Title: QFT III Lecture

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Unitarity in 2d CFTs and

Previously: conf. blocks $\rightsquigarrow$ correlation functions for generic c $(h, \bar{h})$

$Q = (S)^{-1}$, $S = \langle \chi, h | \mu, h \rangle$

minimal models: $\leftarrow$ $\det S = 0$

models with finite number of primaries.

breakdown of unitarity.

"naïvely"

e.g. Ising CFT.

in "good" QFT: $\langle \chi | \chi \rangle \leq 0$

and singular vectors

The Hilbert space of CFT: $\mathcal{H} = \bigoplus_{(h,\bar{h})} V_{h,c} \otimes V_{\bar{h},c}$ $\quad |\mu| = \sum_{i=1}^{n} \mu_i$

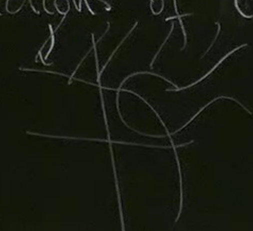
$S_{\lambda,\mu}(h,c) = \langle \mu, h | \lambda, h' \rangle \neq 0$ if $h = h'$; $|\mu| = |\lambda|$

level $|\mu| = |\lambda|$.

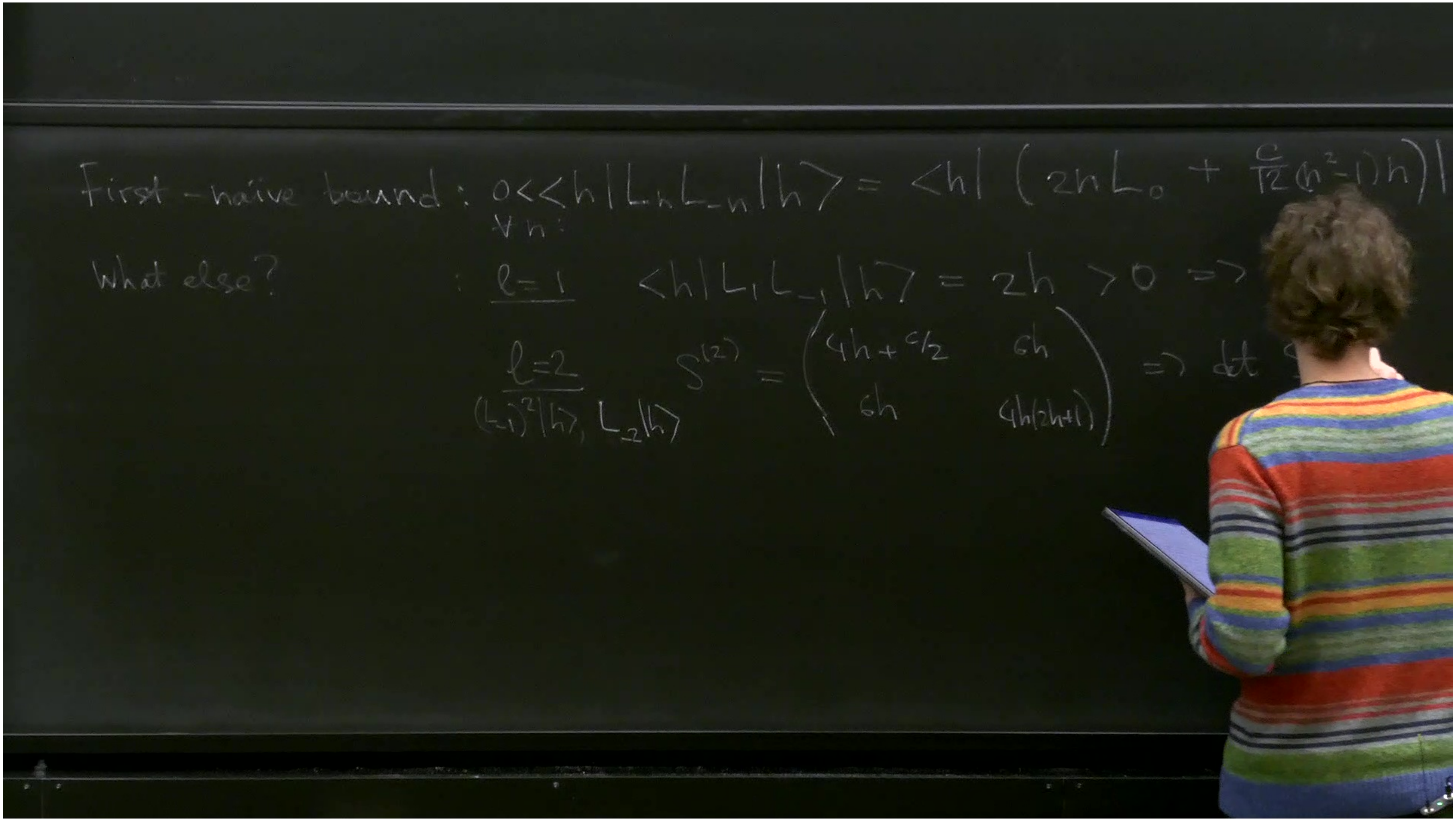
Existence of negative norm state $\Leftrightarrow$ existence of negative eigenvalue of $S^{(\ell)} = S|_{\text{level } \ell}$

$$S^{(\ell)} = U^\dagger \Lambda U, \quad \Lambda = \mathrm{diag}(\Lambda_1^{(\ell)}, \dots, \Lambda_{p(\ell)}^{(\ell)})$$

We can look for $\det S^{(\ell)}$ as functions of c and h. Any eigenvalue can change sign



$\det S^{(\ell)} = 0$


First - naive bound : $0 < \langle h|L_n L_{-n}|h\rangle = \langle h|\,(2nL_0 + \frac{c}{12}(n^2-1)h)\,|$
$\forall n$:
What else?
: $\ell = 1$ $\langle h|L_1 L_{-1}|h\rangle = 2h > 0 \Rightarrow$
$\ell = 2$ $S^{(2)} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \Rightarrow$ det
$(L_{-1})^2|h\rangle$, $L_2|h\rangle$

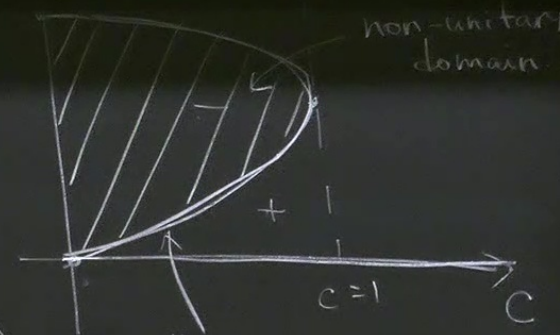
$h > 0$

$$D^{(2)} = \left(4h + \frac{c}{2}\right)\left(4h(2h+1)\right) - 36h^2 = h\left(32h^2 + 4h(c-5) + 2c\right) =$$

$$h\left(h - h_{1,2}(c)\right)\left(h - h_{2,1}(c)\right)$$

$$h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right)$$

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right)$$



there exists null state
i.e. State of zero norm.

$\frac{c}{12}(n^2-1)h\Big)|h\rangle = 2nh + \frac{1}{12}cn(n^2-1) \Rightarrow c>0.$

$\Rightarrow h>0$

$\det S^{(2)} = \left(4h+\frac{c}{2}\right)\left(4h(2h+1)\right) - 36h^2 = h\left(32h^2+4h(c-5)+2c\right) =$

$= h(h-h_{1,2}(c))(h-h_{2,1}(c))$

$h_{1,2} = \frac{1}{16}\left(5-c-\sqrt{(1-c)(25-c)}\right)$

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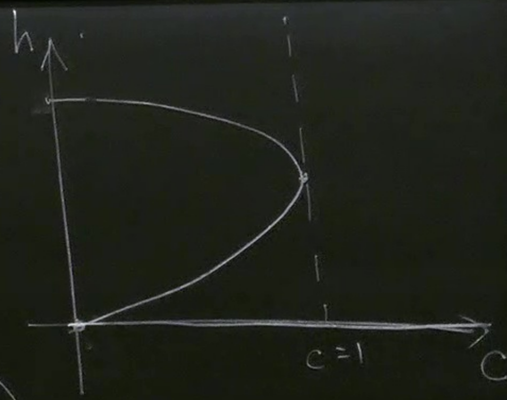

We can repeat the same for higher levels ℓ. apparentely,

$$\det S^{(\ell)} = A_\ell \prod_{\substack{r,s\geq 1 \\ r\cdot s\leq \ell}} \left[h - \underline{h_{r,s}(c)}\right]^{p(\ell - r\cdot s)} \quad \text{where}$$

there is general formula (Kac determinant formula):

- $p(n)$ is a number of Young diagrams with n boxes.
- The prefactor $A_\ell = \prod_{\substack{r,s\geq 1 \\ rs\leq \ell}} \left[(2r)!\, s!\right]^{p(\ell - rs) - p(\ell - r(s+1))}$
- $h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$, $\quad h_0 = \frac{1}{24}(c-1)$, $\quad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$

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where

• $p($

• The

• h

There are two qualitativelly different regions:

• $0<c<1$ – almost all points give non-unitar modules.

• $c>1$ – all modules are unitary.

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...modules.

$\underline{c>1}$ $\quad 1<c<25, \quad c>25.$ $\quad h_{r,s}(c) = \frac{1-c}{96}\{[(r+s)+(r-s)$

In the region $c>1$ there are no curves $C_{r,s} = \{h_{r,s}(c) -$

$\cdot\ 1<c<25$ $\quad h_{r,s}$ is real $\Leftrightarrow$ $r=s$: $h_{r,s}(c) = \frac{1-c}{96}\{(2r)^2 -$

$\cdot\ c>25$: $\sqrt{\frac{25-c}{1-c}} = k > 0$: $[(r+s)+(r-s)k]^2 = [r(1+k)+s(1$

$1>k>0$

$\Rightarrow$ if at any point of R module is unitary - it is unitary everywh

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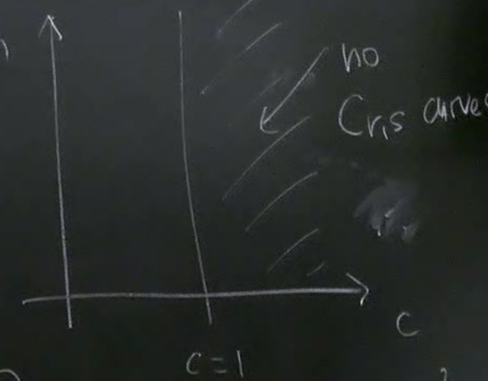
To see that theory is unitary in R – consider $h \to \infty$ limit.

$$= \left\{ \left[(r+s) + (r-s)\sqrt{\frac{25-c}{1-c}} \right]^2 - 4 \right\}$$

$$C_{r,s} = \{ h_{r,s}(c) - h = 0 \}$$

$$(c) = \underbrace{\frac{1-c}{96}}_{<0} \underbrace{\{ (2r)^2 - 4 \}}_{>0} \Rightarrow h_{r,s}(c) < 0$$

$$) k]^2 = [r(1+k) + s(1-k)]^2 \geq 4 \quad \text{for } r,s \in \mathbb{Z}_{\geq 1} \Rightarrow h_{r,s}(c) < 0$$



$$R = \{ h > 0,\ c > 1 \}$$

t is unitary everywhere in R

n → ∞ limit. There one can see that eigenvalues of $S^{(2)}$ are determined by diagonal matrix element.

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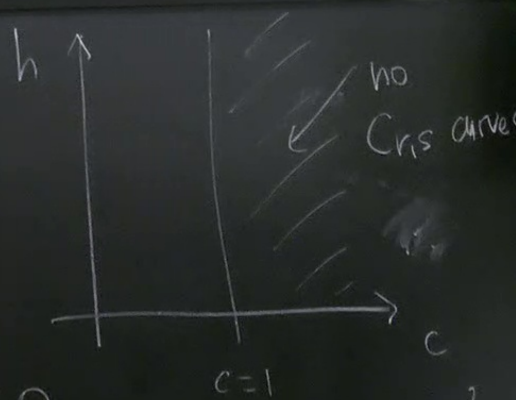

$R = \{ h > 0,\ c > 1 \}$

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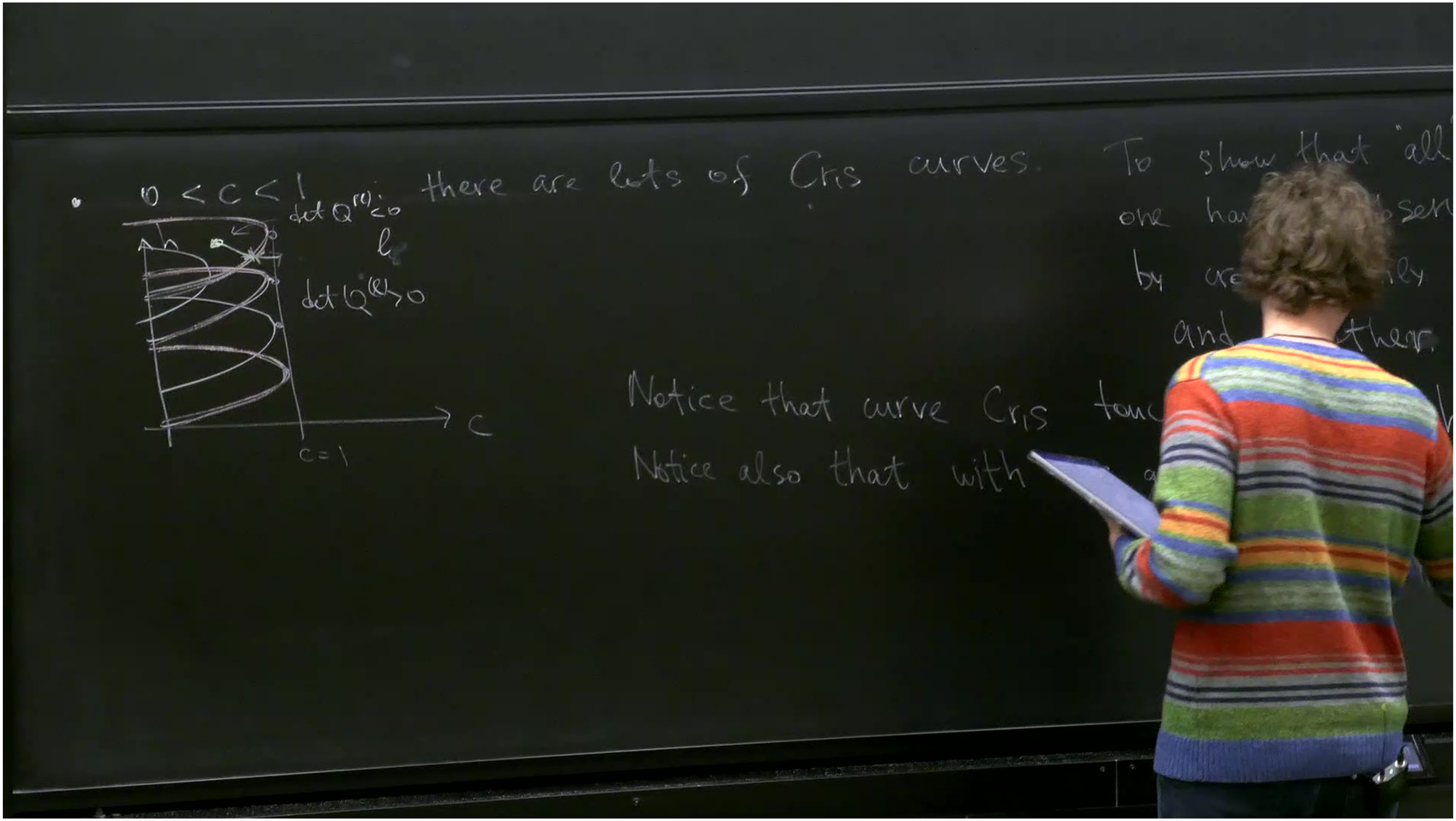
$n \to \infty$ limit. There one can see that eigenvalues of $S^{(2)}$ are determined by diagonal matrix elements, and they are positive e lots of $C_{r,s}$ curves. To show that "all" the points correspond to non-u

one have to observe that any (c,h) can be connec

by crossing only one $C_{r,s}$ contributing with o

and no other $C_{r,s}$ contributing to this $Q^{(\ell)}$,

- Notice that curve $C_{r,s}$ touches $c=1$ at $h_{r,s} = \frac{1}{4}(r-s)^2$.
- Notice also that with $r.s$ growing, the curve $C_{r,s}$ becomes more "wide" (and $r-s$ fixed)
- The multiplicity of $h_{r,s}-h$ at level $\ell = r.s$ is one.



0 < c < 1
there are lots of Cr,s curves.
To show that "all"
det Q(l) < 0
det Q(l) > 0
c
c = 1
Notice that curve Cr,s
Notice also that with

at "all" the points correspond to non-unitary theories

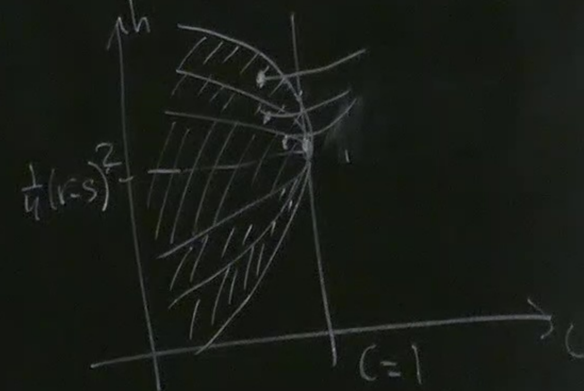
observe that any (c,h) can be connected to $c>1$ region

only one $C_{r,s}$ contributing with odd multiplicity to some $\det Q^{(\ell)}$,

other $C_{r,s}$ contributing to this $Q^{(\ell)}$,

at $h_{r,s} = \frac{1}{4}(r-s)^2$.

curve $C_{r,s}$ becomes more "wide"



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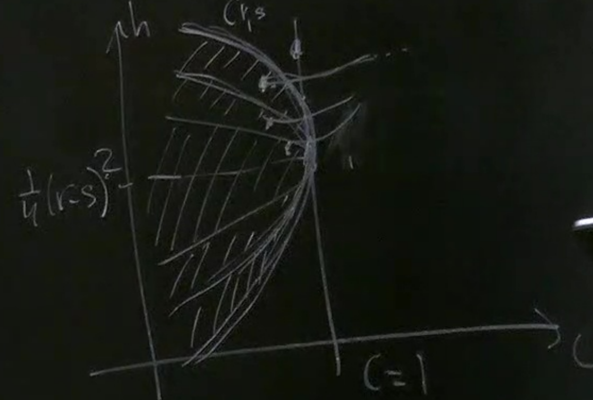
xed)

evel $\ell = r\cdot s$ is one, which is odd number.



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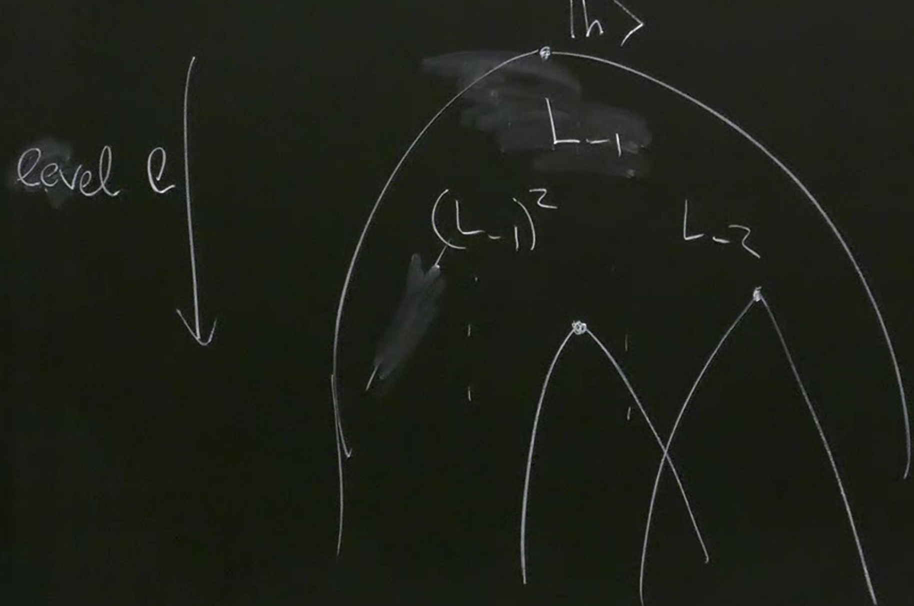
ules.

However, even in $0 < c < 1$ region there are p
$h > 1$

Instead of $V_{h,c}$, we



For $c = 1 - \dfrac{6}{m(m+1)}$, h

e are points, whose modules can be made unitary:

h,c , we take their quotients $M_{h,c} = V_{h,c} / (\text{singular vectors})$

) , $h_{r,s}(m) = \dfrac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$ $M_{h,c}$ are unitary. (unitary minimal models)

$1 \le r < m$, $1 \le s < r$