Title: SAGBI bases and mirror constructions for Kronecker moduli spaces

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Series: Mathematical Physics

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Abstract: One way of constructing mirror partners to Fano varieties is via toric degenerations. The case in which this is best understood is the Grassmannian, using the well-known SAGBI basis of the Plucker coordinate ring indexed by semi-standard Young tableaux (SSYT). The mirror construction goes back to work of Eguchi--Hori--Xiong, however its geometry and combinatorics still plays an important role in current mirror constructions. In this talk, I will give an overview of this story, then turn to the question of what can be generalized for Kronecker moduli spaces. Like Grassmannians (which they generalize), Kronecker moduli spaces are high Fano index Picard rank 1 smooth Fano varieties. I will introduce linked SSYT pairs, which play the analogous role of SSYT for Grassmannians in understanding the coordinate ring of the Kronecker moduli space. This is joint work with Liana Heuberger.

Zoom link

SAGBI bases of Cox rings

toric degenerations,

and mirror symmetry

Grassmannians & Kronecker moduli

(jt work with L. Heuberger)

Plucker coordinates, straightening laws

SSYT

Gelfand–Cetlin polytope

ladder diagrams

EHX mirror

Plucker coordinate mirror

§1. Construction of Kronecker moduli

$\rightarrow$ quiver moduli $\{1, \ldots, \rho\}$

Let $Q = (Q_0, Q_1)$ be an acyclic directed graph

vertices, arrows

$s, t : Q_1 \to Q_0$.

quiver

Dimension vector $r = (r_1, \ldots, r_\rho)$

Let $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}})$.

$G = \prod_{i=1}^{p} GL(r_i)$ acts by change of basis

Note: not effective

$\lambda_0 : \mathbb{C}^* \xrightarrow[\text{diagonal}]{} G$

Want to study $V /\!/ G$ ← GIT quotient

Need a stability condition $\theta \in G$, $\langle \theta, \lambda_0 \rangle = 0$

quiver moduli space $V /\!/_\theta G = M_\theta(Q, r)$

King: If $V^s_\theta = V^{ss}_\theta$ and r is indivisible

then this a fine moduli space for θ stable reps of Q

Assume V^{us} has codim ≥ 2

SAGBI bases of Cox rings, toric degenerations, and mirror symmetry

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Kronecker quivers

$$\underset{r_1}{\bullet} \xrightarrow{\;n\;} \underset{r_2}{\bullet} \qquad r=(r_1,r_2), \quad r_1<r_2$$

$\gcd(r_1,r_2)=1$.

Up to a positive scalar, there's only one interesting stability condition

$$\theta = (-r_2, r_1)$$

Define $K^n_{r_1 r_2} = V/\!/_\theta$.

Facts:

- all assumptions $\longrightarrow$ hold
- $K^n_{r_1 r_2}$ is a smooth Picard rank 1 Fano variety of dim $n r_1 r_2 - r_1^2 - r_2^2 + 1$ and Fano index n.

Grassmannians: $r_1 = 1$:

$$V = \bigoplus_{i=1}^{n} \mathrm{Mat}(r_2, 1)$$

$$= \mathrm{Mat}(r_2 \times n)$$

$V^{ss} = V^{s}$ = full rank matrices

$V /\!/ G$ = rank r_2 quotients of $\mathbb{C}^n$ $= Gr(r_2, n)$

= rank r_2 subspaces of $(\mathbb{C}^n)^\vee$

quiver moduli space $V /\!/_\theta G = M_\theta(Q, r)$

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Cox rings of quiver moduli spaces

$\mathrm{Cox}(X)$ = generated by all $s \in \Gamma(X, L)$
of all line bundles

$\rightarrow$ generalization: homogeneous coord ring of a toric variety

Line bundles $\quad \alpha \in \chi(G) \quad \langle \alpha, \lambda_c \rangle = 0$

$L_\alpha = V^{ss} \times \mathbb{C}/G$ $\leftarrow$ line bundle

$\downarrow$

V^{ss}/G

s section of trivial line bundles that is G-equivariant $\rightsquigarrow s \in \Gamma(L_\alpha)$.

Defn: A polynomial $f \in \mathbb{C}[V]$
is a semi-inv of weight α
if $\forall g \in G$
$f(g.v) = \theta(g) s(v)$

$SI^G_\alpha(V) = \{$ sem-inv of weight $\alpha \}$

$SI^G(V) = \bigoplus_\alpha SI^G_\alpha(V)$ = semi-invariant ring

Under assumptions
Thm (Hausel, Fisac-Rapinda(?), Schiffmann(?)) $\mathbb{C}\alpha(V/\!/G) \simeq SI^G(V)$

Semi-invariants of $V = Rep(Q, r)$

Schofield, Derksen-Weyman, Domokos-Zubkov
~ skew tableau

BI bases of Cox rings,

degenerations,

mirror symmetry

Grassmannians & Kronecker moduli

(with L. Heiberger)

Let $T \subset G$ be the max diagonal torus

$$\mathcal{X}(G) = \mathcal{X}(T)^W$$

$$f \in SI^G_\alpha(V) \qquad \alpha \in \mathcal{X}(G)$$

then every monomial of f is an element of $SI^T_\alpha(V)$

Goal: work backwards

monomial in $SI^T_\alpha(V)$, $\alpha \in \mathcal{X}(T)^W$ $\xrightarrow[\text{symmetrize}]{}$ $f \in SI^G_\alpha(V)$

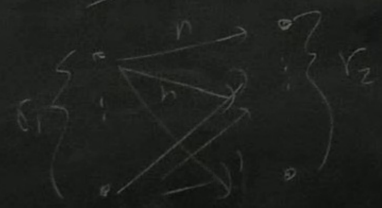
$G \curvearrowright V$

$T \curvearrowright V \rightsquigarrow V/\!/_\theta T$ ← quiver moduli space

$M_\theta(Q^{ab}, (1,\dots,1))$

Q

e.g. $r_1 \xrightarrow{n} r_2$



monomial $m \in SI^T_\alpha(V)$ $\alpha \in \mathbb{Z}(T)^\vee$

↕

collection of arrows in Q^{ab} s.t. outdeg − indeg is constant along lifts

↑

linked pair of tableaux ← determinants for $K^n_{r_1 r_2}$

BI bases of Cox rings

degenerations,

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Grassmannians & Kronecker moduli

(with L. Heuberger)

Def'n: Fix n, r_1, r_2 $\quad \alpha = \begin{pmatrix} -\alpha_1 \\ \alpha_2 \end{pmatrix}$ satisfying $\alpha_1 r_1 = \alpha_2 r_2$

A linked pair of tableaux of weight α is a pair of tableaux T^+, T^- and a link σ

$$T^+ = r_2 \underbrace{\overbrace{\boxed{\quad \boxed{ij} \quad}}^{\alpha_2}}_{\substack{1 \le i \le n \\ 1 \le j \le r_1}} \qquad T^- = r_1 \underbrace{\overbrace{\boxed{\quad \boxed{ik} \quad}}^{\alpha_1}}_{\substack{1 \le i \le n \\ 1 \le k \le r_2}}$$

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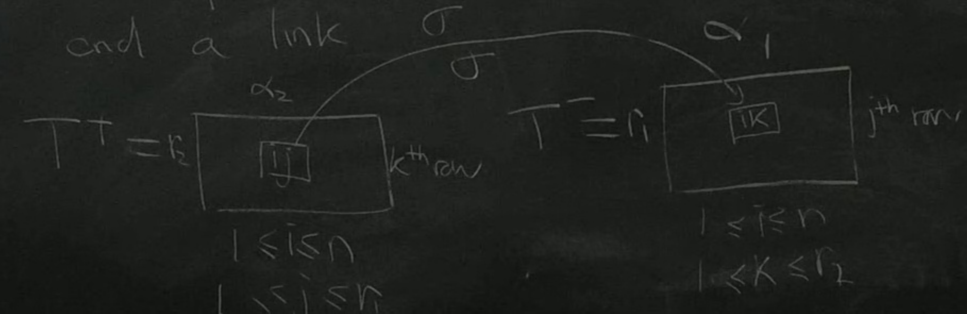

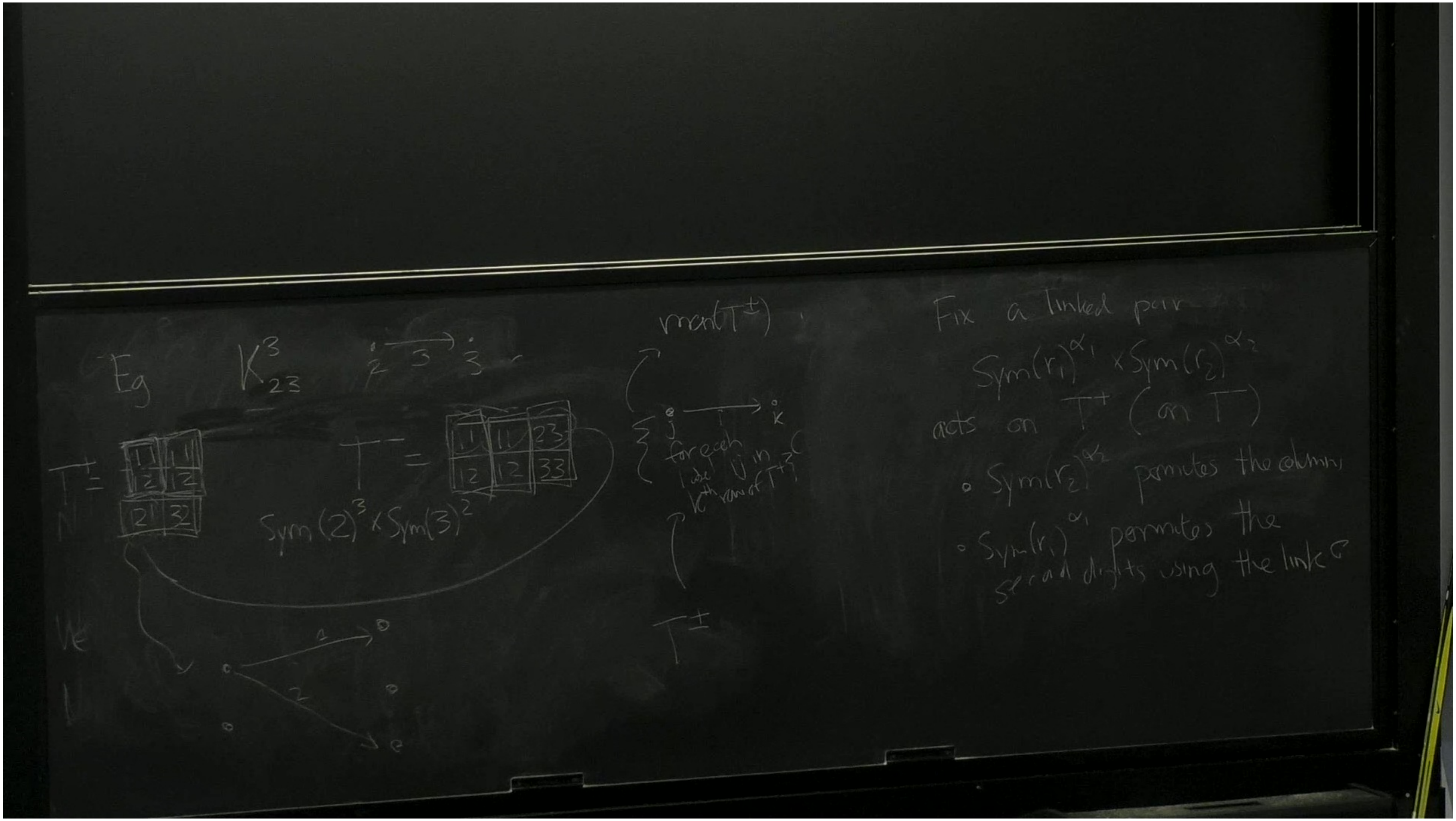
Eg
Sym(2)^3 × Sym(3)^2
Fix a linked pair
Sym(r1)^α1 × Sym(r2)^α2
acts on T+ (on T−)
Sym(r2)^α2 permutes the columns
Sym(r1)^α1 permutes the second digits using the link

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(with L. Herberger)

Def'n: To a linked pair $(T^{\pm}, \sigma)$

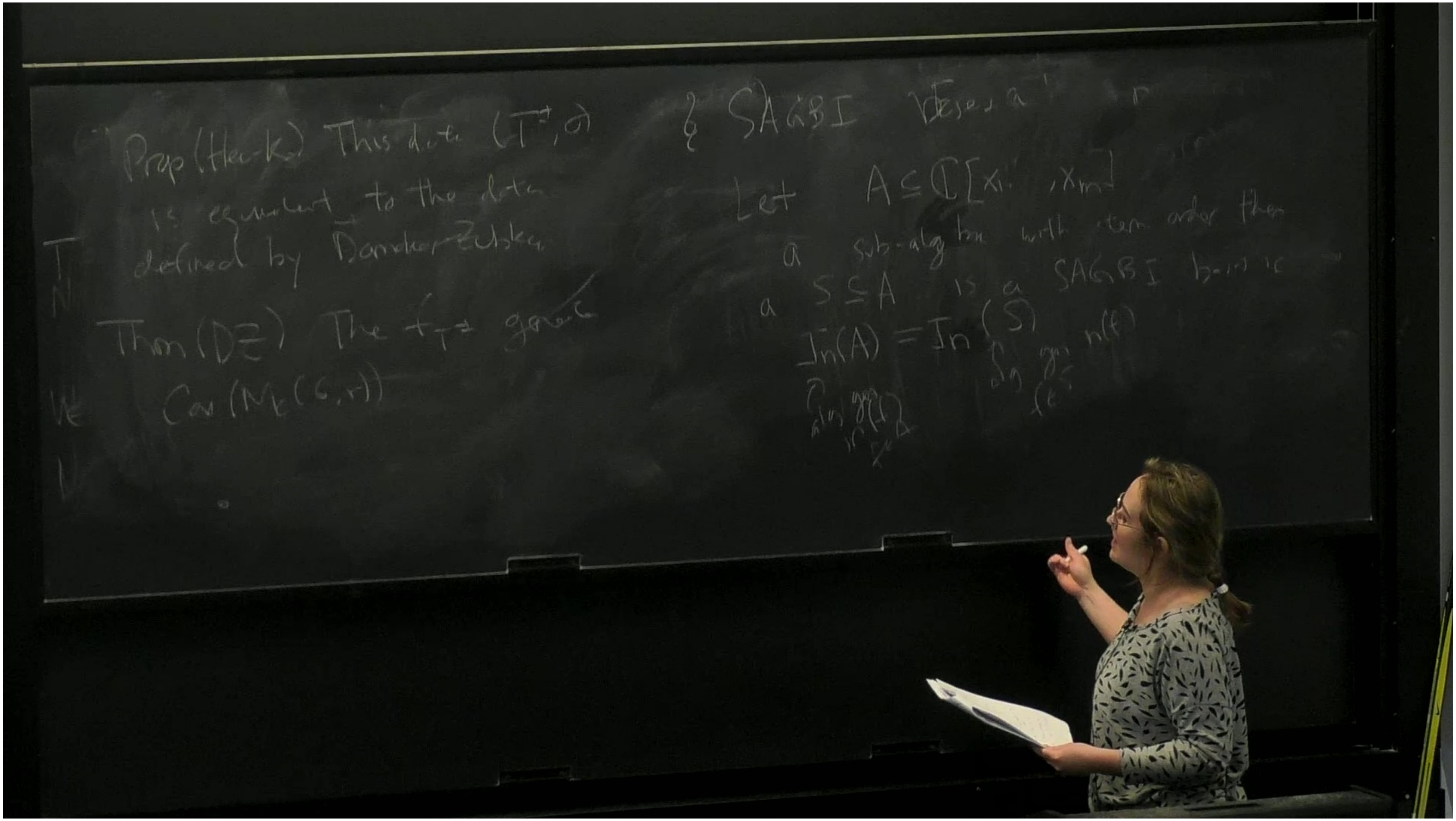
$$f_{T^{\pm}, \sigma} = \sum_{\tau \in \mathrm{Sym}(r_1)^{\alpha_1} \times \mathrm{Sym}(r_2)^{\alpha_2}} \mathrm{sign}(\tau)\, \mathrm{mon}(\tau \cdot T^{+})$$

e.g. $Gr(r_2, n)$

$T_{+} =$ [111 … / … / r_1]

$f_{T^{\pm}} = i_1 \cdots i_r$ minor of matrix of vars of size $r_2 \times n$

$\to$ Plücker coordinates


SAGBI
Let
is equivalent to the data
defined by
Thm
In(A) = In(S)

Thm (DZ) The $f_{T^\pm}$ generate $\mathrm{Cox}(M_t(G,r))$

a $S \subseteq A$ is a SAGBI ...

$\mathrm{In}(A) = \mathrm{In}(S)$

alg gen $\mathrm{in}(f)$ $f \in A$; alg gen $\mathrm{in}(f)$ $f \in S$

Semi-std tableau (weakly inc, str inc)

For $Gr(r,n)$, if T^+ is S.S, then $f_T = \sum c_i f_{T_i}$ (T_i S.S of lower order)

Key step → Plücker coord are a SAGBI basis for $\mathrm{Cox}(Gr(r,n))$

SAGBI basis ⇝ toric degeneration ⇝ Gelfand-Cetlin toric degeneration (Gonciulea-Lakshmibai, Batyrev-Ciocan-Fontanine-Kim-van Straten)

Thm (Hu-K). $f_{T^\pm}$ (for any $M_t(G,n)$) satisfy straightening inherited from the Grassmannian

For $Gr(r,n)$

$$f_{\square\square} = f_{\square} \cdot f_{\square} \quad \leftarrow \text{does not work for } K^n_{r,n}$$

Def'n: A SS tableux pair $T^{\pm}$ is primitive if there are no SS pairs $T_1^{\pm}, T_2^{\pm}$ s.t.

$$\mathrm{mon}(T_1^{\pm})\,\mathrm{mon}(T_2^{\pm}) = \mathrm{mon}(T^{\pm})$$

For $r_1 = r_2 = 2$, only finitely many

Thm: There are only finitely many primitive SS tableux pairs.

(Cor $\to$) SAGBI basis

mirror symmet

...assmannians & k ... moduli

... with L. He

$\tau \in \mathrm{Sym}(r_1)^{\alpha_1} \times \mathrm{Sym}(r_2)^{\alpha_2}$

e.g. $Gr(r_2, n)$

$T_\tau = \boxed{\begin{matrix} i_1 \\ \vdots \\ i_{r_2} \end{matrix}}$

$f_{T_\tau} = i_1 \cdots i_r$ minor of matrix of vars of size $r_2 \times n$

$\to$ Plücker coordinates

e.g. K^3_{23} all primitives are of deg $\binom{-3}{2}$ $\to$ 20 of them

		#prim	# SST	
eg K^4_{23}	$\binom{-3}{2}$	126	126	$\to$ no more primitive; $\leadsto$ construct a toric degeneration
	$2\binom{-3}{2}$	86	4032	
	$3\binom{-3}{2}$	210	59268	

Def'n: A SS tableaux pair $T^{\pm}$ is primitive if there are no SS pairs $T_1^{\pm}, T_2^{\pm}$ s.t.

$$\mathrm{mon}(T_1^{\pm})\,\mathrm{mon}(T_2^{\pm}) = \mathrm{mon}(T^{\pm})$$

primitive SS tableaux pairs (Cor.) → SAGBI basis

Why toric degenerations?

$K^n_{r_1 r_2}$

$Gr(r_1, nr_2)$ $Gr(r_2, nr_1)$

Motto: extend the combinatorial control for toric varieties to the $\mathbb{R}^{nr_1r_2}$ spaces

1) Integrable system ~ $K^n_{r_1r_2}$ ~ integrable system whose image is the intersection of the GC polytopes of the 2 Grassmannians

$f_{T_1^{\pm}} f_{T_2^{\pm}}$

$\boxed{T_1 | T_2}$

2) Mirror symmetry

eg K^3_{23} → Laurent poly

(Batyrev) mirror

→ checked first 26 terms of the period sequence are correct

$K^n_{r_1r_2} = Gr(r_1, r_2^n)//GL(r_2)$

Harada–Kaveh