Title: Cohomological description of contextual measurement-based quantum computations â€” the temporally ordered case

Speakers: Robert Raussendorf

Collection: Foundations of Quantum Computational Advantage

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Abstract: It is known that measurement-based quantum computations (MBQCs) which compute a non-linear Boolean function with sufficiently high probability of success are contextual, i.e., they cannot be described by a non-contextual hidden variable model. It is also known that contexuality has descriptions in terms of cohomology [1,2]. And so it seems in range to obtain a cohomological description of MBQC. And yet, the two connections mentioned above are not easily strung together. In a previous work [3], the cohomological description for MBQC was provided for the temporally flat case. Here we present the extension to the general temporally ordered case.

[1] S. Abramsky, R. Barbosa, S. Mansfield, The Cohomology of Non-Locality and Contextuality, EPTCS 95, 2012, pp. 1-14
[2] C. Okay, S. Roberts, S.D. Bartlett, R. Raussendorf, Topological proofs of contextuality in quantum mechanics, Quant. Inf. Comp. 17, 1135-1166 (2017).
[3] R. Raussendorf, Cohomological framework for contextual quantum computations, Quant. Inf. Comp. 19, 1141-1170 (2019)

This is jount work with Polina Feldmann and Cihan Okay

Putting contradictions to work,

now in a temporally ordered fashion

Robert Raussendorf
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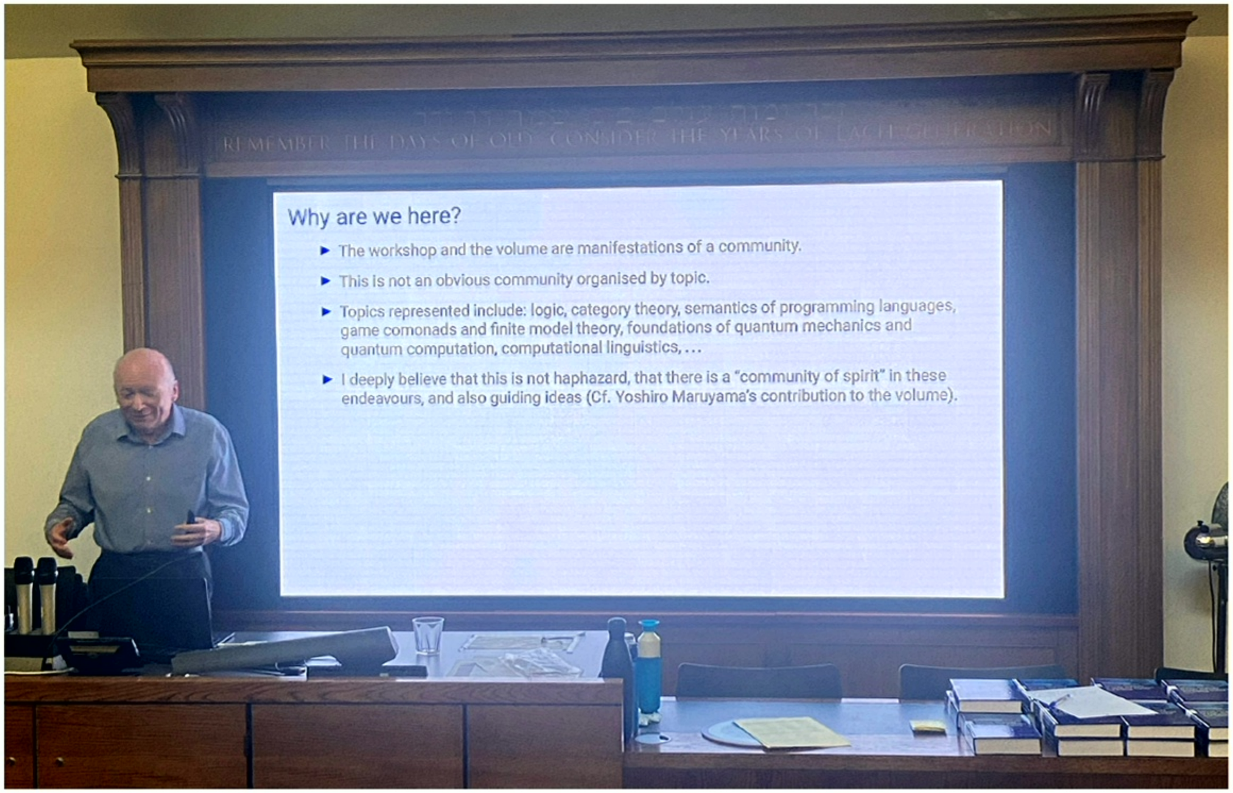
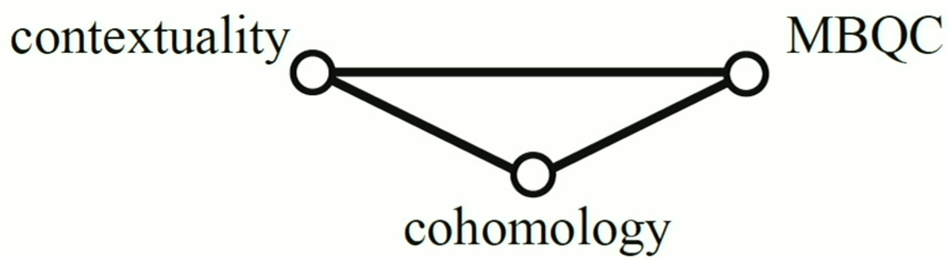
Joint work with Polina Feldmann (UBC) and Cihan Okay (Bilkent)


REMEMBER THE DAYS OF OLD CONSIDER THE YEARS OF EACH GENERATION
Why are we here?
The workshop and the volume are manifestations of a community.
This is not an obvious community organised by topic.
Topics represented include: logic, category theory, semantics of programming languages, game comonads and finite model theory, foundations of quantum mechanics and quantum computation, computational linguistics, ...
I deeply believe that this is not haphazard, that there is a "community of spirit" in these endeavours, and also guiding ideas (Cf. Yoshiro Maruyama's contribution to the volume).

Outline



1. Review: The MBQC-contextuality-cohomology triangle for the case of flat temporal order

2. New: The same for the case with proper temporal order

What's the triangle all about?

A question we ask:

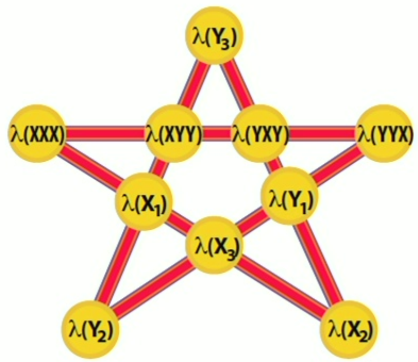
- The Boolean Algebra is at the foundation of classical digital computation.
- *Which structures are at the foundation of quantum computation?*

We don't really know, but
Mermin's star is an example of those foundational structures: it computes, it is contextual, and it is described by cohomology.

The MBQC–contextality–cohomology triangle generalizes the structure present in Mermin's star to all MBQC.

Mermin's star

.. a simple proof of the KS Theorem in dimension $d \geq 8$.



Is there a consistent value assignment $\lambda(\cdot) = \pm 1$ *for all observables in the star?*

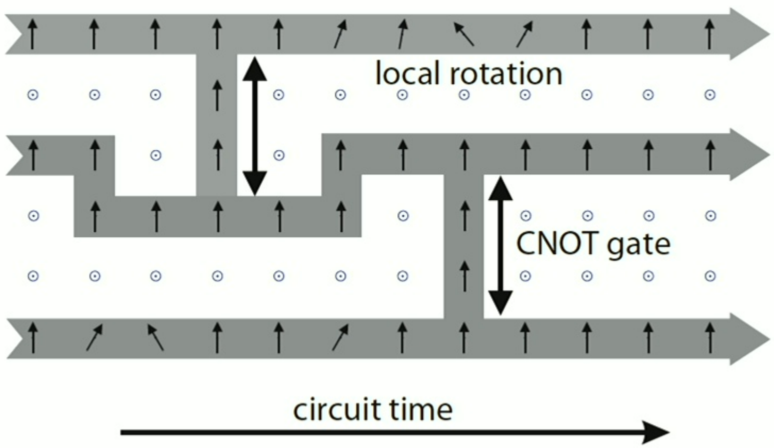
- No consistent non-contextual value assignment λ exists.

  Any attempt to assign values leads to an algebraic contradiction.

.. but there is no temporal order in Mermin's star.

N.D. Mermin, RMP 1992.

Quantum computation by measurement



- Information written onto a cluster state, processed and read out by one-qubit measurements only.
- The resulting computational scheme is *universal*.

R. Raussendorf and H.J. Briegel, PRL 2001.

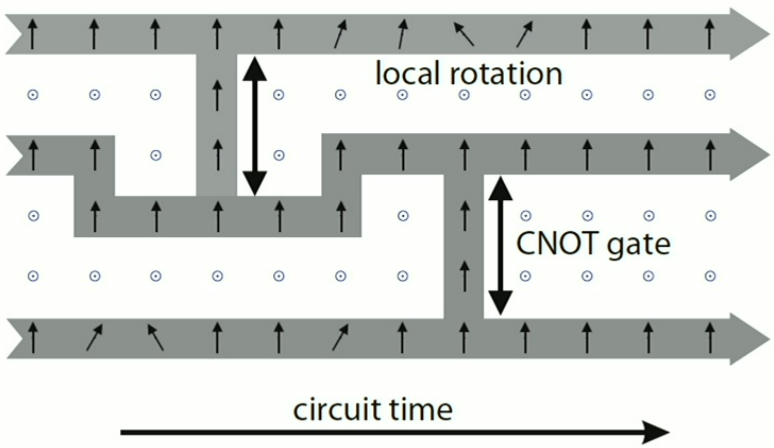
Classical side processing in MBQC

Every measurement outcome is individually random. Classical processing required in the following places:

1. Extract correlations and to obtain computational output.
2. Adapt measurement bases

Classical side-processing is all linear mod 2.

Quantum computation by measurement



- Information written onto a cluster state, processed and read out by one-qubit measurements only.
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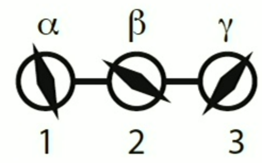
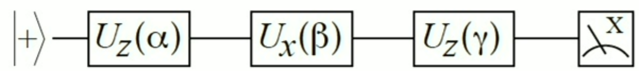
How temporal order comes about

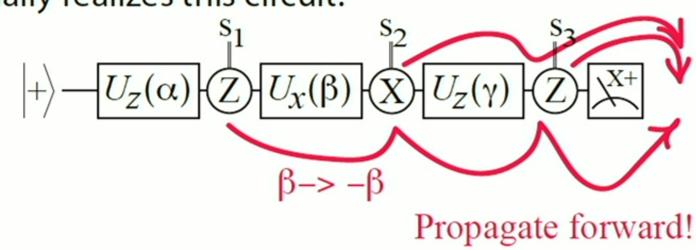
This MBQC on a 3-qubit cluster state



can simulate this circuit:



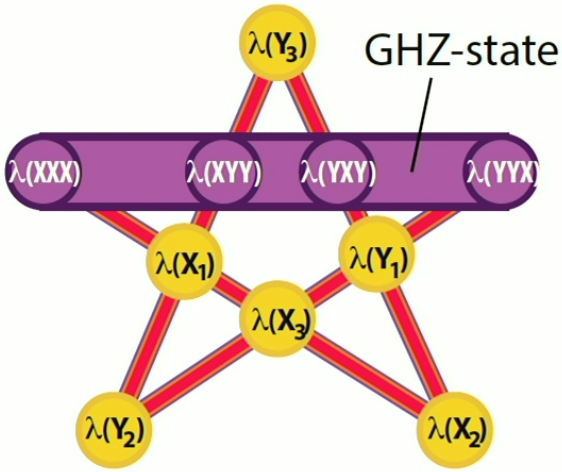
but actually realizes this circuit:



output $o = s_1 \oplus s_3$, basis choice $q_2 = s_1$, $q_3 = s_2$.

Contextuality

Mermin's star has a state-dependent version.



The state-dependent version invokes

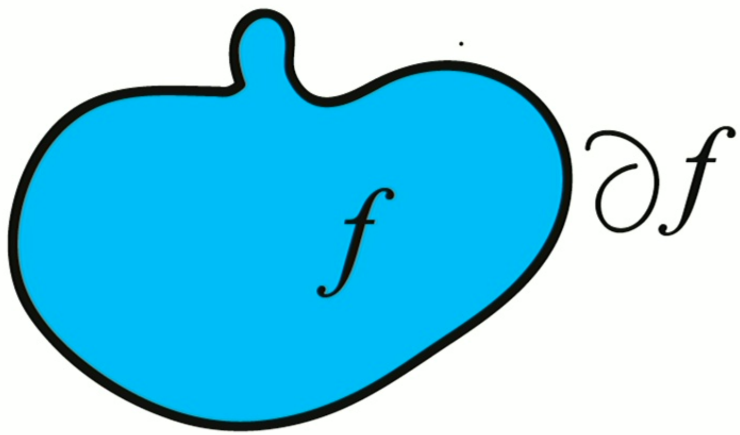
- A GHZ-state
- Only local observables

.

- Still no consistent value assignment λ for the remaining local observables.

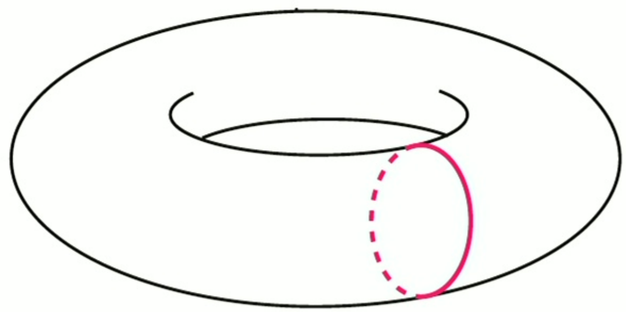
N.D. Mermin, RMP 1992.

Homology and cohomology



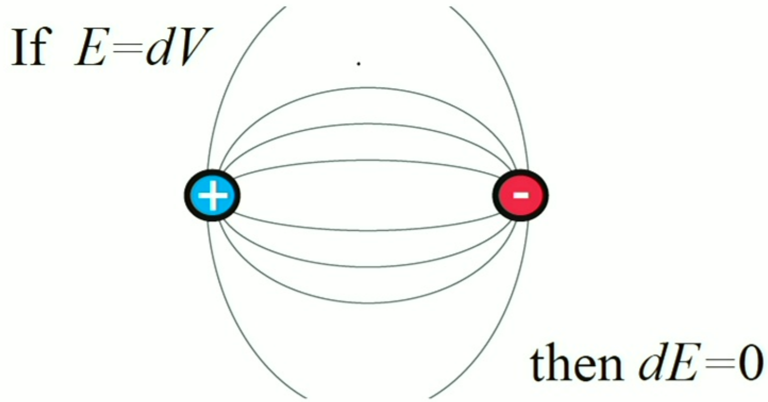
Geometric objects, such as surfaces, have boundaries.

Homology and cohomology



Not every chain with vanishing boundary is itself a boundary of something.

Homology and cohomology

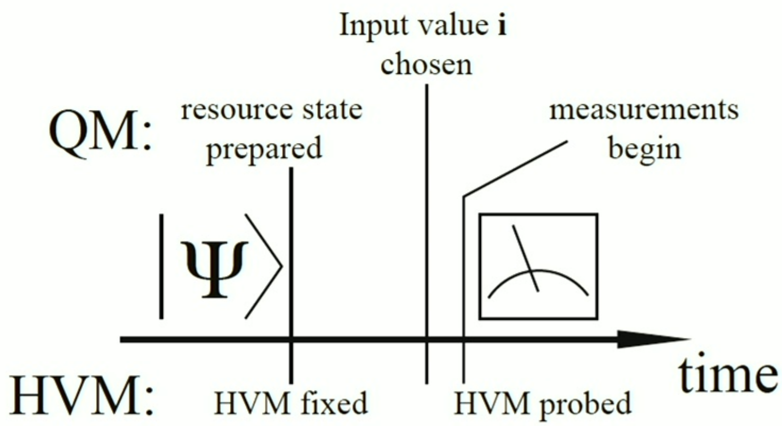


If a field is the gradient of a potential, then its curl vanishes.

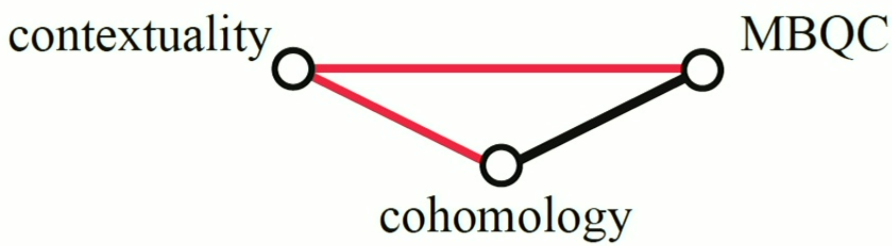
$(ddV = 0)$

Contextuality and MBQC

Our notion of hidden-variable model:

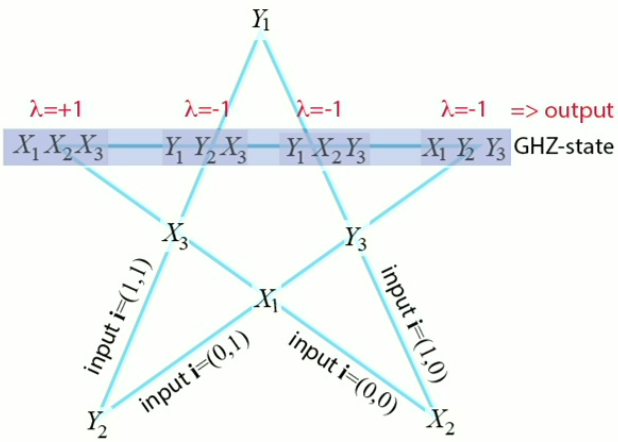


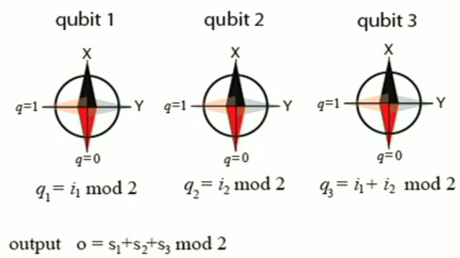
1B: Edges of the triangle



- Contextuality in MBQC
- The cohomology of contextuality

Mermin's KS proof computes!





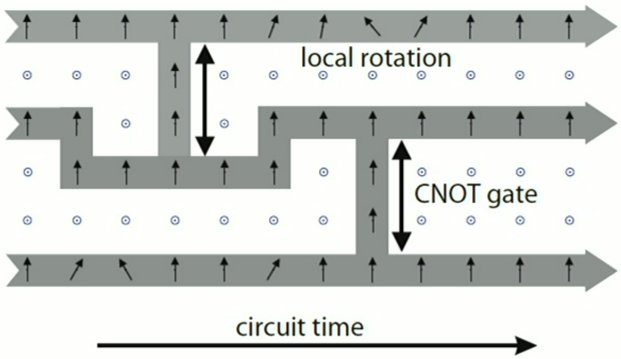
* Use GHZ state as computational resource
* Compute OR-gate

- Classical processing all *linear*, computed OR-gate *non-linear*.

⇒ Classical control computer promoted to classical universality.

J. Anders and D. Browne, PRL 102, 050502 (2009).

Contextuality and MBQC



In MBQC, quantumness is required in the form of contextuality

Theorem 1: An MBQCs that deterministically computes a non-linear Boolean function is contextual.

J. Anders and D. Browne, PRL 102, 050502 (2009).

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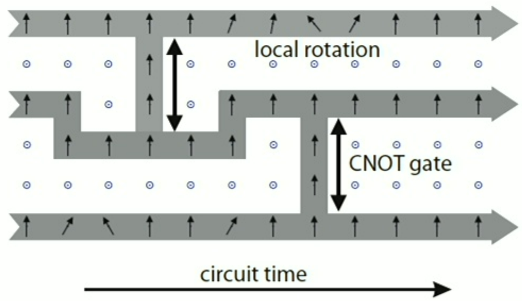
Contextuality and MBQC



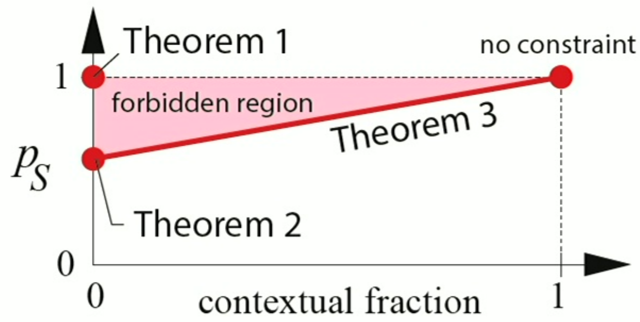
Theorem 2. Consider an MBQC computing a Boolean function $o : \mathbb{Z}_2^m \longrightarrow \mathbb{Z}_2$ with an average success probability p_S. If

$$p_S > 1 - \frac{\mathbb{H}(o)}{2^m},$$

with $\mathbb{H}(o)$ the Hamming distance of o to the closest linear function, then this MBQC is contextual.

R. Raussendorf, PRA, 2013.

Contextuality and MBQC



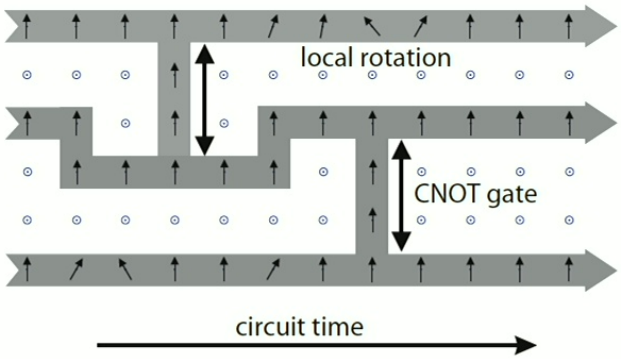
Theorem 3. Consider an MBQC $\mathcal{M}$ characterized by a contextual fraction $CF(\mathcal{M})$ computing a Boolean function $o : \mathbb{Z}_2^m \longrightarrow \mathbb{Z}_2$ with an average success probability p_S. Then it holds that

$$p_S \leq 1 - \frac{1 - CF(\mathcal{M})}{2^m} \mathbb{H}(o).$$

- The larger the contextual fraction, the higher p_S can be.

S. Abramsky, R.S. Barbosa, and S. Mansfield, Phys. Rev. Lett. 119, 050504 (2017).

Contextuality and MBQC



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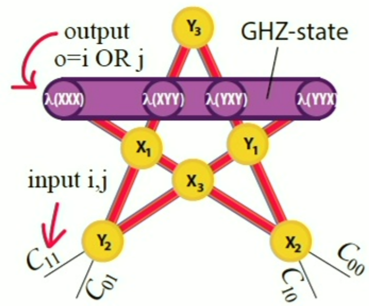
Example: Mermin's star



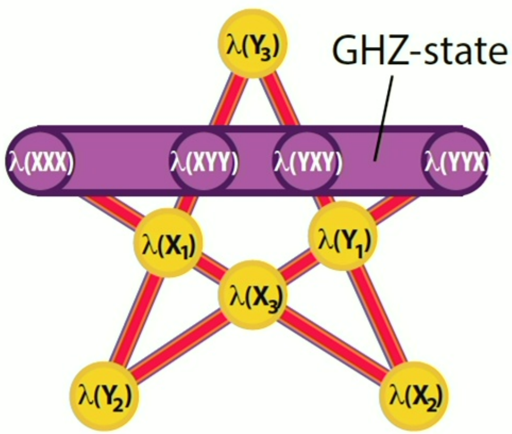
Theorem 2. Consider an MBQC computing a Boolean function $o : \mathbb{Z}_2^m \longrightarrow \mathbb{Z}_2$ with an average success probability p_S. If $p_S > 1 - \mathbb{H}(o)/2^m$, then this MBQC is contextual.

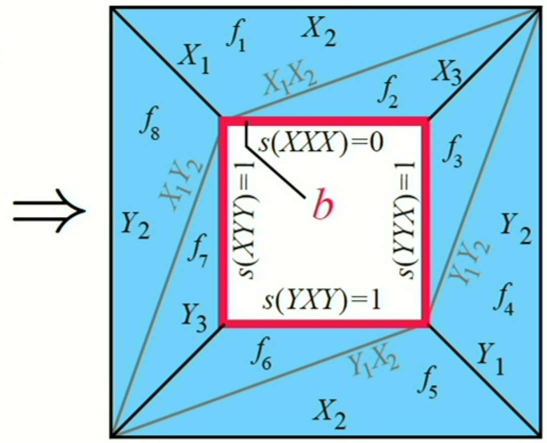
Here, $m = 2$ and $\mathbb{H}(\text{OR}) = 1$, hence the threshold is

$$p_{S,crit} = \frac{3}{4}.$$

This coincides with the Mermin inequality

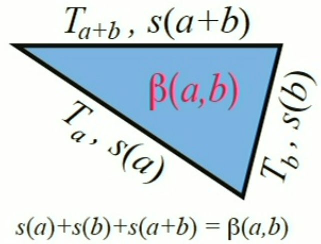
The cohomology of contextuality





- Convert Mermin's star into a chain complex.

Parity proofs–cohomological version



- β is a function defined on the faces,

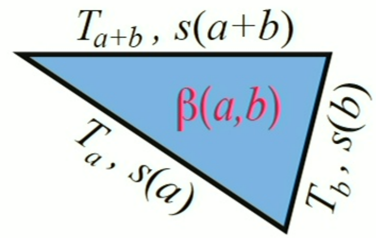
$$T_a T_b T_{a+b} = (-1)^{\beta(a,b)} I, \quad [T_a, T_b] = 0 \text{ etc.} \qquad (1)$$

- β contains all relevant information about the observables T_x

- β is a 2-cochain, $\beta : C_2 \longrightarrow \mathbb{Z}_2$.
  In fact, β is a 2-cocycle, $d\beta = 0$. (follows from $(T_a T_b) T_c = T_a (T_b T_c)$.)

- If β is a non-trivial cocycle ($\beta \neq d\chi$ for any χ), then the setting is contextual.
  (No consistent context-independent value assignment exists.)

Cohomological parity proofs



$s(a)+s(b)+s(a+b) = \beta(a,b)$

- Recall: $T_aT_bT_{a+b} = (-1)^{\beta(a,b)}I$, for all faces (a, b).

- Any ncHVM value assignment s is a 1-cochain, $s : C_1 \longrightarrow \mathbb{Z}_2$. $(-1)^{s(a)}$ is the "measured" eigenvalue of T_a, for all $a \in E$.

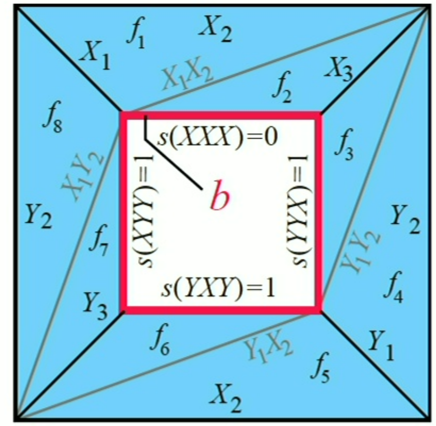
- Eq. (1) implies a relation between β and s (mod 2),

$$\beta = ds.$$

If β is a non-trivial cocycle, then no nc value assignment exists.

C. Okay, S. Roberts, S. Bartlett, R. Raussendorf, Quant. Inf. Comp. 17, 1135-1166 (2017).

Cohomology & Mermin's star



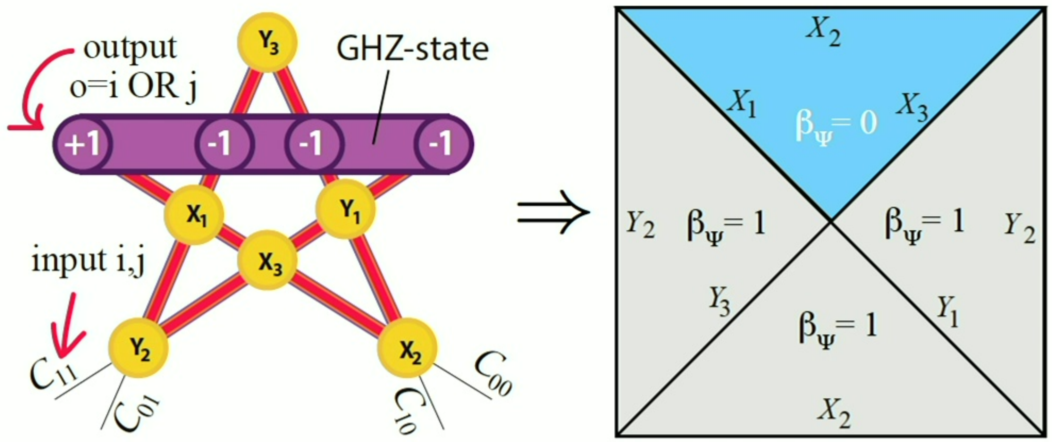
- Two facts:
  - For the faces $f_1, .., f_8$ it holds that $\beta(f_i) = 0$.
  - For the entire surface $F = \sum_{i=1}^{8} f_i$ it holds that $\partial F = b$

Contextuality proof: Assume an nc value assignment exists.

$$\boxed{0 = \int_F \beta = \int_F ds = \int_{\partial F} s = 1 \mod 2}$$

Contradiction.

β_Ψ and computational output

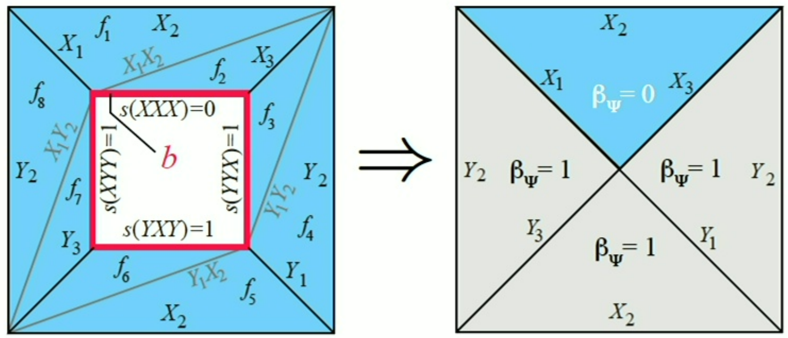


The output function o is contained in β_Ψ,

$$o \subseteq \beta_\Psi.$$

(shown here only for the GHZ-MBQC, but holds in general) ✓

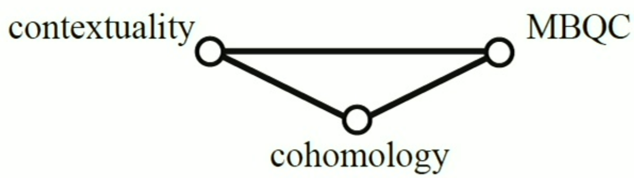
The state-dependent version



- Contract the chain complex into a smaller one.
- Merge cocycle β and partial assignment s_Ψ into a new cocycle $\beta_\Psi := \beta + ds_\Psi \mod 2$.
- If $[\beta_\Psi] \neq 0$ then the MBQC setting is contextual.

  Here: $1 = \int_F \beta_\Psi = \int_F ds = \int_{\partial F} s = 0$. Contradiction.

Summary of the recap



The cocycle class $[\beta_\Psi] \in H^2(\mathcal{C}_R, \mathbb{Z}_2)$ describes temporal flat MBQCs. Namely,

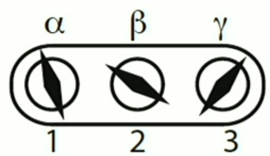
- β_ψ contains the computed function o
- $[\beta_\psi]$ is a contextuality witness
- $[\beta_\psi]$ is a witness for the nonlinearity of the computed function

There is also a probabilistic version of this.

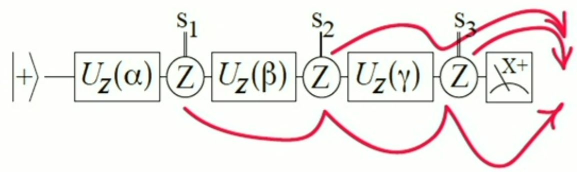
R. Raussendorf, *Cohomological framework for contextual quantum computations,* Quant. Inf. Comp. 19, 1141 - 1170 (2019).)

The new example

Old example: GHZ state

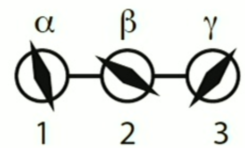


realizes this circuit:

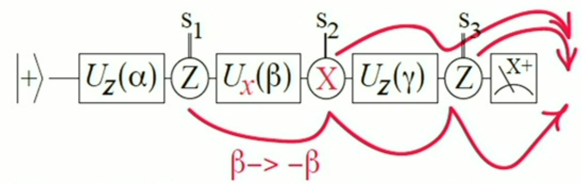


Trivial propagation - flat temporal order

New example: 1D cluster state

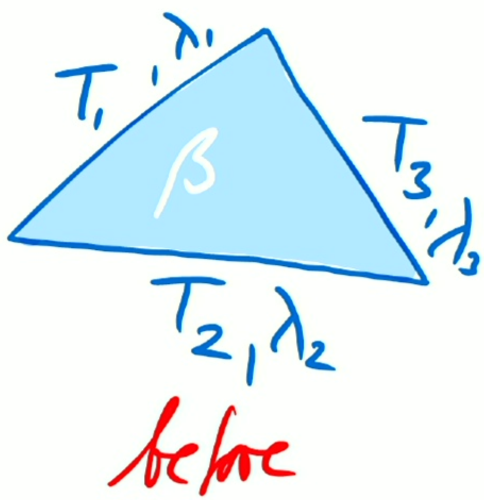


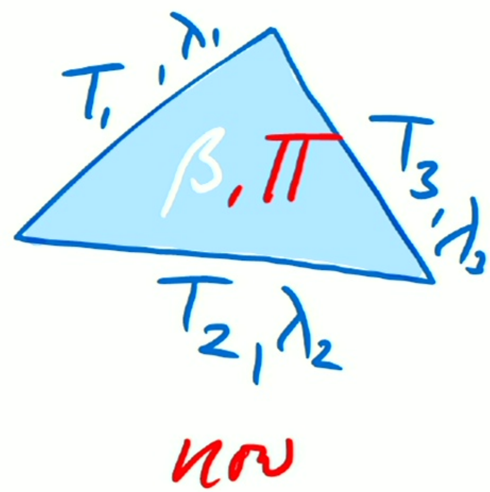
realizes this circuit:



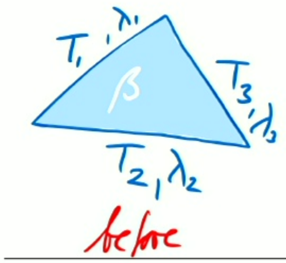
Non-trivial propagation - temporal order

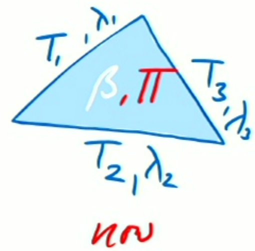
The new ingredient





What changes

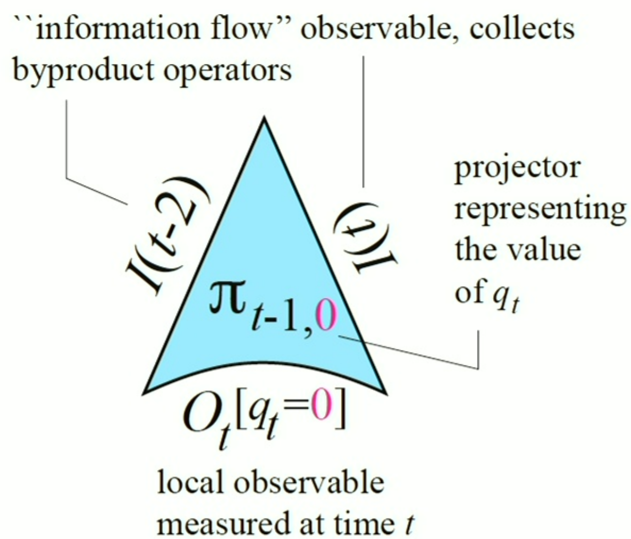




	The observables T_i of interest have block-diagonal form, $$T_i = \Pi T_i \Pi + \overline{\Pi} T_i \overline{\Pi}$$
The observables commute, $$[T_i, T_j] = 0,\ \forall i, j.$$	The observables commute on the subspace given by Π, $$[\Pi T_i \Pi, \Pi T_j \Pi] = 0,\ \forall i, j.$$
The observables are dependent, $$T_1 T_2 T_3 = (-1)^\beta I.$$	The observables are dependent on the subspace given by Π, $$(\Pi T_1 \Pi)(\Pi T_2 \Pi)(\Pi T_3 \Pi) = (-1)^\beta \Pi.$$

The new face describes ..

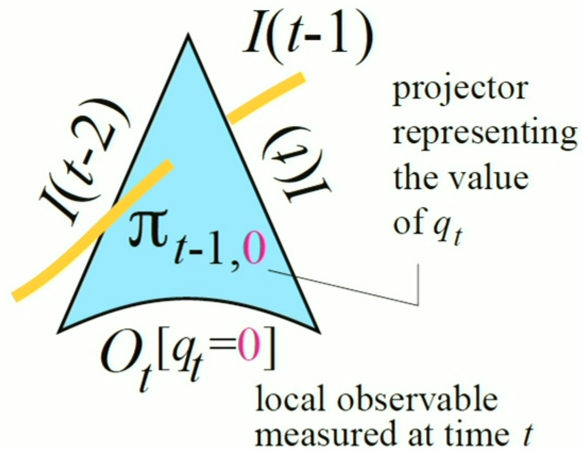
.. an individual act of measurement



The observables $I(t)$ measure the X (Z) component of the byproduct operator at even (odd) times.

Two more things:

(i)

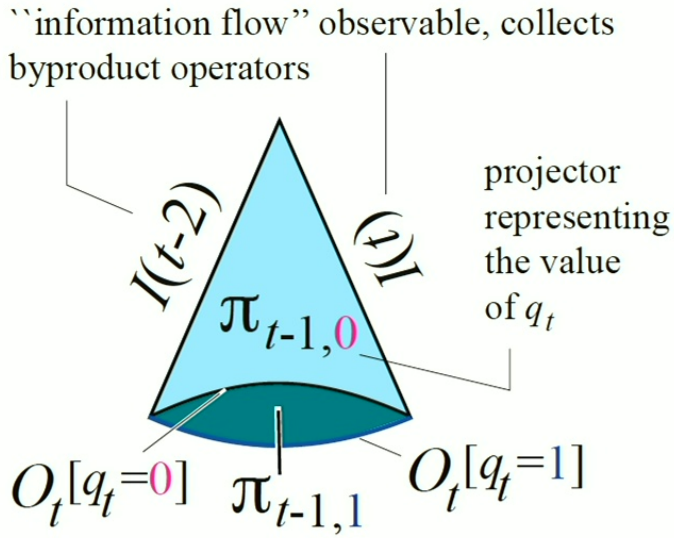


The projectors too are related to the information flow observable,

$$\pi_{t-1,0} = \frac{I + \mathbf{I}(t-1)}{2}, \quad \pi_{t-1,1} = \frac{I - \mathbf{I}(t-1)}{2}$$

Two more things:

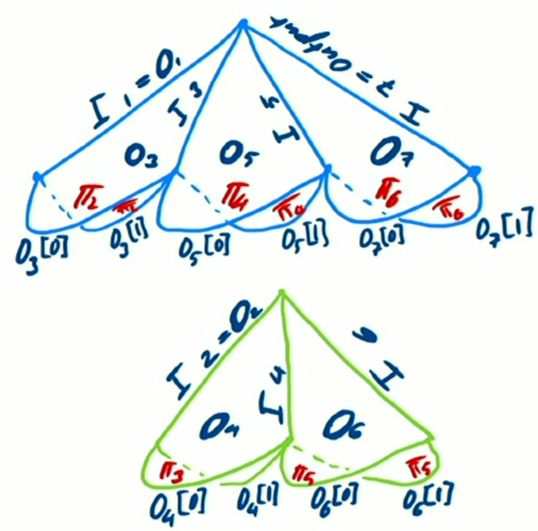
(ii)



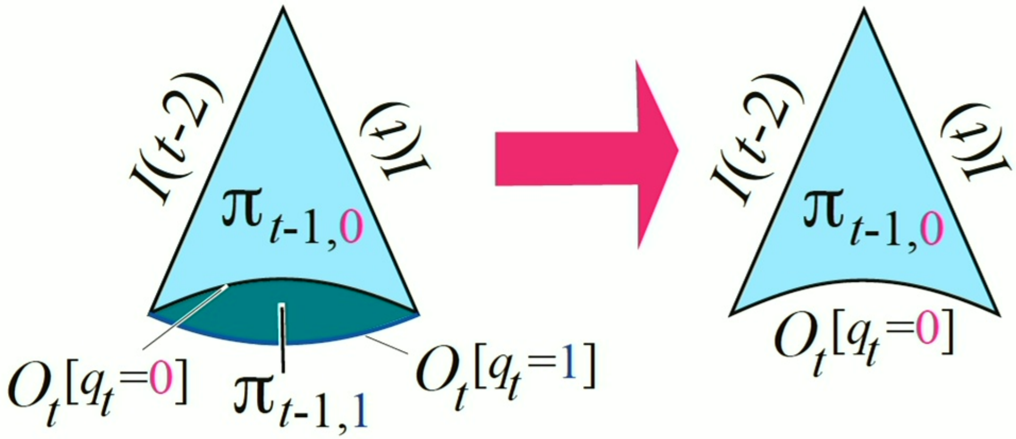
Two faces are needed to describe the act of a single measurement, one for Π_0 and one for Π_1.

What an algorithm now looks like

Let's take the duality out to simplify ..

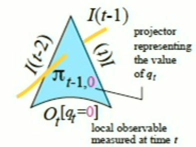


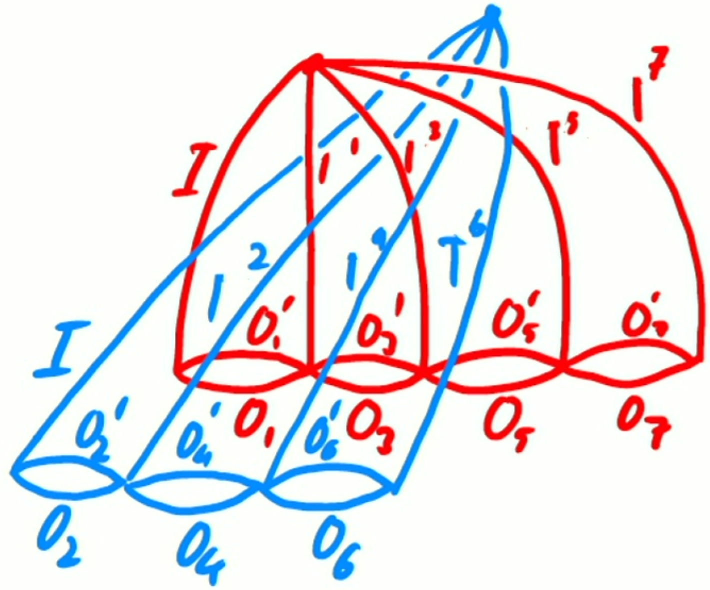
Action of the HVM on the chain complex



At time t, the observables $\mathbf{I}(t-2)$, $\mathbf{I}(t-1)$ already have been assigned values, and $O_t[q]$ have a value straight from the HVM.

What an algorithm now looks like





A pair of interconnected (dual) complexes.