Title: Lagrangian Relations, Half-Densities and BV Fiber Integrals

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Series: Mathematical Physics

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Abstract: Abstract TBA

Zoom link

LAGRANGIAN RELATIONS, $\frac{1}{2}$-DENSITIES & BV FIBER INTEGRALS

2401.06110 with BRAÑO JURČO & MARTIN ZIKA [CHARLES UNI]

INTRODUCTION BV FORMALISM GEOMERIC PERSPECTIVE:

M ← BV space of fields & antifields

odd symplectic supermanifold

$$Z = \int_L e^S \sqrt{\mathcal{D}\phi} \qquad \langle F \rangle = \frac{1}{Z}\int_L F e^S \sqrt{\mathcal{D}\phi}$$

$L \subset M$ Lagrangian

→ Δ-closed $\frac{1}{2}$-density on M, Δ canonical "BV Laplacian" acts on $\frac{1}{2}$-densities

[Ševera]

$$Z = \langle e^S \sqrt{\mathcal{D}\phi} \mid \delta_L \rangle$$

see both as a distributional $\frac{1}{2}$-density

- δ_L is formally Δ-closed, Hamiltonian flows change it by something Δ-exact

Δ-closed

$\langle e^S \sqrt{\mathcal{D}\phi} \mid \delta_L - \delta_{\Phi_{\{H,-\}}}' \rangle = 0$

- Δ is self-adjoint wrt $\langle\ ,\ \rangle$ | $\langle \Delta X \mid \delta_L \rangle$ |

... symplectic supermanifold

[Jevon]: $Z = \langle e^{\frac{S}{\hbar}} \sqrt{\mathcal{D}\phi} \mid \delta_L \rangle$, $L \subset M$ Lagrangian → Δ-closed $\frac{1}{2}$-density on M | Δ canonical "BV Laplacian" acts on $\frac{1}{2}$-densities

see both as a distributional $\frac{1}{2}$-density

- δ_L is formally Δ-closed, Hamiltonian flows change it by something Δ-exact ↙ Δ-closed
- Δ is self-adjoint wrt $\langle \, , \, \rangle$ | $\langle \Delta X \mid \delta_L \rangle$ | $\langle e^{\frac{S}{\hbar}} \sqrt{\mathcal{D}\phi} \mid \delta_L - \delta_{\Phi'_{\{H,-\}}} \rangle = 0$

- extend this idea to form a quantum version of ... symplectic category
  - objects (M, ω)
  - morphisms distributional $\frac{1}{2}$-densities on $\overline{M_1} \times M_2$

$M : \Pi T^* M$

$\mathcal{O}(\Pi T^* M) = PV(M)$

$\mathrm{Dens}^{\frac{1}{2}}(\Pi T^* M) = \Omega^\bullet(M)$

$\Delta = d_{dR}$

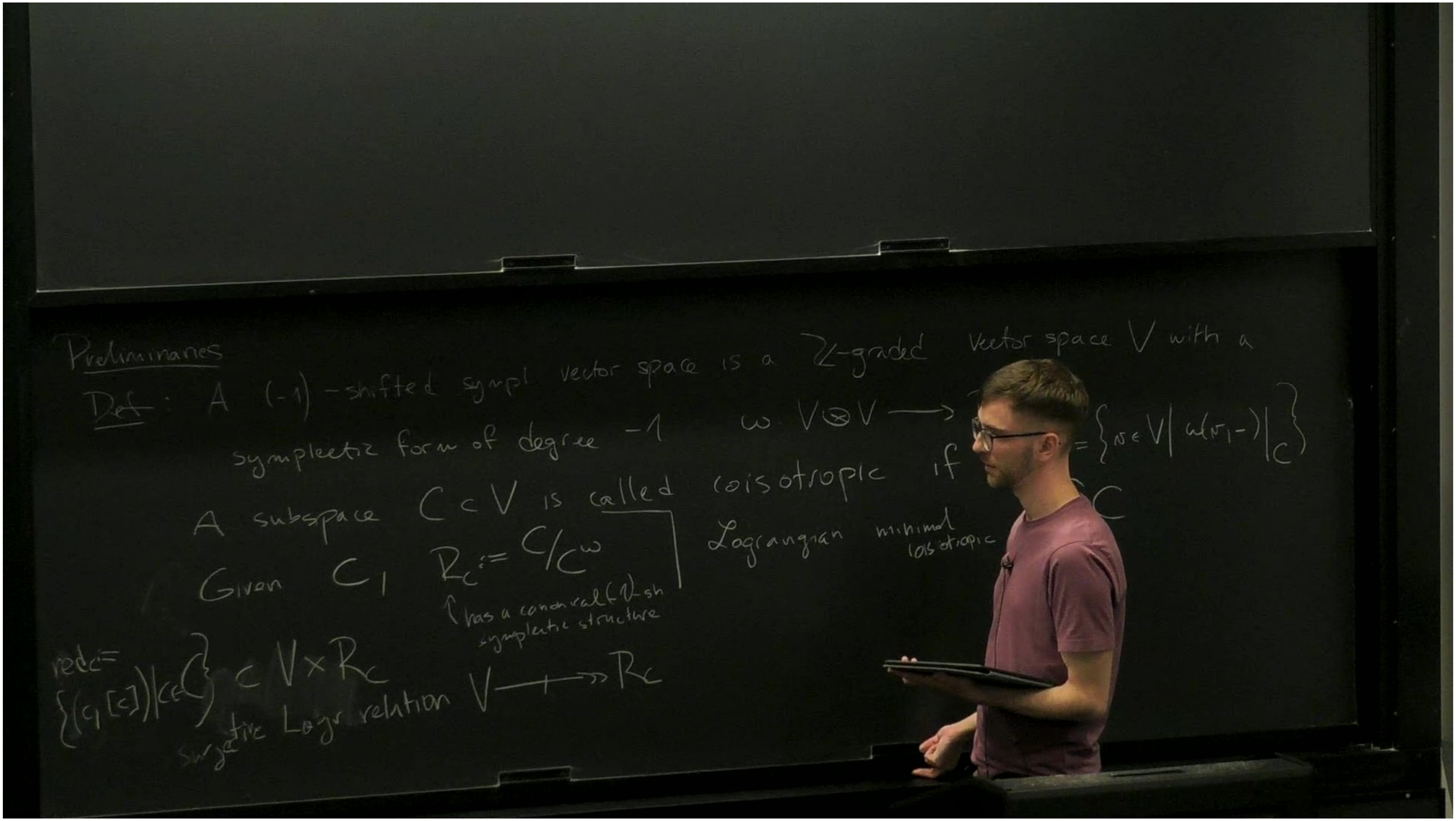
Our work: · we give a rigorous defn of a distr $\frac{1}{2}$-density : finite-dim $\mathbb{Z}$-graded vector spaces

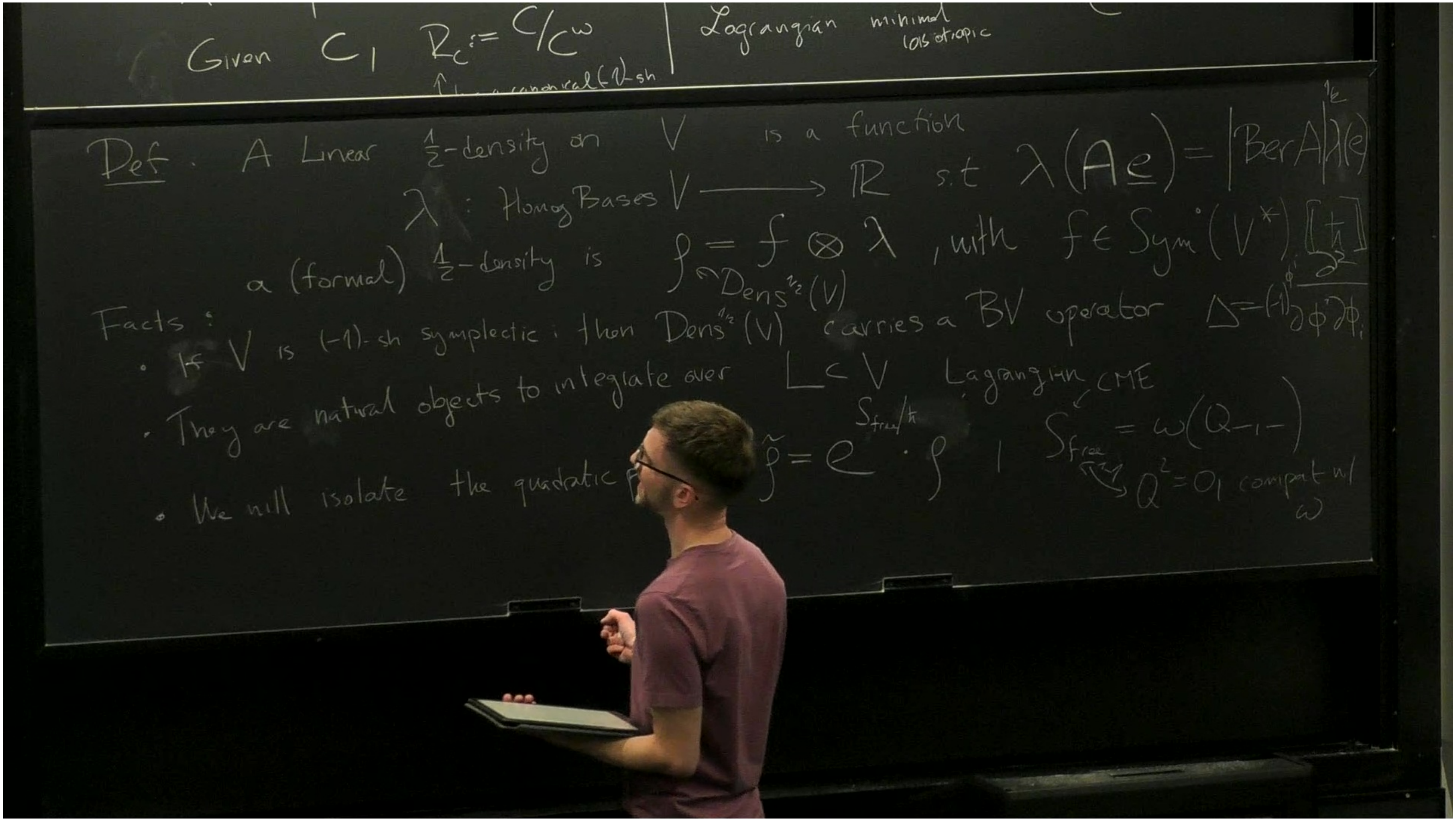
· we define $\mathrm{LinQSymp}_{-1}$ such symplectic category.

Def (ver 0): Let (V, ω) be a (-1)-sh symplectic v.s. A generalized Lagrangian in V is

$V = \Pi T^* V_0$ $\left(C \subset V \text{ coisotropic}, \ \rho \in \mathrm{Dens}^{\frac{1}{2}}(C/C^\omega) \right)$

$(V_0, \lambda \in \mathbb{R}[[\hbar]]) \ d^\bullet v_0$


Preliminaries
Def: A (-1)-shifted sympl vector space is a Z-graded vector space V with a
symplectic form of degree -1
A subspace C ⊂ V is called coisotropic if
Given C, R_C := C/C^ω
has a canonical (-1)-sh symplectic structure
Lagrangian minimal coisotropic
red_C = {(c, [c]) | c ∈ C} ⊂ V × R_C
surjective Lagr relation V → R_C


Given C, R_C := C/C^ω
canonical (-1)-sh
Lagrangian
minimal
isotropic
Def. A Linear ½-density on V is a function
λ : Hom_g Bases V ⟶ ℝ s.t. λ(Ae) = |Ber A|^{1/2} λ(e)
a (formal) ½-density is ρ = f ⊗ λ, with f ∈ Sym(V*)[[ħ]]
Dens^{1/2}(V)
Facts :
• If V is (-1)-sh symplectic, then Dens^{1/2}(V) carries a BV operator Δ = (-1)^{φ^i} ∂²/∂φ^i∂φ_i
• They are natural objects to integrate over L ⊂ V Lagrangian
CME
ρ̃ = e^{S_free/ħ} · ρ | S_free = ω(Q_{-1}, -)
Q² = 0, compat w/ ω
• We will isolate the quadratic

• If V

• They are natural objects to integrate over $L \subset V$ Lagrangian

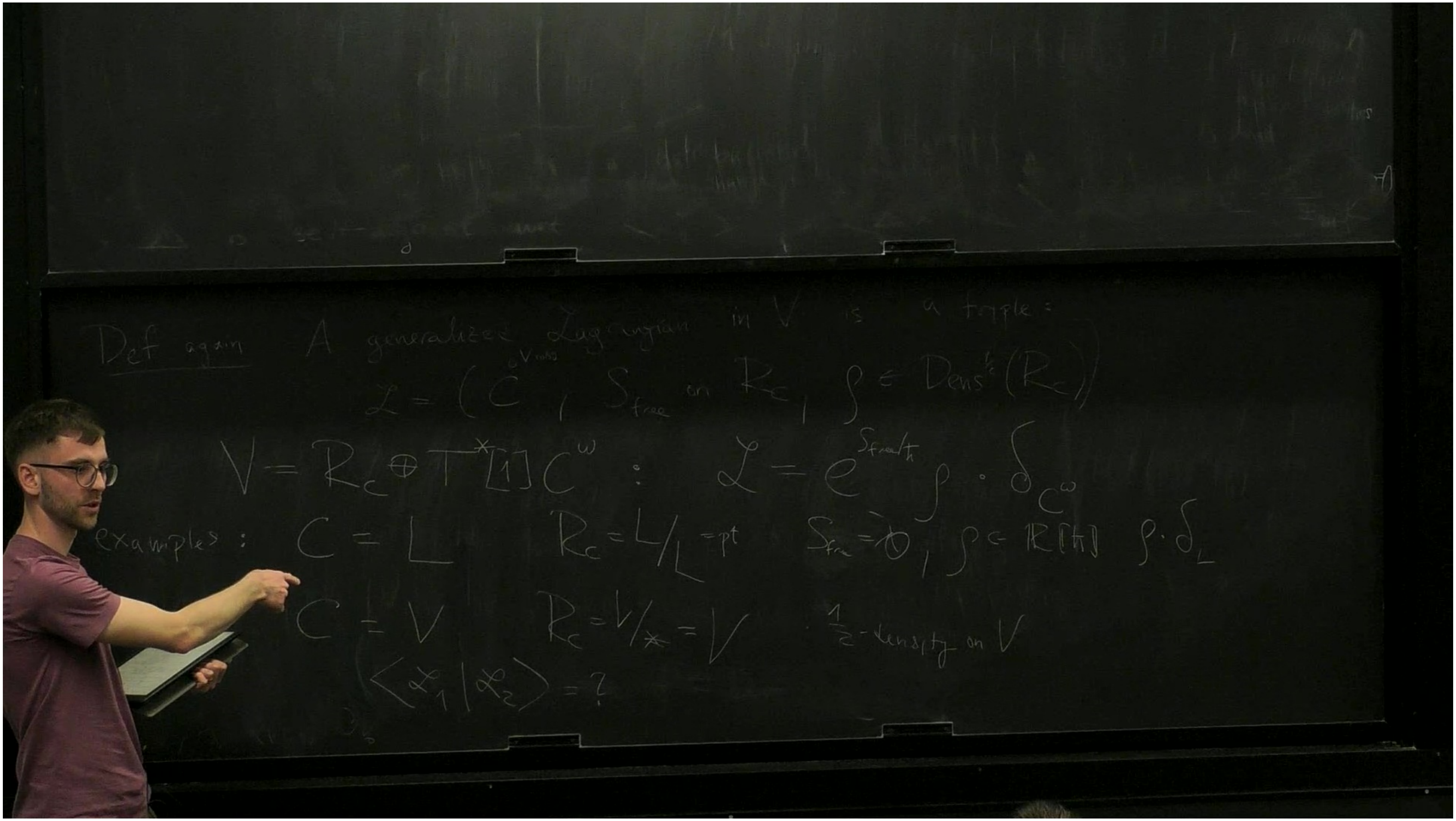
• We will isolate the quadratic part : $\tilde{\rho} = e^{S_{free}/\hbar} \cdot \rho$, $S_{free} = \omega(Q-,-)$ (CME) $\leftrightarrow$ $Q^2 = 0$, compat w/ ω

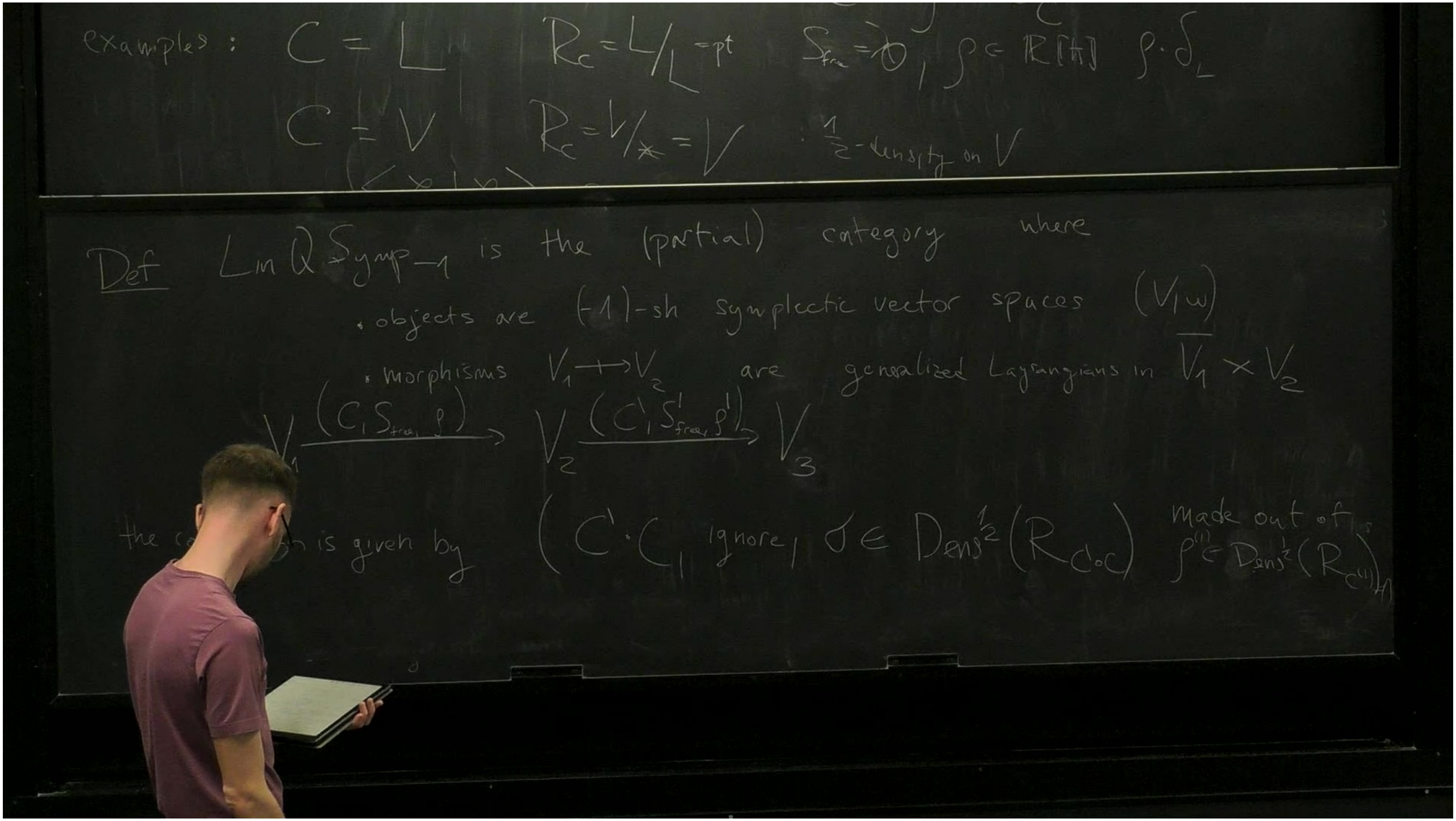
Prop [BV fiber integrals] : take (V, ω, Q), $C \subset V$ s.t. $S_{free} = \omega(Q-,-)$ is nondeg on C^ω. Then we can define a BV fiber integral:

$$\int_{red_C} e^{S_{free}/\hbar} (\ \ ) : Dens^{\frac{1}{2}}(V) \longrightarrow Dens^{\frac{1}{2}}(R_C = C/C^\omega)$$

; restrict to C and integrate out C^ω

; HPL


Def again A generalized Lagrangian in V is a triple:
examples:
C = L
C = V
½-density on V


examples : C = L   R_C = L/L = pt   S_free = 0, ρ ∈ R[[ħ]]   ρ·δ_L
C = V   R_C = V/* = V   : 1/2-density on V
Def LinQSymp_-1 is the (partial) category where
• objects are (-1)-sh symplectic vector spaces (V, ω)
• morphisms V_1 → V_2 are generalized Lagrangians in V̄_1 × V_2
V_1 —(C, S_free, ρ)→ V_2 —(C', S'_free, ρ')→ V_3
the c... is given by (C'∘C, ignore, σ ∈ Dens^1/2(R_C'∘C)   made out of ρ^(1) ∈ Dens^1/2(R_C(1))

$V_1 \xrightarrow{(C,\, S_{free},\, \rho)} V_2 \xrightarrow{(C',\, S'_{free},\, \rho')} V_3$

the composition is given by $\left(C' \circ C,\ \text{ignore},\ \sigma \in \mathrm{Dens}^{\frac{1}{2}}(R_{C' \circ C}) \right)$ made out of $\rho^{(1)} \in \mathrm{Dens}^{\frac{1}{2}}(R_{C^{(1)}})$

Fact there exists a surjective Lagrangian relation $X_{C,C'} : R_C \times R_{C'} \longrightarrow R_{C' \circ C}$

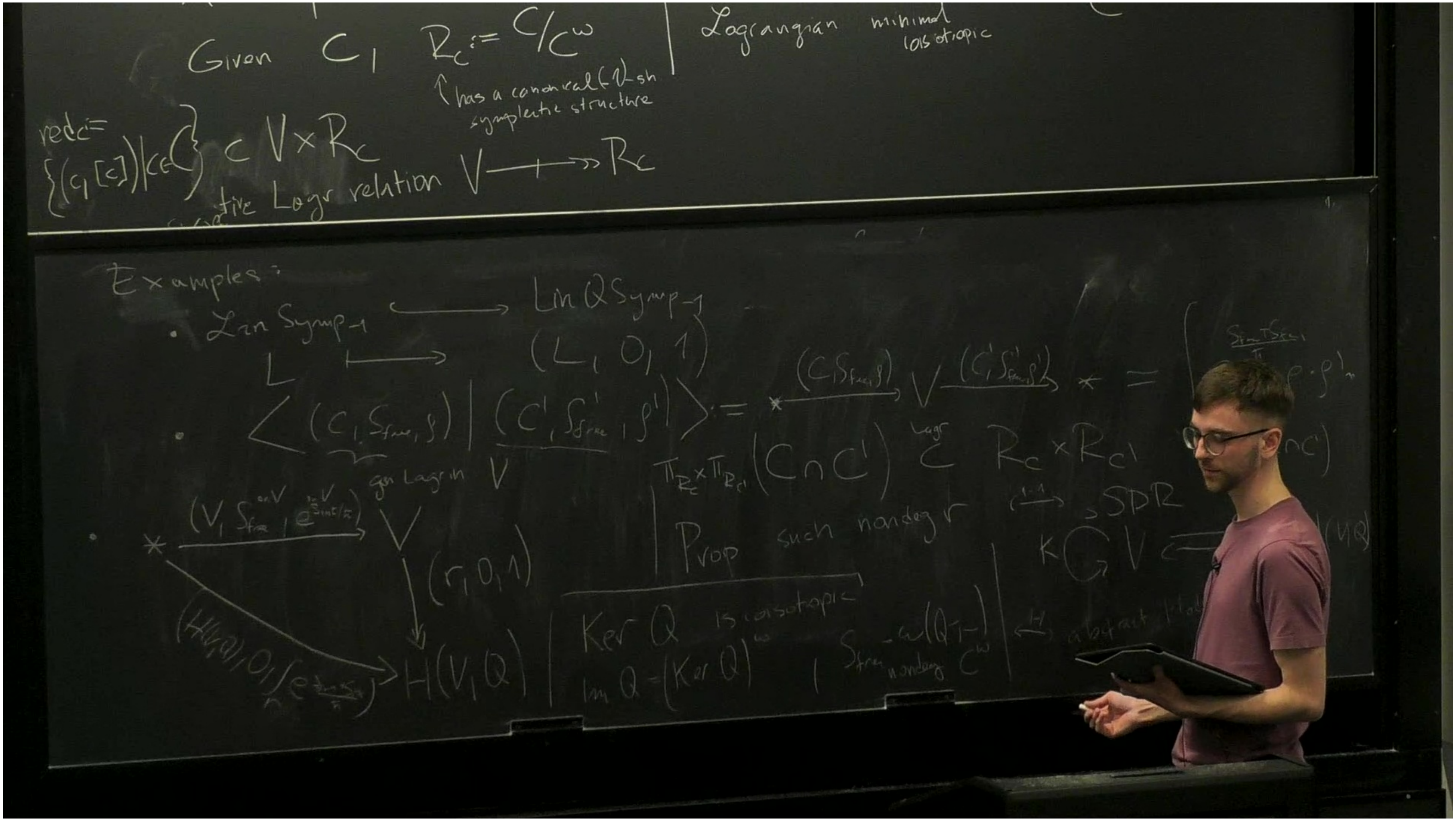
$$X_{C,C'} = \left\{ \left([(v_1, v_2)],\ [(v_2, v_3)],\ [(v_1, v_3)] \right) \;\middle|\; (v_1, v_2) \in C,\ (v_2, v_3) \in C' \right\}$$

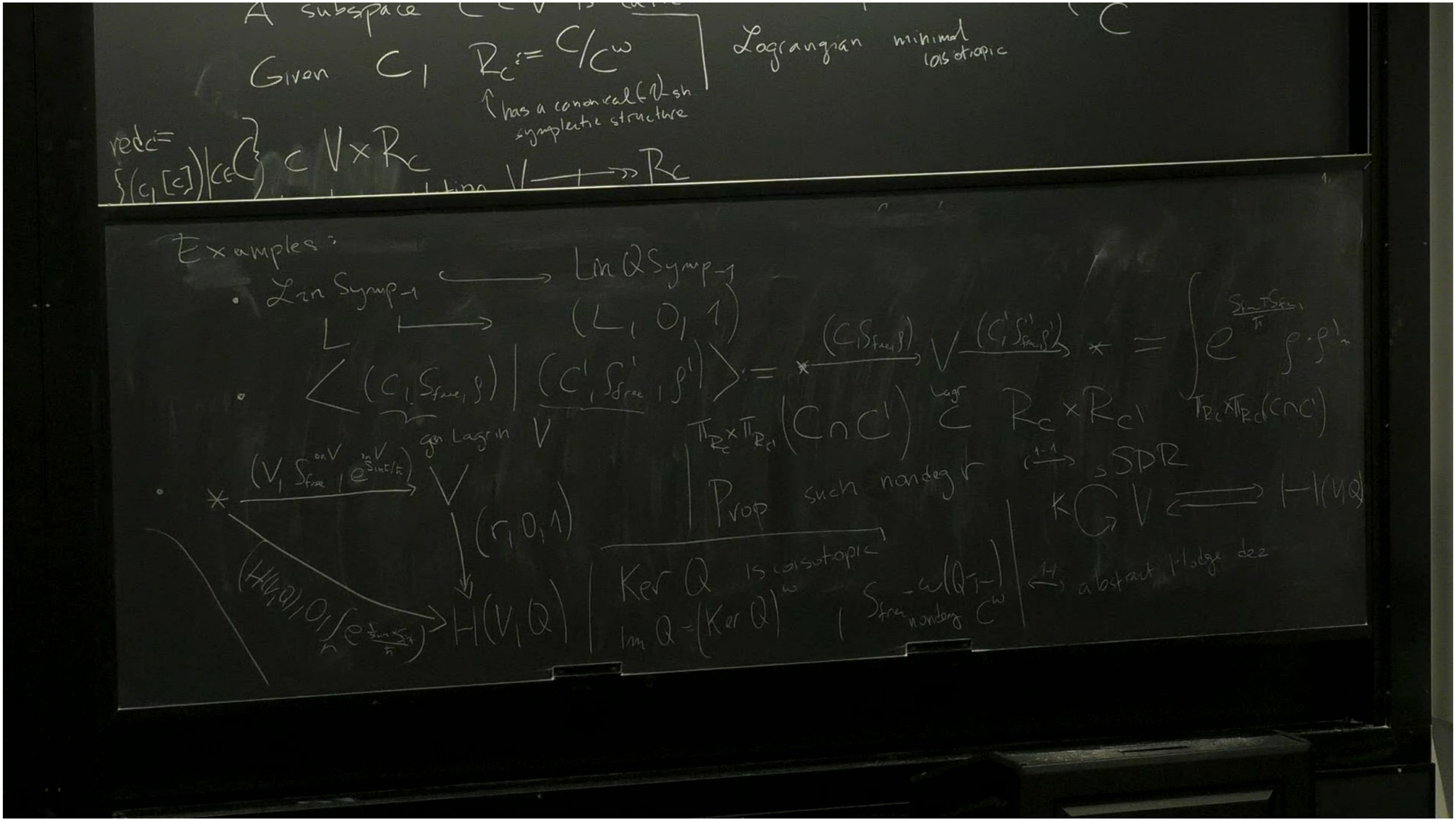
→ define $\sigma = \int_{X_{C,C'}} e^{\frac{S_{free} + S'_{free}}{\hbar}} \rho \otimes \rho'$ (if this converges; i.e. $S_{free} + S'_{free}$ is nondeg on $\operatorname{Ker} X_{C,C'}$)

• $X_{C,C'}$ is a lax structure on the 2-functor $\mathrm{Lin\,Coiso}_{-1} \longrightarrow B\,\mathrm{Lin\,Symp}_{-1}$

$V \longmapsto \bullet$

$C \longmapsto R_C$


Given C, R_C := C/C^ω
has a canonical (?) symplectic structure
Lagrangian minimal coisotropic
red_C := {(c, [c]) | c ∈ C} ⊂ V × R_C
Lagr relation V ⟶ R_C
Examples:
Lin Symp ⟶ Lin Q Symp
L ⟼ (L, 0, 1)
gen Lagrn
P_vop such nondeg
Ker Q is coisotropic
Im Q = (Ker Q)^ω
SDR
abstract


A subspace
Given C, R_C := C/C^ω
has a canonical symplectic structure
Lagrangian
minimal
coisotropic
Examples:
Lin Symp_1
Lin Q Symp_1
(L, 0, 1)
gen Lagrn
Lagr
Prop
such nondeg
SDR
Ker Q is coisotropic
Im Q = (Ker Q)^ω
nondeg
abstract Hodge dec

Def $\mathrm{Lin}\,\mathrm{QSymp}_{-1}$ is the (partial) category

- objects are (-1)-sh symplectic vector spaces (V,ω)
- morphisms $V_1 \to V_2$ are generalized Lagrangians in $\overline{V_1} \times V_2$

$$V_1 \xrightarrow{(C, S_{free}, \rho)} V_2 \xrightarrow{(C', S'_{free}, \rho')} V_3$$

the composition is given by $\left(C' \circ C,\ \text{ignore},\ \sigma \in \mathrm{Dens}^{\frac{1}{2}}(R_{C' \circ C})\right)$ made out of $\rho^{(1)} \in \mathrm{Dens}^{\frac{1}{2}}(R_{C^{(1)}})$

$e^S \quad C$

$$* \xrightarrow{(V,\rho)} V \xrightarrow{(V,\rho')} * \qquad \int_V \rho\rho'$$

$$* \xrightarrow{(L,0)} \qquad \int_L \rho'$$

$$\frac{\ldots}{\mathrm{Maps}(T[1]M, X)_{BF^{(1)}V}}$$

→ define $\sigma = \int_{X_{C,C'}} e^{\frac{\ldots}{\hbar}} \rho \otimes \rho'$ (if this converges, i.e. $S_{free} \cdot S'_{free}$ is nondeg on $\mathrm{Ker}\, X_{C,C'}$)

- $X_{C,C'}$ is a lax structure on the 2-functor $\mathrm{Lin}\,\mathrm{Coiso}_{-1} \longrightarrow B\,\mathrm{Lin}\,\mathrm{Symp}_{-1}$

$$V \longmapsto *$$
$$C \longmapsto R_C$$

$* \quad V$