Title: QFT III Lecture

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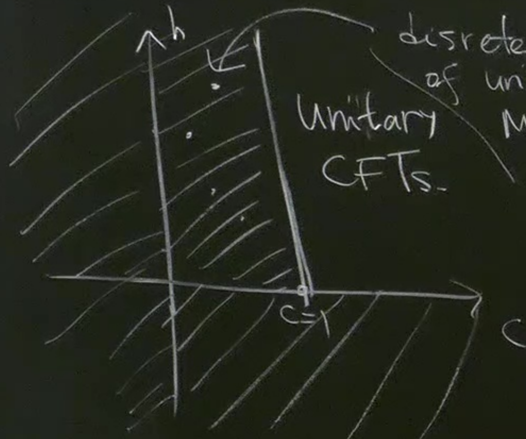
Minimal models & Ising

Last time: unitarity



discrete series of unitary Minimal Models.

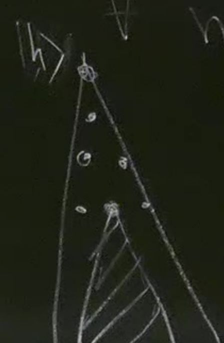
Ising CFT $c = 1/2$.

$\forall\, |\chi\rangle \neq 0 \in \mathcal{H}: \ \langle\chi|\chi\rangle > 0$

$$\mathcal{H} = \bigoplus_{(h,\bar{h})} V_{h,c} \otimes V_{\bar{h},c}$$

How minimal models occur?

This happen because of singular vector!



$\forall\, n > 0 \quad L_n|\chi\rangle = 0 \quad$ for $|\chi\rangle \in V_{h,c}$.

$$\Downarrow$$

$$\langle\chi|\chi\rangle = 0$$

CFT

- Singular vectors are orthogonal to the whole $V_{h,c}$

$$\langle \lambda, h | \chi \rangle = \langle h | L_{\lambda_1} \ldots (L_{\lambda_n} |\chi\rangle) = 0 \rightsquigarrow S_{\lambda,\mu} = \langle \lambda, h | \mu, h \rangle$$

$$|\chi\rangle = \sum c_\mu |\mu, h\rangle \Rightarrow \sum_\mu S_{\lambda,\mu} c_\mu = 0$$

$\Rightarrow \det S^{(\ell)} = 0$ if $|\chi\rangle$ of level ℓ.

- The singular vector gives rise to Verma-submodule $V_{h+\ell,c} \subset V_{h,c}$.

since $L_n|\chi\rangle = 0 \ \forall n > 0$, there are descendants $L_{-\lambda_n} \ldots L_{-\lambda_1}|\chi\rangle$.

- Moreover, every vector of this submodule is $\perp$ to the whole Verma module.

$$\langle \mu, h | (L_{-\lambda_n} \ldots L_{-\lambda_1}|\chi\rangle) = \langle h | L_{\mu_m} \ldots L_{\mu_1} L_{-\lambda_n} \ldots L_{-\lambda_1} |\chi\rangle = \sum \ldots \langle h | \ldots \underset{k>0}{(L_k|\chi\rangle)} = 0$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} \ldots$$

$$\Rightarrow \sum_i \lambda_i + \ell = \sum_j \mu_j \Rightarrow \sum \mu_i - \sum \lambda_i > 0$$

This is also helpful to explain Kac determinant formula

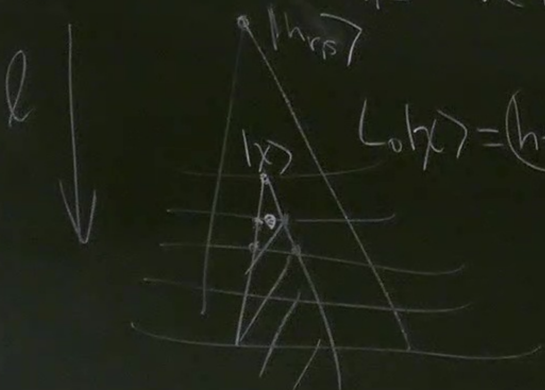
$$\det S^{(\ell)} = A_\ell \prod_{\substack{r,s\geq 1 \\ r\cdot s \leq \ell}} [h - h_{r,s}(c)]^{p(\ell - rs)}$$

$p(0)=1$
$p(1)=1$
$p(2)=2$

where $h_{r,s} = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$

when $h = h_{r,s}$ we have singular on level $r\cdot s$

$h_0 = \frac{1}{24}(c-1)$, $\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$



$L_0|\chi\rangle = (h + r\cdot s)|h_{r,s}\rangle$

By quotiening singulars and their desc. out of Verma modules we get irreducible factors $M_{h,c} = V_{h,c}/(\text{singular submod})$

Surprsingly, "decoupling condition", is very restrictive for C_{h,h_1,h_2}!

First non-trivial example is at $h = h_{2,1}$, $V_{h_{2,1},c} \ni |\chi\rangle = \left(L_{-2} - \frac{3}{2(h+1)}(L_{-1})^2\right)|h_{2,1}\rangle$

$L_n|\chi\rangle = 0 \quad \forall\, n > 0.$

In terms of fields: $\chi(z) = (\underset{\shortparallel}{L_{-2}}\phi)(z) - \frac{3}{2(2h+1)}\partial_z^2\phi(z)$, where $\phi(z)$ corresp. to $|h_{2,1}\rangle$

$\oint_z (w-z)^0 T(w)\phi(z)$

$\langle \chi(z)\, X\rangle = 0$ $\qquad X = \phi_1(z_1)\cdots\phi_n(z_n)$

$$\Rightarrow (*) \left\{\sum_{i=1}^{n}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right\}\langle \phi(z)\, X\rangle = 0$$

- If we pick 2pt function $\langle\phi(z)\phi(z_1)\rangle$ – there is no extra constraints.
- For 3pt function $\langle\phi(z)\,\phi_1(z_1)\,\phi_2(z_2)\rangle = \frac{C_{h,h_1,h_2}}{(z-z_1)^{h+h_1-h_2}(z-z_2)^{h+h_2-h_1}(z_1-z_2)^{h_1+h_2-h}}$

$(*)$ gives: $2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1)$ or $C_{h,h_1,h_2} = 0$

It can be solved by $h_2 = \frac{1}{6} + \frac{1}{3}h + h_1 \pm \frac{2}{3}\sqrt{h^2 + 3hh_1 - \frac{1}{2}h + \frac{3}{2}h_1 + \frac{1}{16}}$

Using nice param.: $h(\alpha) = \frac{1}{24}(c-1) + \frac{1}{4}\alpha^2 \Rightarrow$

$$\alpha_2 = \alpha_1 \pm \alpha_+ \quad \text{for} \quad h = h_{2,1}$$
$$\alpha_2 = \alpha_1 \pm \alpha_- \quad \text{for} \quad h = h_{1,2}.$$

For OPEs it can be drawn as

$$\phi_{(2,1)} \times \phi_{(\alpha)} = \phi_{(\alpha-\alpha_+)} + \phi_{(\alpha+\alpha_+)}$$
$$\phi_{(1,2)} \times \phi_{(\alpha)} = \phi_{(\alpha-\alpha_-)} + \phi_{(\alpha+\alpha_-)}$$

fusion rules.

(**) gives: $2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1)$ or $C_{h h_1 h_2} = 0$

general formula: $$\phi_{(r,s)} \times \phi_{(\alpha)} = \sum_{\substack{k=1-r \\ k+r \equiv 1 \bmod 2}}^{k=1+r} \; \sum_{\substack{l=1-s \\ l+s \equiv 1 \bmod 2}}^{l=1+s} \phi_{(\alpha + k\alpha_+ + l\alpha_-)}$$

$h_{r,s}$

This have nice consequences.

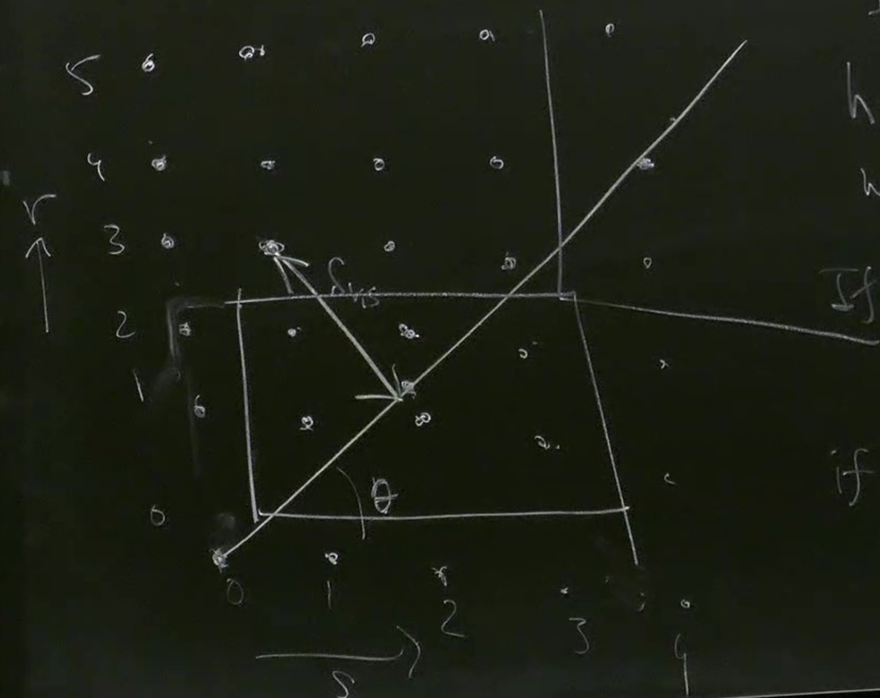
- Primaries with dim $h_{r,s}$ form a closed operator algebra.

$$\phi_{(1,2)} \times \phi_{(r,s)} = \phi_{(r,s-1)} + \phi_{(r,s+1)} \quad , \quad \phi_{(r,s)} = \phi_{(\alpha_+ r + \alpha_- s)}$$

- Commutativity of fusion rules restricts them.

$$\phi_{(1,2)} \times \phi_{(2,1)} = \phi_{(2,0)} + \underline{\phi_{(2,2)}}$$
$$\phi_{(2,1)} \times \phi_{(1,2)} = \phi_{(0,2)} + \underline{\phi_{(2,2)}} \quad \Rightarrow \quad \phi_{(1,2)} \times \phi_{(2,1)} = \phi_{(2,2)}$$

Minimal models are models where degenerate fields form closed and <u>finite</u> operator algebra.



$$\tan\theta = -\frac{\alpha_+}{\alpha_-}(c)$$

$$h_{r,s} = h_0 + \frac{1}{4}\delta^2(\alpha_+^2 + \alpha_-^2)$$

wher δ is a distance between line and (r,s)

If $\tan\theta \notin \mathbb{Q} \Rightarrow$ we have ∞ number of different $h_{r,s}$, arbitrarill cose to h_0

if $0 < R < 1 \Rightarrow h_0 < 0 \Rightarrow h_{r,s} < 0$ for (r,s)
$\Rightarrow$ unitarity is broken.

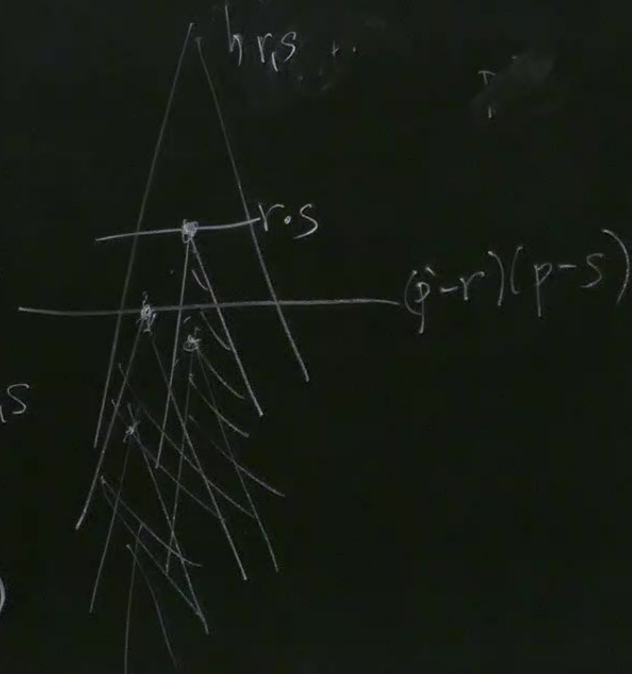
Let t

- $h_{r,s} =$
- $h_{r,s} +$
- $h_{r,s} +$

$\Rightarrow$

h

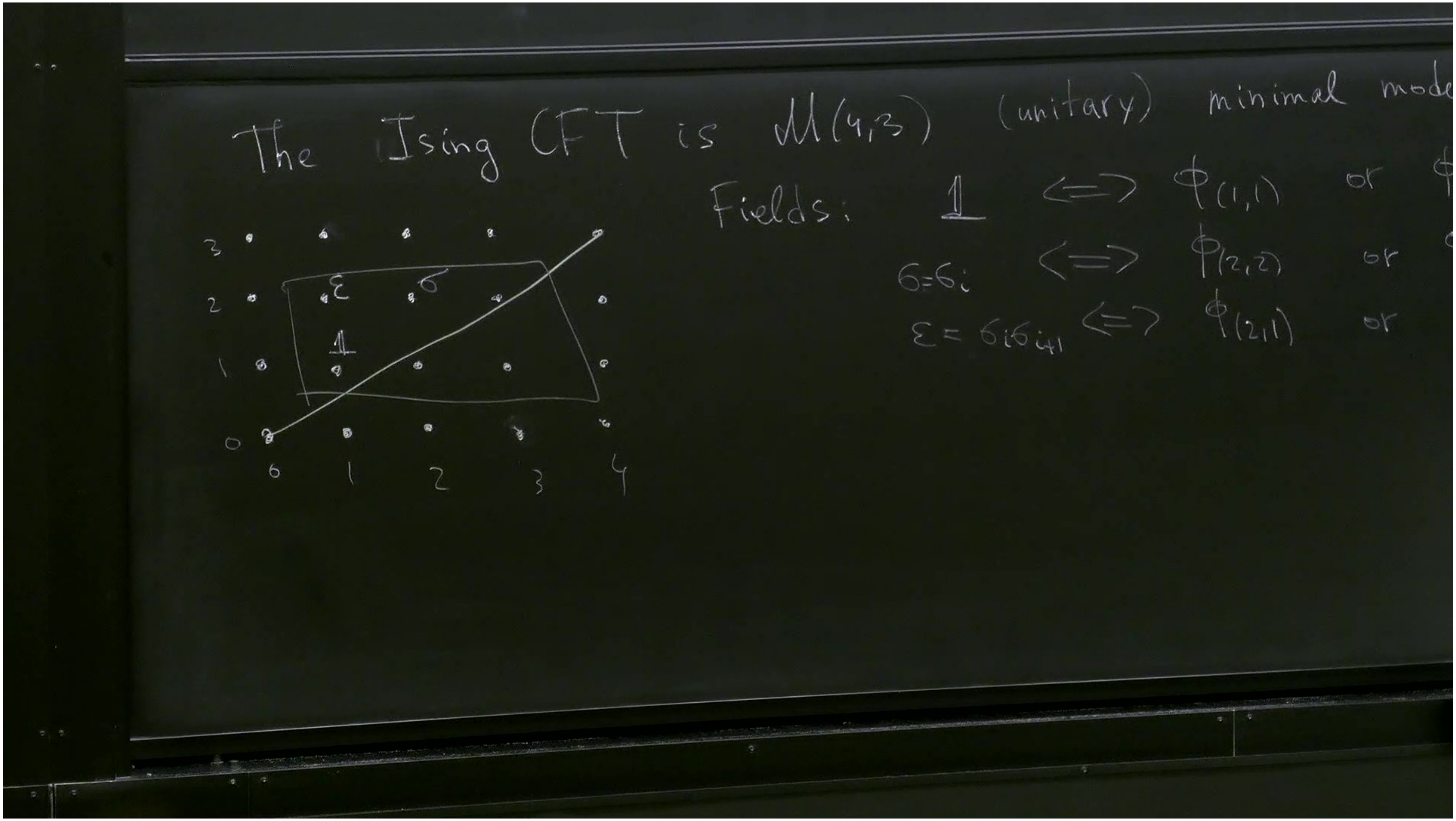
Let $\tan\theta = \frac{p}{p'} \in \mathbb{Q}$, $p > p'$ $\Rightarrow$ $c = 1 - 6\frac{(p-p')^2}{pp'}$, $h_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$



- $h_{r,s} = h_{r+p', s+p}$
- $h_{r,s} + r\cdot s = h_{p'+r, p+s} = h_{p'-r, p+s}$
- $h_{r,s} + (p'-r)(p-s) = h_{r, 2p-s} = h_{2p'-r, s}$

$\Rightarrow$ $\left.\begin{array}{l} 1 \le r < p' \\ 1 \le s < p \end{array}\right\} \Rightarrow$ we have $\frac{(p-1)(p'-1)}{2}$ independent fields

$h_{r,s} = h_{p'-r, p-s}$


The Ising CFT is M(4,3) (unitary) minimal mode
Fields: 1 ⟺ φ(1,1) or
σ=σi ⟺ φ(2,2) or
ε = σiσi+1 ⟺ φ(2,1) or
ε σ
1
3
2
1
0
0 1 2 3 4

(4,3) (unitary) minimal model, $\boxed{c = 1/2}$.

$\mathbb{1} \Longleftrightarrow \phi_{(1,1)}$ or $\phi_{(2,3)}$

$\sigma = \sigma_i \Longleftrightarrow \phi_{(2,2)}$ or $\phi_{(1,2)}$

$\varepsilon = \sigma_i \sigma_{i+1} \Longleftrightarrow \phi_{(2,1)}$ or $\phi_{(1,3)}$

$(h,\bar{h})_\sigma = \left(\frac{1}{16}; \frac{1}{16}\right) \Rightarrow \Delta_\sigma = \frac{1}{8}$

$(h,\bar{h})_\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \Delta_\varepsilon = 1$

$y_\sigma = 2 - \Delta_\sigma = \frac{15}{8}$

$y_\varepsilon = 2 - \Delta_\varepsilon = 1$

$\sigma \times \sigma = \mathbb{1} + \varepsilon$

$\sigma \times \varepsilon = \sigma$

$\varepsilon \times \varepsilon = \mathbb{1}$

$\Rightarrow \boxed{\alpha = 0,\ \beta = 1/8,\ \gamma = 7/4,\ \delta = 15,\ \nu = 1,\ \eta = 1/4}$