background:

$$\underbrace{\frac{\tilde{D}^2\gamma^\mu}{d\tau^2}}_{\text{wrt } \tilde{g}} = 0 \quad \Leftrightarrow \underbrace{\frac{D^2\gamma^\mu}{d\tau^2}}_{\text{wrt } g_{(0)}} = F^\mu \quad (7)$$

- The self-force is schematically:

$$F^\mu(z) \propto \nabla^\mu \int G\big[z, \gamma(\tau')\big] d\tau' \tag{8}$$

- Written in terms of a 2-point function $G(x, x')$ sourced by the complete worldline $\gamma^\mu(\tau)$ of the object.



Object can emit GW's that scatter off the curvature and are reabsorbed $\rightarrow$ Integral over full past history.

Leading order conservative/dissipative split of the self-force

At first order, different pieces of the self-force are constructed using different Green's functions.

- **Retarded/Advanced** $G^{\pm}(x,y)$
- **Conservative** $G^{C}(x,y) = \frac{1}{2}\left[G^{+}(x,y) + G^{-}(x,y)\right]$: Symmetric under $x \leftrightarrow y$ and even under time-reversal.
- **Dissipative** $G^{D}(x,y) = \frac{1}{2}\left[G^{+}(x,y) - G^{-}(x,y)\right]$: antisymmetric under $x \leftrightarrow y$ and odd under time-reversal.

Summary of self-force effects

Name of effect	Energy scaling	Phase shift after inspiral	Hamiltonian?
Geodesic motion	m	$1/\epsilon$	✓
1st Order dissipative SF	$m\epsilon$	$1/\epsilon$	✗
1st Order conservative SF	$m\epsilon$	1	✓ Flanagan and FMB 2205.01667
2nd Order dissipative SF	$m\epsilon^2$	1	✗
2nd Order conservative SF	$m\epsilon^2$	ϵ	?
Leading spin-curvature coupling	$\frac{S}{M} \sim m\epsilon$	1	✓ Witzany et al. 1808.06582
First order dissipative spin-induced SF	$\frac{Sm}{M^2} \sim m\epsilon^2$	1	✗
First order conservative spin-induced SF	$\frac{Sm}{M^2} \sim m\epsilon^2$	ϵ	✓ Flanagan and FMB 2302.10233

How to build your own Hamiltonian

Simplest example of a perturbation F^μ that is Hamiltonian is when it's the gradient of a potential:

$$F^\mu(x) = \nabla^\mu V(x) \tag{9}$$

The Problem with Tail terms

The self-force at the particle's position z is schematically

$$F^\mu(z) \propto \nabla^\mu \int G[z, z_{\tau'}] d\tau' \tag{10}$$

where the gradient acts on the local z but not on its functional dependence $z_{\tau'}$.

Problem: It looks like the gradient of a potential $\nabla^\mu V(z)$ but it's actually $\nabla^\mu V(z;[z])$! The Hamiltonian should be a local function of phase space variables (z, p), not a functional!

Derivation from action principle

Unperturbed motion can be derived from an action principle

$$S_0[z,p] = \int p_\mu dz^\mu - \int H_0(z,p)ds \qquad (11)$$

The 1st order dynamics can be derived from a **non-local action principle**

$$S[z,p] = S_0[z,p] - \frac{1}{2}\int dsds' G\Big[z(s), z(s')\Big] \qquad (12)$$

Important: The two integrals pick only the symmetric piece of $G(x,y)$: We get the **conservative** dynamics.

General result in dynamical systems

Any non-local Hamiltonian system with phase space coordinate $Q^A = (z^\mu, p_\mu)$

$$S[z,p] = S_0[z,p] - \sum_{n=2}^{N} \frac{\epsilon_n}{n} \int ds_1 \dots ds_n G_n\Big[Q(s_1), \dots, Q(s_n)\Big] \tag{13}$$

admits a local Hamiltonian description to any finite order in the $\{\epsilon_n\}$ with Hamiltonian and symplectic form

$$H(Q) = H_0(Q) + \sum_{n=2}^{N} \epsilon_n H_n(Q) \tag{14a}$$

$$\Omega(Q) = \Omega_0(Q) + \sum_{n=2}^{N} \epsilon_n \Omega_n(Q) \tag{14b}$$

Sketch of the proof

1. The self-force is:

$$F_A(Q,[Q]) = \epsilon \partial_A \int G[Q, Q_{\tau'}] d\tau' + O(\epsilon^2) \tag{15}$$

2. Order-reduction: Replace $Q_{\tau'} \to Q^{(0)}_{\tau'}(Q)$. Now self-force is local

$$F_A(Q) = \partial^{(1)}_A \int G[Q, Q^{(0)}_{\tau'}(Q)] d\tau' + O(\epsilon^2) \tag{16}$$

3. Define Hamiltonian

$$H(Q) = H_0 + \int G[Q, Q^{(0)}_{\tau'}(Q)] d\tau' + O(\epsilon^2) \tag{17}$$

4. Add correction to symplectic form such that it cancels extra derivative in H.

Canonical Coordinates to leading order in ϵ_n

Find new coordinates

$$\tilde{Q}^A = Q^A + \sum_{n=2}^{N} \epsilon_n \xi_n^A + O(\epsilon_n^2) \tag{18}$$

to put symplectic form in canonical form. We get

$$H(\tilde{Q}) = H_0(\tilde{Q}) \tag{19a}$$
$$+ \frac{1}{2} \sum_{n=2}^{N} \epsilon_n \int ds_2 \dots ds_n \mathcal{G}_n \left[\tilde{Q}, \tilde{Q}_{s_2}^{(0)}(\tilde{Q}), \dots, \tilde{Q}_{s_n}^{(0)}(\tilde{Q})\right] + O(\epsilon_n^2)$$
$$\Omega(\tilde{Q}) = \delta \tilde{p}_\mu \wedge \delta \tilde{x}^\mu + O(\epsilon_n^2) \tag{19b}$$

Effect of the self-force on Integrability

- Zach Nasipak (2207.02224) showed that the conservative **scalar** self-force breaks integrability.
- With our formalism, we found the conditions that would need to be satisfied to keep integrability. Need numerics to check!

Adding spin of the secondary

- In 2302.10233 (Flanagan, FMB) we added the effect of the secondary's spin to first order.
- Adding spin modifies the zeroth order dynamics, as well as the N-point functions in the self-force, but the proof still applies!

Second order self-force... finally

If the 1st order self-force looks like:

$$F^\mu(z) \propto \varepsilon \nabla^\mu \int G\left[z, z_{\tau'}\right] d\tau' \tag{20}$$

then the 2nd order self-force *should look like*:

$$\begin{aligned} F^\mu(z) \propto\ & \varepsilon \nabla^\mu \int G_{\text{II}}\left[z, z_{\tau'}\right] d\tau' \\ & + \varepsilon^2 \nabla^\mu \int G_{\text{III}}\left[z, z_{\tau'}, z_{\tau''}\right] d\tau' d\tau'' \end{aligned} \tag{21}$$

Subtleties at 2nd Order: What's the conservative piece?

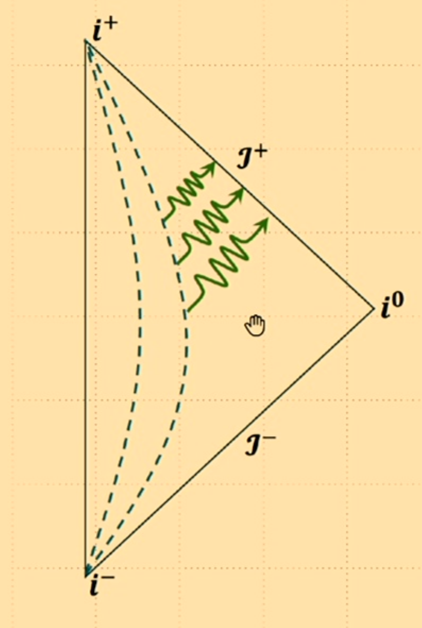
- At 1st order, the **conservative** piece of G^+ is simply $G^C = \frac{1}{2}\Big[G^+ + G^-\Big]$.
- But Einstein's equations are non-linear! 2nd order self-force will have products $G^+ \times G^+ \times G^+$.
- We get combinations

$$G^C \times G^C \times G^C, \quad G^D \times G^D \times G^C, \quad G^D \times G^D \times G^D \ldots$$

  What's the **prescription** for picking the conservative piece?
- **Solution**: We pick the piece that is **symmetric** and **even**!

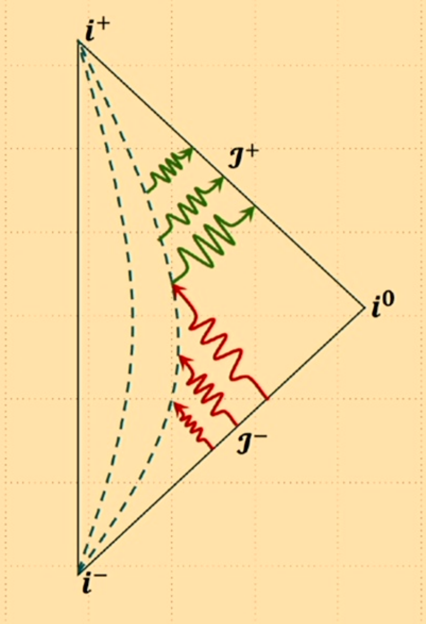
Subtleties at 2nd Order: IR divergences

When we use the retarded Green's function G^+ the system emits radiation to future infinity:



Subtleties at 2nd Order: IR divergences

When we use the conservative Green's function G^C, the system has standing waves up to i^0!



- At 1st order this is not a problem...
- But the 1st order waves are sources for the 2nd order perturbations!
- $\rightarrow$ Divergent energy at infinity.

Subtleties at 2nd order: Don't even have a 3-point function!

Current methods for calculating the 2nd order self-force don't give a simple closed expression for the 3-point function G_{III}.

Recent work

We are working with Abraham Harte on a new formalism to calculate the second-order self-force in a more "elegant" way.

- So far only applied to **scalar self-forces**.
- It doesn't have point-particle divergences to deal with.
- Gives simple expression for the self-force in terms of N-point functions to any order.
- It's fully symmetric piece is given by G^C and products of G^C at all orders.
- It still has IR divergences! So it only applies to unbound orbits for now.

Toy model with a scalar field ϕ coupled to matter ρ

$$S[\phi,\rho] = \int dV \left\{ L_{\text{matter}}(\rho) + \left[\frac{\nabla^\mu\phi\nabla_\mu\phi}{2} - V(\phi) - \rho\phi\right] \right\} \quad (22)$$

with EoM's

$$\Box\phi + V'(\phi) = -\rho \quad (23)$$

$$\nabla^\mu T^{\text{matter}}_{\mu\nu} = \rho\nabla_\nu\phi \quad (24)$$

- As ρ moves through some curved spacetime, it will experience a self-force $\rho\nabla_\nu\phi$.
- ϕ will have divergences on the point-particle limit.
- We want an effective field $\hat{\phi}$ that gives the same self-force but is a vacuum solution.

Renormalized EoM's

- We do a change of coordinates

$$\begin{aligned} \phi(x) &= \Phi_p(x; \hat{\phi}, \rho] \\ &= \hat{\phi} + \Phi_S(x; \hat{\phi}, \rho] \end{aligned} \tag{25}$$

- And define an effective action $S_e[\hat{\phi}, \rho] = S\left[\Phi_p[\hat{\phi}, \rho], \rho\right]$
- We pick Φ_S such that we get renormalized EoM's

$$\Box\hat{\phi} + V'(\hat{\phi}) = 0 \tag{26}$$

$$\nabla^\mu \hat{T}^{\text{matter}}_{\mu\nu} = \hat{\rho}\nabla_\nu\hat{\phi} \tag{27}$$

Expand in powers of ρ and take point particle limit

The scalar charge becomes

$$\hat{\rho}(x) \to q\delta\left[x - \gamma(\tau)\right] \tag{28}$$

and the equations become

$$\nabla^{\mu}\hat{T}^{\text{matter}}_{\mu\nu} = \hat{\rho}\nabla_{\nu}\hat{\phi} \tag{29}$$

$$\downarrow$$

$$\frac{D^2}{ds^2}(m\gamma^{\mu}) = q^2\nabla^{\mu}\int ds' G_{\text{II}}\left[\gamma, \gamma(s')\right] + q^3\nabla^{\mu}\int ds' ds'' G_{\text{III}}\left[\gamma, \gamma(s'), \gamma(s'')\right] + O(q^4) \tag{30}$$

Takeaways

- The conservative piece of the **1st order gravitational** self-force admits a Hamiltonian description (including spin of the secondary).
- Already at first order, the self-force very likely destroys integrability (Need numeric check).
- Developed **nonperturbative self-force approach** that gives self-force in terms of finite N-point functions to any order.
- Unique prescription for conservative sector at second order: **Fully symmetric under exchange of arguments** and **even under time-reversal** piece of the N-point functions.
- The conservative piece of the **2nd order scalar** self-force admits a Hamiltonian description.
- IR divergences of the conservative dynamics are still a problem!
- We are working to generalize all this to the gravitational case.